

## WEIGHTED INTERPOLATION OF WEIGHTED $\ell^p$ SEQUENCES AND CARLESON INEQUALITY

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*Abstract.* Let  $\{z_j\}$  be a sequence in the open unit disc  $D$  and  $\rho_n = \prod_{j \neq n} |z_n - z_j| / |1 - \bar{z}_j z_n| > 0$ . For  $0 < p < \infty$ ,  $H^p$  denotes a Hardy space on  $D$ . For a given  $f$  in  $H^p$ , we study a sequence  $\{(1 - |z_j|^2)^{1/p} f(z_j)\}$ . Then it is related to a Carleson inequality.

### 1. Introduction

Let  $H^p$  ( $0 < p \leq \infty$ ) denote a usual Hardy space in the open unit disc. In this paper, we assume that a sequence  $\{z_j\}$  in  $D$  satisfies that  $\sum_{j=1}^{\infty} (1 - |z_j|) < \infty$  and  $z_j \neq 0$  ( $1 \leq j < \infty$ ), that is, there exists a Blaschke product

$$B(z) = \prod_{j=1}^{\infty} \left( -\frac{\bar{z}_j}{|z_j|} \frac{z - z_j}{1 - \bar{z}_j z} \right).$$

Let

$$\rho_{nk} = \prod_{\substack{j=1 \\ j \neq k}}^n \left| \frac{z_k - z_j}{1 - \bar{z}_j z_k} \right|, \quad 1 \leq k \leq n,$$

$$\rho_k = \prod_{\substack{j=1 \\ j \neq k}}^{\infty} \left| \frac{z_k - z_j}{1 - \bar{z}_j z_k} \right|.$$

Then  $\rho_{nk} \geq \rho_{n+1,k}$  and  $\lim_{n \rightarrow \infty} \rho_{nk} = \rho_k$  for  $k \geq 1$ . In this paper, we assume that  $\rho_j > 0$  ( $1 \leq j < \infty$ ). This hypothesis is equivalent to that there exists a function  $f_k$  in  $H^p(D)$  such that  $f_k(a_j) = \delta_{jk}$ .

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Put  $\ell = [\{w_j\}; w_j \in \mathbb{C}, 1 \leq j < \infty]$  and  $a = \{a_j\}$  where  $a_j > 0$  ( $1 \leq j < \infty$ ). For  $0 < p < \infty$ , suppose

$$\ell^p(a) = [\{w_j\} \in \ell; \sum_{j=0}^{\infty} a_j |w_j|^p < \infty]$$

and

$$\ell^\infty(a) = [\{w_j\} \in \ell; \sup_{0 \leq j < \infty} a_j |w_j| < \infty].$$

For  $a = \{a_j\}$  and  $-\infty < t < \infty$ ,  $a^t$  denotes  $\{a_j^t\}$ . For  $\rho = \{\rho_j\}$ ,  $a = \{a_j\}$  and  $-\infty < t, s < \infty$ ,  $a^s \rho^t$  denotes  $\{a_j^s \rho_j^t\}$ .

Given a sequence  $\{z_j\}$  in  $D$ , let  $T_p$  be the linear operator on  $H^p$  ( $0 < p \leq \infty$ ) defined by

$$T_p(f) = \{(1 - |z_j|^2)^{1/p} f(z_j)\}$$

with  $1/p = 0$  for  $p = \infty$ . It is known (cf. [1]) that  $\inf_j \rho_j > 0$  if and only if  $T_p(H^p) = \ell^p$ .

J. P. Earle [2] showed that  $\ell^\infty(\rho^{-2}) \subset T_\infty(H^\infty)$ . This was pointed out by A. M. Gleason (see [3]). The author [5] showed that  $\ell^\infty(\rho^{-1}) \subset T_\infty(H^\infty)$  if and only if  $\{a_n\}$  is the union of a finite number of uniformly separated sequences. J. Garnett [3] showed that  $T_\infty(H^\infty)$  contains  $\ell^\infty(\rho^{-1-\varepsilon})$  for any  $\varepsilon > 0$ . Hence by Lemma 2 in §2

$$\ell^1(\rho^{-1}) \subseteq T_1(H^1) \subseteq \ell^1(\rho^\varepsilon)$$

for any  $\varepsilon > 0$ .

In this paper, we are interested in the range of  $T_p$  for  $1 < p < \infty$ . We could not generalize results of  $p = \infty$  and  $p = 1$  to  $1 < p < \infty$ . However we show that  $T_p(H^p) \subset \ell^p(\rho^{2+\varepsilon})$  for any  $\varepsilon > 0$  if  $\rho_j > 0$  ( $1 \leq j < \infty$ ). As a result, if  $p \neq \infty$  and  $1/p + 1/q = 1$  then  $\ell^p(\rho^{-(2+\varepsilon+q)p/q}) \subset T_p(H^p)$  for any  $\varepsilon > 0$ .

It should be noted that  $T_p(H^p) \subset \ell^p(\rho^{2+\varepsilon})$  if and only if a Carleson inequality holds, that is,

$$\sum_{j=1}^{\infty} \rho_j^{2+\varepsilon} (1 - |z_j|^2) |f(z_j)|^p \leq \gamma \|f\|_p^p \quad (f \in H^p)$$

for some finite constant  $\gamma$ .

## 2. Lemmas

In this section, we prove two lemmas in order to prove Theorem. Lemma 1 is well known (see [1, p. 142]).

For  $1 \leq j \leq n$ , let

$$B_n(z) = \prod_{j=1}^n \frac{z - z_j}{1 - \bar{z}_j z} \quad \text{and} \quad B_{nj}(z) = B_n(z) \frac{1 - \bar{z}_j z}{z - z_j}$$

If  $b_{nj} = B_{nj}(z_j)$  and

$$f_n(z) = \sum_{j=1}^n b_{nj}^{-1} w_j B_{nj}(z)$$

then  $f_n$  is in  $H^\infty$  and  $f_n(z_j) = w_j$  ( $1 \leq j \leq n$ ). Put

$$\rho_{nj} = |b_{nj}| \quad (1 \leq j \leq n).$$

LEMMA 1. *Let  $1 \leq p \leq \infty$  and  $1/p + 1/q = 1$ . Suppose  $w_j$  is a complex number for  $j = 1, 2, \dots$ . There exists a function  $f$  in  $H^p$  such that  $(1 - |z_j|^2)^{1/p} f(z_j) = w_j$  for  $j = 1, 2, \dots$  if and only if there exists a positive finite constant  $\gamma$  such that for any  $n \geq 1$  and for all  $g$  in  $H^q$ ,*

$$\left| \sum_{j=1}^n \frac{w_j}{b_{nj}} (1 - |z_j|^2)^{1/q} g(z_j) \right| \leq \gamma \|g\|_q.$$

LEMMA 2. *Let  $1 \leq p \leq \infty$  and  $1/p + 1/q = 1$ .*

(1) *When  $1 < p < \infty$ ,  $T_p(H^p) \supseteq \ell^p(a)$  if and only if  $T_q(H^q) \subseteq \ell^q(a^{-q/p} \rho^{-q})$ .*

(2)  *$T_1(H^1) \supseteq \ell^1(a)$  if and only if  $T_\infty(H^\infty) \subseteq \ell^\infty(a^{-1} \rho^{-1})$ .*

(3)  *$T_\infty(H^\infty) \supseteq \ell^\infty(a)$  if and only if  $T_1(H^1) \subseteq \ell^1(a^{-1} \rho^{-1})$ .*

*Proof.* (1) For the ‘only if’ part, since  $[\{(1 - |z_j|^2)^{1/p} f(z_j)\}; f \in H^p] \supset \ell^p(a)$ , by Lemma 1 there exists a positive finite constant  $\gamma$  such that for any  $n \geq 1$

$$\sup_{\substack{w \in \ell^p(a) \\ \|w\| \leq 1}} \left| \sum_{j=1}^n \frac{w_j}{b_{nj}} (1 - |z_j|^2)^{1/q} g(z_j) \right| \leq \gamma \|g\|_q \quad (g \in H^q)$$

where  $w = \{w_j\}$  and  $\|w\| = (\sum_{j=1}^\infty a_j |w_j|^p)^{1/p}$ . Hence for any  $n \geq 1$

$$\left( \sum_{j=1}^n (a_j^{1/p} \rho_{nj})^{-q} (1 - |z_j|^2) |g(z_j)|^q \right)^{1/q} \leq \gamma \|g\|_q \quad (g \in H^q).$$

Assuming  $\|g\|_q = 1$ ,

$$\sum_{j=1}^n (a_j^{1/p} \rho_{nj})^{-q} (1 - |z_j|^2) |g(z_j)|^q \leq \gamma^q.$$

For any  $\varepsilon > 0$ , there exists a positive integer  $s(j)$  for each  $j$  such that for all  $k \geq s(j)$

$$(a_j^{1/p} \rho_j)^{-q} (1 - |z_j|^2) |g(z_j)|^q - \frac{\varepsilon}{2j} \leq (a_j^{1/p} \rho_{kj})^{-q} (1 - |z_j|^2) |g(z_j)|^q$$

because  $\lim_{n \rightarrow \infty} \rho_{nj} = \rho_j$ . Hence for any  $n \geq 1$

$$\begin{aligned} & \sum_{j=1}^n (a_j^{1/p} \rho_j)^{-q} (1 - |z_j|^2) |g(z_j)|^q - \sum_{j=1}^n \frac{\varepsilon}{2j} \\ & \leq \sum_{j=1}^n (a_j^{1/p} \rho_{s(j)j})^{-q} (1 - |z_j|^2) |g(z_j)|^q \\ & \leq \sum_{j=1}^k (a_j^{1/p} \rho_{kj})^{-q} (1 - |z_j|^2) |g(z_j)|^q \leq \gamma^q \end{aligned}$$

where  $k = \max(s(1), \dots, s(n))$ . Thus for any  $\varepsilon > 0$

$$\sum_{j=1}^{\infty} (a_j^{1/p} \rho_j)^{-q} (1 - |z_j|^2) |g(z_j)|^q - \varepsilon \leq \gamma^q.$$

This implies the ‘only if’ part. For the ‘if’ part, by Lemma 1 it is sufficient to show that there exists a finite positive constant  $\gamma$  such that for all  $n \geq 1$

$$\sup_{\substack{w \in \ell^p(a) \\ \|w\| \leq 1}} \sup_{\|g\|_q \leq 1} \left| \sum_{j=1}^n \frac{w_j}{b_{nj}} (1 - |z_j|^2)^{1/q} g(z_j) \right| \leq \gamma < \infty$$

In fact, for all  $n \geq 1$

$$\begin{aligned} & \sup_{\substack{w \in \ell^p(a) \\ \|w\| \leq 1}} \sup_{\|g\|_q \leq 1} \left| \sum_{j=1}^n \frac{w_j}{b_{nj}} (1 - |z_j|^2)^{1/q} g(z_j) \right| \\ & \leq \sup_{\|g\|_q \leq 1} \left( \sum_{j=1}^n (a_j^{1/p} \rho_{nj})^{-q} (1 - |z_j|^2) |g(z_j)|^q \right)^{1/q} \\ & \leq \sup_{\|g\|_q \leq 1} \left( \sum_{j=1}^{\infty} (a_j^{1/p} \rho_j)^{-q} (1 - |z_j|^2) |g(z_j)|^q \right)^{1/q}. \end{aligned}$$

(2) For the ‘only if’ part, since  $\{(1 - |z_j|)f(z_j)\}; f \in H^1\} \supset \ell^1(a)$ , by Lemma 1 there exists a positive finite constant  $\gamma$  such that for any  $n \geq 1$  and  $\|g\|_{\infty} = 1$

$$\max_{1 \leq j \leq n} \frac{1}{a_j \rho_{nj}} |g(z_j)| \leq \gamma.$$

For any  $\varepsilon > 0$ , there exists a positive integer  $s(j)$  for each  $j$  such that for all  $k \geq s(j)$

$$\frac{1}{a_j \rho_j} |g(z_j)| - \varepsilon \leq \frac{1}{a_j \rho_{kj}} |g(z_j)| \leq \gamma$$

because  $\lim_{n \rightarrow \infty} \rho_{nj} = \rho_j$ . This implies that  $\{(f(z_j)\}; f \in H^{\infty}\} \subset \ell^{\infty}(a^{-1} \rho^{-1})$ . For the ‘if’ part, we can prove it as in (1).

(3) We can prove (3) as in (1).  $\square$

LEMMA 3. For  $k = 1, 2, \dots$

$$\sum_{j=1}^{\infty} \frac{(1 - |z_j|^2)(1 - |z_k|^2)}{|1 - \bar{z}_j z_k|^2} \leq 1 - 2 \log \rho_k.$$

*Proof.* See [1, p. 150].  $\square$

LEMMA 4. Let  $a_{jk}$  ( $j, k = 1, 2, \dots$ ) be complex numbers such that  $a_{kj} = \overline{a_{jk}}$  and

$$\sum_{j=1}^n |a_{jk}| \leq M \quad (1 \leq k \leq n).$$

Then for any numbers  $x_1, \dots, x_n$ ,

$$\left| \sum_{j,k=1}^n a_{jk} x_j \bar{x}_k \right| \leq M \sum_{j=1}^n |x_j|^2 \quad (1 \leq k \leq n).$$

*Proof.* See ([1, p. 150], [7, p. 42]).  $\square$

### 3. Theorem

By (1) of Lemma 2, when  $1 < p < \infty$  and  $1/p + 1/q = 1$ ,  $T_p(H^p) \supseteq \ell^p(\rho^{-p})$  if and only if  $T_q(H^q) \subseteq \ell^q$ .  $T_\infty(H^\infty) \supseteq \ell^\infty(\rho^{-1})$  if and only if  $T_1(H^1) \subseteq \ell^1$ .  $T_1(H^1) \supseteq \ell^1(\rho^{-1})$  if and only if  $T_\infty(H^\infty) \subseteq \ell^\infty$ . Hence for  $1 < p \leq \infty$ , this is equivalent to that  $\{z_j\}_{j=1}^\infty$  is the union of a finite number of uniformly separated sequence. See [6] for  $1 < p < \infty$  and [5] for  $p = \infty$ . It is known [3] that  $T_\infty(H^\infty) \supseteq \ell^\infty(\rho^{-1-\varepsilon})$  for any  $\varepsilon > 0$ . It is interesting to know that for  $1 < p < \infty$ ,  $T_p(H^p) \supseteq \ell^p(\rho^{-p-\varepsilon})$  for any  $\varepsilon > 0$ . Unfortunately we can not prove it. In this section, we show that  $T_2(H^2) \supseteq \ell^2(\rho^{-4-\varepsilon})$  for any  $\varepsilon > 0$ .

THEOREM.

- (1)  $\ell^2(\rho^{-4-\varepsilon}) \subset T_2(H^2)$  for any  $\varepsilon > 0$ .
- (2)  $T_2(H^2) \subset \ell^2(\rho^{2+\varepsilon})$  for any  $\varepsilon > 0$ .

*Proof.* (1) Suppose  $w = (w_k) \in \ell^2(\rho^{-4-\varepsilon})$  for any  $\varepsilon > 0$ . Put

$$F_{nk}(z) = (1 - |z_k|^2)^{3/2} [B_n(z)]^2 (z - z_k)^{-2}$$

and

$$f_n(z) = \sum_{k=1}^n w_k [B_{nk}(z_k)]^{-2} F_{nk}(z).$$

Then  $(1 - |z_k|^2)^{1/2} f_n(z_k) = w_k \quad (1 \leq k \leq n)$  and for any  $\varepsilon > 0$

$$\begin{aligned} \|f_n\|_2^2 &= (f_n, f_n) = \sum_{j,k=1}^n w_j [B_{nj}(z_j)]^{-2} \cdot \overline{w_k [B_{nk}(z_k)]^{-2}} \cdot (F_{nj}, F_{nk}) \\ &\leq \sum_{j,k=1}^n \rho_j^{-2} |w_j| \cdot \rho_k^{-2} |w_k| \cdot |(F_{nj}, F_{nk})| \\ &= \sum_{j,k=1}^n \rho_j^{-2-\varepsilon} |w_j| \cdot \rho_k^{-2-\varepsilon} |w_k| \cdot |(F_{nj}, F_{nk})| \rho_j^\varepsilon \rho_k^\varepsilon. \end{aligned}$$

Since  $|(F_{nj}, F_{nk})| \leq 2(1 - |z_j|^2)(1 - |z_k|^2)|1 - z_j \bar{z}_k|^2$ , by Lemma 3

$$\sum_{j=1}^n |(F_{nj}, F_{nk})| \rho_j^\varepsilon \rho_k^\varepsilon \leq 2\rho_k^\varepsilon (1 - 2\log \rho_k) \quad (1 \leq k \leq n)$$

By Lemma 4

$$\|f_n\|_2^2 \leq \left(\max_{1 \leq k \leq n} 2\rho_k^\varepsilon (1 - 2\log \rho_k)\right) \sum_{j=1}^n \rho_j^{-4-2\varepsilon} |w_j|^2.$$

and  $\sup_{1 \leq k < \infty} \rho_k^\varepsilon (1 - 2\log \rho_k) < \infty$ . This implies that if  $w = \{w_k\} \in \ell^2(\rho^{-4-2\varepsilon})$  then  $(f_n)$  is a normal family and so a subsequence tends uniformly in each disc  $|z| \leq r < 1$  to a function  $f \in H^2$  for which  $T_2(f) = w$ .

(2) Put  $a_j = \rho_j^{-4-2\varepsilon}$  then  $a_j^{-1} \rho_j^{-2} = \rho_j^{4+2\varepsilon} \cdot \rho_j^{-2} = \rho_j^{2+2\varepsilon}$ , then Lemma 2 implies (2).  $\square$

**COROLLARY 1.** *Let  $0 < p < \infty$ , then  $T_p(H^p) \subset \ell^p(\rho^{2+\varepsilon})$  for any  $\varepsilon > 0$ .*

*Proof.* If  $f \in H^p$  then  $f = Bg^{2/p}$  where  $B$  is a Blaschke product and  $g$  is nonvanishing  $H^2$  function. By (2) of Theorem

$$\begin{aligned} \sum_{j=1}^{\infty} \rho^{2+\varepsilon} (1 - |z_j|^2) |f(z_j)|^p &\leq \sum_{j=1}^{\infty} \rho^{2+\varepsilon} (1 - |z_j|^2) |g(z_j)|^2 \\ &\leq \gamma \|g\|_2^2 = \gamma \|f\|_p^p \end{aligned}$$

where  $\gamma$  is a finite positive constant depending  $p$ .  $\square$

**COROLLARY 2.** *Let  $1 \leq p < \infty$ , then  $\ell^p(\rho^{-(2+\varepsilon+q)p/q}) \subset T_p(H^p)$  for any  $\varepsilon > 0$ .*

*Proof.* This is a result of Lemma 2 and Corollary 1.  $\square$

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