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SUFFICIENT CONDITIONS FOR INTEGRAL OPERATOR DEFINED BY BESSEL FUNCTIONS

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Abstract. The main object of the present paper is to derive several sufficient conditions for integral operator defined by Bessel functions of the first kind to be convex and strongly convex of given order in the open unit disk.

1. Introduction and definitions

Let $\mathscr{A}$ denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{D} = \{z : |z| < 1\}$. A function $f(z)$ belonging to $\mathscr{A}$ is said to be convex of order γ if it satisfies

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \gamma \qquad (z \in \mathbb{D})$$

for some $\gamma (0 \leqslant \gamma < 1)$. We denote by $\mathscr{C}(\gamma)$ the subclass of $\mathscr{A}$ consisting of functions which are convex of order γ in $\mathbb{D}$. If $f(z) \in \mathscr{A}$ satisfies

$$\left|\arg\left(1 + \frac{zf''(z)}{f'(z)} - \gamma\right)\right| < \frac{\pi}{2}\beta \quad (z \in \mathbb{D})$$

for some $\gamma (0 \leqslant \gamma < 1)$ and $\beta (0 < \beta \leqslant 1)$, then $f(z)$ is said to be strongly convex of order β and type γ in $\mathbb{D}$, and we denote by $\overline{\mathscr{C}}_\gamma(\beta)$ the class of such functions. It is clear that $\overline{\mathscr{C}}_0(\beta) \equiv \overline{\mathscr{C}}(\beta)$ the class of strongly convex of order β in $\mathbb{D}$ and $\overline{\mathscr{C}}_0(1) \equiv \mathscr{C}$ the class of all convex functions in $\mathbb{D}$.

The Bessel function of the first kind of order ν is defined by the infinite series

$$J_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{2n+\nu}}{n!\Gamma(n+\nu+1)},$$

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where Γ stands for the Euler gamma function, $z \in \mathbb{C}$ and $\nu \in \mathbb{R}$. Recently, Szász and Kupán [8] investigated the univalence of the normalized Bessel function of the first kind $g_\nu : \mathbb{D} \to \mathbb{C}$, defined by

$$g_\nu(z) = 2^\nu \Gamma(\nu+1) z^{1-\nu/2} J_\nu(z^{1/2}) = z + \sum_{n=1}^{\infty} \frac{(-1)^n z^{n+1}}{4^n n!(\nu+1)\dots(\nu+n)}. \tag{1.2}$$

Baricz and Frasin [1] have obtained various sufficient conditions for the univalence of the following integral operators defined by Bessel functions of the first kind:

$$F_{\nu_1,\dots,\nu_n,\alpha_1,\dots,\alpha_n,\beta}(z) = \left\{ \beta \int_0^z t^{\beta-1} \prod_{i=1}^{n} \left(\frac{g_{\nu_i}(t)}{t} \right)^{1/\alpha_i} dt \right\}^{1/\beta},$$

$$F_{\nu_1,\dots,\nu_n,\alpha,n}(z) = \left\{ (n\alpha+1) \int_0^z \prod_{i=1}^{n} (g_{\nu_i}(t))^\alpha \, dt \right\}^{1/(n\alpha+1)}$$

and

$$F_{\nu,\gamma}(z) = \left\{ \gamma \int_0^z t^{\gamma-1} \left(e^{g_\nu(t)} \right)^\gamma dt \right\}^{1/\gamma}.$$

In this paper we are mainly interested on some integral operators of the following types which involve the normalized Bessel function of the first kind:

$$F_{\nu_1,\dots,\nu_n,\alpha_1,\dots,\alpha_n}(z) = \int_0^z \prod_{i=1}^{n} \left(\frac{g_{\nu_i}(t)}{t} \right)^{\alpha_i} dt \tag{1.3}$$

More precisely, we would like to obtain some sufficient conditions for $F_{\nu_1,\dots,\nu_n,\alpha_1,\dots,\alpha_n}(z)$ to be in the classes $\mathscr{C}(\gamma)$. Also, we prove new results, which involves strongly convexity of the integral operator of the type (1.3) when $n = 1$. In particular, we obtain simple sufficient conditions for some integral operators which involve the sine and cosine functions.

For the integral operators of the form (1.3) which involve the normalized analytic function of the form (1.1) see the references [2, 3, 4, 5, 6, 7].

In the proofs of our results we need the following result based on [8].

LEMMA 1.1. *Let $\nu > (-5+\sqrt{5})/4$ and consider the normalized Bessel function of the first kind $g_\nu : \mathbb{D} \to \mathbb{C}$, defined by $g_\nu(z) = 2^\nu \Gamma(\nu+1) z^{1-\nu/2} J_\nu(z^{1/2})$, where J_ν stands for the Bessel function of the first kind. Then the following inequality hold for all $z \in \mathbb{D}$*

$$\left| \frac{z g_\nu'(z)}{g_\nu(z)} - 1 \right| \leqslant \frac{\nu+2}{4\nu^2+10\nu+5} \tag{1.4}$$

2. Convexity of integral operators involving Bessel functions

Our first result provides sufficient conditions for integral operator of the type (1.3) to be convex of given order δ.

THEOREM 2.1. *Let n be a natural number and let $\nu_1, \nu_2, \ldots, \nu_n > (-5+\sqrt{5})/4$. Consider the functions $g_{\nu_i} : \mathbb{D} \to \mathbb{C}$, defined by*

$$g_{\nu_i}(z) = 2^{\nu_i}\Gamma(\nu_i+1)z^{1-\nu_i/2}J_{\nu_i}(z^{1/2}). \tag{2.1}$$

Let $\nu = \min\{\nu_1, \nu_2, \ldots, \nu_n\}$ and $\alpha_1, \alpha_2, \ldots, \alpha_n$ be positive real numbers. Moreover, suppose that these numbers satisfy the following inequality

$$0 \leqslant 1 - \frac{2+\nu}{4\nu^2+10\nu+5}\sum_{i=1}^{n}\alpha_i < 1.$$

Then the function $F_{\nu_1,\ldots,\nu_n,\alpha_1,\ldots,\alpha_n} : \mathbb{D} \to \mathbb{C}$, defined by

$$F_{\nu_1,\ldots,\nu_n,\alpha_1,\ldots,\alpha_n}(z) = \int_0^z \prod_{i=1}^{n}\left(\frac{g_{\nu_i}(t)}{t}\right)^{\alpha_i} dt, \tag{2.2}$$

is in $\mathscr{C}(\delta)$, where

$$\delta = 1 - \frac{2+\nu}{4\nu^2+10\nu+5}\sum_{i=1}^{n}\alpha_i.$$

Proof. First observe that, since for all $i \in \{1, 2, \ldots, n\}$ we have $g_{\nu_i} \in \mathscr{A}$, i.e. $g_{\nu_i}(0) = g'_{\nu_i}(0) - 1 = 0$, clearly $F_{\nu_1,\ldots,\nu_n,\alpha_1,\ldots,\alpha_n} \in \mathscr{A}$, i.e. $F_{\nu_1,\ldots,\nu_n,\alpha_1,\ldots,\alpha_n}(0) = F'_{\nu_1,\ldots,\nu_n,\alpha_1,\ldots,\alpha_n}(0) - 1 = 0$. On the other hand, it is easy to see that

$$F'_{\nu_1,\ldots,\nu_n,\alpha_1,\ldots,\alpha_n}(z) = \prod_{i=1}^{n}\left(\frac{g_{\nu_i}(z)}{z}\right)^{\alpha_i}$$

and

$$\frac{zF''_{\nu_1,\ldots,\nu_n,\alpha_1,\ldots,\alpha_n}(z)}{F'_{\nu_1,\ldots,\nu_n,\alpha_1,\ldots,\alpha_n}(z)} = \sum_{i=1}^{n}\alpha_i\left(\frac{zg'_{\nu_i}(z)}{g_{\nu_i}(z)} - 1\right).$$

or, equivalently,

$$1 + \frac{zF''_{\nu_1,\ldots,\nu_n,\alpha_1,\ldots,\alpha_n}(z)}{F'_{\nu_1,\ldots,\nu_n,\alpha_1,\ldots,\alpha_n}(z)} = \sum_{i=1}^{n}\alpha_i\left(\frac{zg'_{\nu_i}(z)}{g_{\nu_i}(z)}\right) + 1 - \sum_{i=1}^{n}\alpha_i. \tag{2.3}$$

Taking the real part of both terms of (2.3), we have

$$\mathrm{Re}\left\{1 + \frac{zF''_{\nu_1,\ldots,\nu_n,\alpha_1,\ldots,\alpha_n}(z)}{F'_{\nu_1,\ldots,\nu_n,\alpha_1,\ldots,\alpha_n}(z)}\right\} = \sum_{i=1}^{n}\alpha_i\mathrm{Re}\left(\frac{zg'_{\nu_i}(z)}{g_{\nu_i}(z)}\right) + \left(1 - \sum_{i=1}^{n}\alpha_i\right). \tag{2.4}$$

Now, by using the inequality (1.4) for each ν_i, where $i \in \{1,2,\ldots,n\}$, we obtain

$$\begin{aligned}\operatorname{Re}\left\{1+\frac{zF''_{\nu_1,\ldots,\nu_n,\alpha_1,\ldots,\alpha_n}(z)}{F'_{\nu_1,\ldots,\nu_n,\alpha_1,\ldots,\alpha_n}(z)}\right\} &= \sum_{i=1}^{n}\alpha_i \operatorname{Re}\left(\frac{zg'_{\nu_i}(z)}{g_{\nu_i}(z)}\right)+\left(1-\sum_{i=1}^{n}\alpha_i\right)\\ &> \sum_{i=1}^{n}\alpha_i\left(1-\frac{\nu_i+2}{4\nu_i^2+10\nu_i+5}\right)+\left(1-\sum_{i=1}^{n}\alpha_i\right)\\ &= 1-\sum_{i=1}^{n}\alpha_i\left(\frac{\nu_i+2}{4\nu_i^2+10\nu_i+5}\right)\\ &> 1-\frac{2+\nu}{4\nu^2+10\nu+5}\sum_{i=1}^{n}\alpha_i\end{aligned}$$

for all $z \in \mathbb{D}$ and $\nu,\nu_1,\nu_2,\ldots,\nu_n > (-5+\sqrt{5})/4$. Here we used that the function $\varphi : ((-5+\sqrt{5})/4,\infty) \to \mathbb{R}$, defined by

$$\varphi(x) = \frac{x+2}{4x^2+10x+5},$$

is decreasing and consequently for all $i \in \{1,2,\ldots,n\}$ we have

$$\frac{\nu_i+2}{4\nu_i^2+10\nu_i+5} \leqslant \frac{\nu+2}{4\nu^2+10\nu+5}. \tag{2.5}$$

Because $0 \leqslant 1-\frac{2+\nu}{4\nu^2+10\nu+5}\sum_{i=1}^{n}\alpha_i < 1$, we get $F_{\nu_1,\ldots,\nu_n,\alpha_1,\ldots,\alpha_n}(z) \in \mathscr{C}(\delta)$, where $\delta = 1-\frac{2+\nu}{4\nu^2+10\nu+5}\sum_{i=1}^{n}\alpha_i$. This completes the proof. □

Choosing $\alpha_1=\alpha_2=\ldots=\alpha_n=\alpha$ in Theorem 2.1, we have the following result.

COROLLARY 2.2. *Let the numbers $\nu,\nu_1,\ldots,\nu_n$ be as in Theorem 2.1 and let $\alpha_1,\alpha_2,\ldots,\alpha_n$ be positive real numbers. Moreover, suppose that the functions $g_{\nu_i} \in \mathscr{A}$ defined by (2.1) and the following inequality*

$$0 \leqslant 1-\frac{(2+\nu)n\alpha}{4\nu^2+10\nu+5} < 1.$$

is valid. Then the function $F_{\nu_1,\ldots,\nu_n,\alpha_1,\ldots,\alpha_n}(z)$ defined by (2.2) is in $\mathscr{C}(\zeta)$, where

$$\zeta = 1-\frac{(2+\nu)n\alpha}{4\nu^2+10\nu+5}.$$

Observe that $g_{1/2}(z)=\sqrt{z}\sin\sqrt{z}$ and $g_{-1/2}(z)=z\cos\sqrt{z}$. Thus, taking $n=1$ in Theorem 2.1 or in Corollary 2.2, we immediately obtain the following result.

COROLLARY 2.3. *Let $\nu > (-5+\sqrt{5})/4$ and $\alpha > 0$ be a real number. Moreover, suppose that these numbers satisfy the following inequality*

$$0 \leqslant 1-\frac{(2+\nu)\alpha}{4\nu^2+10\nu+5} < 1.$$

Then the function $F_{\nu,\alpha}:\mathbb{D}\to\mathbb{C}$, *defined by*

$$F_{\nu,\alpha}(z)=\int_0^z\left(\frac{g_\nu(t)}{t}\right)^\alpha dt,$$

is in $\mathscr{C}(\eta)$, *where*

$$\eta=1-\frac{(2+\nu)\alpha}{4\nu^2+10\nu+5}.$$

In particular, if $0<\alpha\leqslant 22/5$, *then the function* $F_{1/2,\alpha}:\mathbb{D}\to\mathbb{C}$, *defined by*

$$F_{1/2,\alpha}(z)=\int_0^z\left(\frac{\sin\sqrt{t}}{\sqrt{t}}\right)^\alpha dt,$$

is in $\mathscr{C}(\xi)$, *where* $\xi=1-(5/22)\alpha$. *Moreover, if* $0<\alpha\leqslant 2/3$, *then the function* $F_{-1/2,\alpha}:\mathbb{D}\to\mathbb{C}$, *defined by*

$$F_{-1/2,\alpha}(z)=\int_0^z\left(\cos\sqrt{t}\right)^\alpha dt,$$

is in $\mathscr{C}(\lambda)$, *where* $\lambda=1-(3/2)\alpha$.

3. Strongly convexity

In this section, we prove the following results, which involves strongly convexity of the integral operator of the type (1.3) when $n=1$.

THEOREM 3.1. *Let* $\nu>(-9+\sqrt{33})/8$. *Then the function* $F_{\nu,\alpha}:\mathbb{D}\to\mathbb{C}$, *defined by*

$$F_{\nu,\alpha}(z)=\int_0^z\left(\frac{g_\nu(t)}{t}\right)^\alpha dt,$$

is in $\overline{\mathscr{C}}_\rho(1)=\mathscr{C}(\rho)$, *where* $\rho=1-\alpha$; $0<\alpha\leqslant 1$.

Proof. It follows from (2.3), (1.4) and (2.5) that

$$\begin{aligned}\left|\arg\left(1+\frac{zF''_{\nu,\alpha}(z)}{F'_{\nu,\alpha}(z)}-(1-\alpha)\right)\right|&=\left|\arg\alpha\left(\frac{zg'_\nu(z)}{g_\nu(z)}\right)\right|=\left|\arg\frac{zg'_\nu(z)}{g_\nu(z)}\right|\\&\leqslant\arcsin\left(\frac{2+\nu}{4\nu^2+10\nu+5}\right)\\&\leqslant\arcsin 1=\frac{\pi}{2},\end{aligned}$$

so that $F_{\nu,\alpha}(z)\in\mathscr{C}(\rho)$, where $\rho=1-\alpha$; $0<\alpha\leqslant 1$, which proves Theorem 3.1. □

COROLLARY 3.2. *The function* $F_{1/2,\alpha} : \mathbb{D} \to \mathbb{C}$, *defined by*

$$F_{1/2,\alpha}(z) = \int_0^z \left(\frac{\sin\sqrt{t}}{\sqrt{t}} \right)^\alpha dt,$$

is in $\mathscr{C}(\rho)$, *where* $\rho = 1-\alpha$; $0 < \alpha \leqslant 1$. *In particular, when* $\alpha = 1$, *the function* $\int_0^z \frac{\sin\sqrt{t}}{\sqrt{t}} dt$ *is convex in* $\mathbb{D}$.

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