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SEVERAL INTEGRAL INEQUALITIES ON TIME SCALES

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Abstract. In this paper, we provide some new integral inequalities of Qi type on time scales by using elementary analytic methods.

1. Introduction

Under what conditions does the inequality

$$\int_a^b f^p(x)\,\mathrm{d}x \geqslant \left[\int_a^b f(x)\,\mathrm{d}x\right]^{p-1} \tag{1.1}$$

hold for $p>1$? This problem was posed by Qi in [6]. Hereafter, this problem simulated many mathematicians to study.

In [1, p. 124, Theorem C], M. Akkouchi proved the following results.

THEOREM 1.1. *Let $[a,b]$ be a closed interval of $\mathbb{R}$ and $p>1$. For any continuous function $f(x)$ such that $f(a)\geqslant 0$ and $f'(x)\geqslant p$ on $[a,b]$, we have*

$$\int_a^b f^{p+2}(x)\,\mathrm{d}x \geqslant \frac{1}{(b-a)^{p-1}}\left[\int_a^b f(x)\,\mathrm{d}x\right]^{p+1}. \tag{1.2}$$

For $b>0$, $a=bq^n$, and $n\in\mathbb{N}$, we write

$$[a,b]_q=\{bq^k, 0\leqslant k\leqslant n\} \quad \text{and} \quad (a,b]_q=[q^{-1}a,b]_q.$$

The q-derivative $D_q f$ of a function f is given by

$$D_q f(x)=\begin{cases}\dfrac{f(x)-f(qx)}{(1-q)x}, & x\neq 0\\ f'(0), & x=0\end{cases}$$

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provided $f'(0)$ exists. The q-Jackson integral from 0 to a is defined by

$$\int_0^a f(x)\,\mathrm{d}_q x = (1-q)a\sum_{n=0}^{\infty} f(aq^n)q^n$$

provided the sum converges absolutely. The q-Jackson integral in a generic interval $[a,b]$ is given by

$$\int_a^b f(x)\,\mathrm{d}_q x = \int_0^b f(x)\,\mathrm{d}_q x - \int_0^a f(x)\,\mathrm{d}_q x.$$

The q-analogue of the above-mentioned Theorem 1.1 was established in [3, Proposition 3.5] as follows.

THEOREM 1.2. *Let $p>1$ be a real number and $f(x)$ be a function defined on $[a,b]_q$ satisfying $f(a)\geqslant 0$ and $D_q f(x)\geqslant p$ for all $x\in(a,b]_q$. Then*

$$\int_a^b f^{p+2}(x)\,\mathrm{d}_q x \geqslant \frac{1}{(b-a)^{p-1}}\left[\int_a^b f(qx)\,\mathrm{d}_q x\right]^{p+1}. \tag{1.3}$$

Recently, Krasniqi and Shabani obtained some sufficient conditions for Qi type h-integral inequalities in [4].

For more information, please refer to the list of references in [5, 7, 8].

The main aim of this paper is to generalize the above results on time scales.

2. Notations and lemmas

For advancing smoothly, we recite some basic notations and lemmas.

2.1. Notations

A time scale $\mathbb{T}$ is a non-empty closed subset of the real numbers $\mathbb{R}$. The forward and backward jump operators $\sigma,\rho:\mathbb{T}\to\mathbb{T}$ are respectively defined by

$$\sigma(t)=\inf\{s\in\mathbb{T}:s>t\} \tag{2.1}$$

and

$$\rho(t)=\sup\{s\in\mathbb{T}:s<t\}, \tag{2.2}$$

where the supremum of the empty set is defined to be the infimum of $\mathbb{T}$.

A point $t\in\mathbb{T}$ is said to be right-scattered if $\sigma(t)>t$ and to be right-dense if $\sigma(t)=t$. On the other hand, a point $t\in\mathbb{T}$ with $t>\inf\mathbb{T}$ is said to be left-scattered if $\rho(t)<t$ and to be left-dense if $\rho(t)=t$. A function $g:\mathbb{T}\to\mathbb{R}$ is said to be rd-continuous provided that g is continuous at right-dense points and has finite left-sided limits at left-dense points in $\mathbb{T}$. In what follows, the set of all rd-continuous functions from $\mathbb{T}$ to $\mathbb{R}$ is denoted by $C_{rd}(\mathbb{T},\mathbb{R})$. The graininess function μ for a time scales $\mathbb{T}$

is defined by $\mu(t) = \sigma(t) - t$. For $f : \mathbb{T} \to \mathbb{R}$, the notation f^σ means the composition $f \circ \sigma$.

We also need below the set $\mathbb{T}^\kappa$ which is derived from the time scale $\mathbb{T}$ as follows. If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^\kappa = \mathbb{T} - \{m\}$. Otherwise, $\mathbb{T}^\kappa = \mathbb{T}$. Throughout this paper, we make the blanket assumption that a and b are points in $\mathbb{T}$. Often we assume $a \leqslant b$. We then define the interval $[a,b]$ in $\mathbb{T}$ by

$$[a,b]_{\mathbb{T}} = \{t \in \mathbb{T} : a \leqslant t \leqslant b\}.$$

For a function $f : \mathbb{T} \to \mathbb{R}$, the (delta) derivative $f^\Delta(t)$ at $t \in \mathbb{T}$ is defined to be the number, if it exists, such that for all $\varepsilon > 0$, there is a neighborhood U of t with

$$\left|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)\right| < \varepsilon|\sigma(t) - s| \tag{2.3}$$

for all $s \in U$. If the (delta) derivative $f^\Delta(t)$ exits for all $t \in \mathbb{T}$, then we say that f is (delta) differentiable on $\mathbb{T}$.

The (delta) derivatives of the product fg and the quotient $\frac{f}{g}$ of two (delta) differentiable functions f and g may be formulated respectively as

$$(fg)^\Delta = f^\Delta g + f^\sigma g^\Delta = fg^\Delta + f^\Delta g^\sigma \tag{2.4}$$

and

$$\left(\frac{f}{g}\right)^\Delta = \frac{f^\Delta g - fg^\Delta}{gg^\sigma}, \tag{2.5}$$

where $gg^\sigma \neq 0$.

For $b, c \in \mathbb{T}$ and a (delta) differentiable function f, Cauchy integral of f^Δ is defined by

$$\int_b^c f^\Delta(t)\Delta t = f(c) - f(b) \tag{2.6}$$

and infinite integrals are defined as

$$\int_b^\infty f(t)\Delta t = \lim_{c\to\infty} \int_b^c f(t)\Delta t. \tag{2.7}$$

An integration-by-part formula reads that

$$\int_b^c f(t)g^\Delta(t)\Delta t = f(t)g(t)\big|_b^c - \int_b^c f^\Delta(t)g(\sigma(t))\Delta t. \tag{2.8}$$

Note that in the case $\mathbb{T} = \mathbb{R}$, we have $\sigma(t) = \rho(t) = t$, $\mu(t) = 0$, $f^\Delta(t) = f'(t)$, and

$$\int_b^c f^\Delta(t)\Delta t = \int_b^c f'(t)\,\mathrm{d}t, \tag{2.9}$$

and that in the case $\mathbb{T} = q\mathbb{Z}$, we have $\sigma(t) = t + q$, $\rho(t) = t - q$, $\mu(t) \equiv q$, and

$$f^\Delta(t) = \frac{f(t+q) - f(t)}{q}. \tag{2.10}$$

If $\mathbb{T} = q^{\mathbb{Z}}$ and $q > 1$, we have $\sigma(t) = qt$, $\rho(t) = \frac{t}{q}$, $\mu(t) = (q-1)t$, and

$$f^{\Delta}(t) = \frac{f(qt) - f(t)}{(q-1)t}, \quad t \neq 0. \tag{2.11}$$

A continuous function $f : \mathbb{T} \to \mathbb{R}$ is called pre-differentiable with D, provided $D \subset \mathbb{T}^{\kappa}$, $\mathbb{T}^{\kappa} \setminus D$ is countable and contains no right-scattered elements of $\mathbb{T}$, and f is differentiable at each $t \in D$. Let f be rd-continuous. Then there exists a function F which is pre-differentiable with region of differentiation D such that $F^{\Delta}(x) = f(t)$ holds for all $t \in D$. We define the Cauchy integral by

$$\int_b^c f(t)\Delta t = F(c) - F(b), \tag{2.12}$$

where F is a pre-antiderivative of f and $b, c \in \mathbb{T}$. The existence theorem in [2, p. 27, Theorem 1.74] reads as follows: Every rd-continuous function has an antiderivative. In particular, if $t_0 \in \mathbb{T}$, then F defined by $F(t) = \int_{t_0}^{t} f(\tau)\Delta\tau$ is an antiderivative of f.

If f is (delta) differentiable, then f is continuous and rd-continuous. We easily know that

$$\sigma, \quad \rho, \quad f^{\rho}(x), \quad f^{\sigma}(x), \quad [f^{\sigma}(x)]^p, \quad [f^{\rho}(x)]^p$$

for $p \in \mathbb{N}$ are rd-continuous by using property of rd-continuous function. Thus, all integrals involving main results of this paper are meaningful.

2.2. Lemmas

The following lemmas are useful and some of them can be found in the book [2].

LEMMA 2.1. ([2, p. 28, Theorem 1.76]) *If $f^{\Delta}(x) \geqslant 0$, then $f(x)$ is nondecreasing.*

LEMMA 2.2. ([2, p. 32, Theorem 1.90. Chain Rule]) *Let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable and suppose $g : \mathbb{T} \to \mathbb{R}$ is delta differentiable. Then $f \circ g : \mathbb{T} \to \mathbb{R}$ is delta differentiable and the formula*

$$(f \circ g)^{\Delta}(t) = \left\{ \int_0^1 \left[f'(g(t) + h\mu(t)) g^{\Delta}(t) \right] \mathrm{d}h \right\} g^{\Delta}(t) \tag{2.13}$$

holds.

LEMMA 2.3. ([2, p. 9, Exercise 1.23]) *Assume that $f : \mathbb{T} \to \mathbb{R}$ is delta differentiable at $t \in \mathbb{T}^{\kappa}$. Then*

$$[f^n(x)]^{\Delta} = \left\{ \sum_{k=0}^{n-1} f^k(x) [f^{\sigma}(x)]^{n-1-k} \right\} f^{\Delta}(x). \tag{2.14}$$

LEMMA 2.4. ([2, p. 5, Theorem 1.16]) *Assume that* $f:\mathbb{T}\to\mathbb{R}$ *is delta differentiable at* $t\in\mathbb{T}^{\kappa}$. *Then*

$$f^{\sigma}(x)=f(x)+\mu(x)f^{\Delta}(x). \tag{2.15}$$

For more discussion on time scales, we refer the reader to [2]. Next, we give a new and useful lemma.

LEMMA 2.5. *Let* $a,b\in\mathbb{T}$ *and* $p>1$. *Assume* $g:\mathbb{T}\to\mathbb{R}$ *is delta differentiable at* $t\in\mathbb{T}^{\kappa}$ *and non-negative and increasing function on* $[a,b]_{\mathbb{T}}$. *Then*

$$pg^{p-1}(x)g^{\Delta}(x)\leqslant[g^{p}(x)]^{\Delta}\leqslant p[g^{\sigma}(x)]^{p-1}g^{\Delta}(x). \tag{2.16}$$

Proof. Using Lemma 2.2, we have

$$\begin{aligned}[g^{p}(x)]^{\Delta}&=\left\{p\int_0^1\left[g(x)+h\mu(x)g^{\Delta}(x)\right]^{p-1}\mathrm{d}h\right\}g^{\Delta}(x)\\&\geqslant\left\{p\int_0^1[g(x)]^{p-1}\mathrm{d}h\right\}g^{\Delta}(x)\\&=pg^{p-1}(x)g^{\Delta}(x).\end{aligned}$$

On the other hand, by Lemma 2.3, we obtain

$$\begin{aligned}[g^{p}(x)]^{\Delta}&=\left\{\sum_{k=0}^{n-1}g^{k}(x)[g^{\sigma}(x)]^{n-1-k}\right\}g^{\Delta}(x)\\&\leqslant\left\{\sum_{k=0}^{n-1}[g^{\sigma}(x)]^{k}[g^{\sigma}(x)]^{n-1-k}\right\}g^{\Delta}(x)\\&=p[g^{\sigma}(x)]^{p-1}g^{\Delta}(x).\end{aligned}$$

The proof of Lemma 2.5 is complete. □

REMARK 2.1. If letting $\mathbb{T}=h\mathbb{Z}$ and $p>1$ in Lemma 2.5, we deduce Lemma 2 in [4].

REMARK 2.2. If taking $\mathbb{T}=q^{\mathbb{Z}}$ and $p>1$ in Lemma 2.5, we deduce

$$pg^{p-1}(x)g^{\Delta}(x)\leqslant[g^{p}(x)]^{\Delta}\leqslant pg^{p-1}(qx)g^{\Delta}(x). \tag{2.17}$$

The formula (2.17) is a simple deformation of Lemma 3.1 in [3].

3. Main results

In this section, we will state and prove our main results.

THEOREM 3.1. *Let $a,b \in \mathbb{T}$ and $p \geqslant 3$. Assume $f,\sigma : \mathbb{T} \to \mathbb{R}$ be delta differentiable at $t \in \mathbb{T}^\kappa$. If f is a non-negative and increasing function on $[a,b]_\mathbb{T}$ and satisfies*

$$f^{p-2}(x)f^\Delta(x) \geqslant (p-2)\left[f^{\sigma^2}(x)\right]^{p-2}[\sigma^2(x)-a]^{p-3}\sigma^\Delta(x) \tag{3.1}$$

and

$$f(a) \geqslant (p-1)\mu^{p-2}(a), \tag{3.2}$$

where $\sigma^2(x) = \sigma(\sigma(x))$. Then

$$\int_a^b f^p(x)\Delta x \geqslant \left[\int_a^b f(x)\Delta x\right]^{p-1}. \tag{3.3}$$

Proof. Define

$$F(x) = \int_a^x f^p(t)\Delta t - \left[\int_a^x f(t)\Delta t\right]^{p-1}$$

and $g(x) = \int_a^x f(t)\Delta t$. It is easy to see that $g^\Delta(x) = f(x)$. Using Lemma 2.5, it follows that

$$F^\Delta(x) = f^p(x) - [g^{p-1}(x)]^\Delta \geqslant f^p(x) - (p-1)[g^\sigma(x)]^{p-2}g^\Delta(x) = f(x)G(x),$$

where

$$G(x) = f^{p-1}(x) - (p-1)[g^\sigma(x)]^{p-2}.$$

Using Lemmas 2.4 and 2.5 again, we have

$$G^\Delta(x) \geqslant (p-1)f^{p-2}(x)f^\Delta(x) - (p-1)(p-2)\left[g^{\sigma^2}(x)\right]^{p-3}f^\sigma(x)\sigma^\Delta(x)$$

where

$$\begin{aligned}
[g^\sigma(x)]^\Delta &= \left[\int_a^{\sigma(x)} f(t)\Delta t\right]^\Delta \\
&= \left[\int_a^x f(t)\Delta t + \int_x^{\sigma(x)} f(t)\Delta t\right]^\Delta \\
&= \left[\int_a^x f(t)\Delta t\right]^\Delta + \{f(x)[\sigma(x)-x]\}^\Delta \\
&= f^\sigma(x)\sigma^\Delta(x).
\end{aligned}$$

Since f is a non-negative and increasing function, then

$$g^{\sigma^2}(x) = \int_a^{\sigma^2(x)} f(t)\Delta t \leqslant f^{\sigma^2}(x)[\sigma^2(x)-a]. \tag{3.4}$$

Hence,

$$\begin{aligned} G^{\Delta}(x) &\geqslant (p-1)\{f^{p-2}(x)f^{\Delta}(x)-(p-2)\left[f^{\sigma^2}(x)\right]^{p-3}f^{\sigma}(x)[\sigma^2(x)-a]^{p-3}\sigma^{\Delta}(x)\} \\ &\geqslant (p-1)\{f^{p-2}(x)f^{\Delta}(x)-(p-2)\left[f^{\sigma^2}(x)\right]^{p-2}[\sigma^2(x)-a]^{p-3}\sigma^{\Delta}(x)\} \\ &\geqslant 0. \end{aligned}$$

By Lemma 2.1, we conclude that $G(x)$ is an increasing function. Hence,

$$G(x) \geqslant G(a) = f^{p-2}(a)[f(a)-(p-1)\mu^{p-2}(a)] \geqslant 0$$

which means that $F^{\Delta}(x) \geqslant 0$. So $F(x) \geqslant F(a) = 0$. The proof is complete. □

REMARK 3.1. If $\mathbb{T} = h\mathbb{Z}$ and $p > 3$ in Theorem 3.1, we deduce Theorem 3 in [4].

THEOREM 3.2. *Let $a, b \in \mathbb{T}$ and $p \geqslant 3$. Assume $f, \sigma : \mathbb{T} \to \mathbb{R}$ be delta differentiable at $t \in \mathbb{T}^{\kappa}$. If f is a non-negative and increasing function on $[a,b]_{\mathbb{T}}$ and satisfies*

$$f^{p-3}(x)f^{\Delta}(x) \geqslant (p-2)(f^{\sigma}(x))^{p-3}[\sigma^2(x)-a]^{p-3}\sigma^{\Delta}(x) \tag{3.5}$$

and

$$f^{\rho}(a) \geqslant (p-1)\mu^{p-2}(a), \tag{3.6}$$

then

$$\int_a^b f^p(x)\Delta x \geqslant \left(\int_a^b f^{\rho}(x)\Delta x\right)^{p-1}. \tag{3.7}$$

Proof. Define

$$F(x) = \int_a^x f^p(t)\Delta t - \left(\int_a^x f^{\rho}(t)\Delta t\right)^{p-1}$$

and $g(x) = \int_a^x f^{\rho}(t)\Delta t$. It is easy to see that $g^{\Delta}(x) = f^{\rho}(x)$. Using Lemma 2.5, it follows that

$$F^{\Delta}(x) = f^p(x) - [g^{p-1}(x)]^{\Delta} \geqslant f^p(x) - (p-1)[g^{\sigma}(x)]^{p-2}g^{\Delta}(x) = f(x)G(x),$$

where

$$G(x) = f^{p-1}(x) - (p-1)[g^{\sigma}(x)]^{p-2}.$$

Using Lemma 2.5 and (3.4) again, we have

$$\begin{aligned} G^{\Delta}(x) &\geqslant (p-1)f^{p-2}(x)f^{\Delta}(x) - (p-1)(p-2)\left[g^{\sigma^2}(x)\right]^{p-3}f(x)[\sigma(x)]^{\Delta} \\ &\geqslant (p-1)f(x)[f^{p-3}(x)f^{\Delta}(x)-(p-2)f^{\sigma}(x)]^{p-3}[\sigma^2(x)-a]^{p-3}\sigma^{\Delta}(x) \\ &\geqslant 0. \end{aligned}$$

By Lemma 2.1, we conclude that $G(x)$ is an increasing function. Hence,

$$\begin{aligned} G(x) &\geqslant G(a) \\ &= f^{p-1}(a) - (p-1)[g^{\sigma}(a)]^{p-2} \\ &\geqslant (f^{\rho}(a))^{p-2}[f^{\rho}(a) - (p-1)\mu^{p-2}(a)] \\ &\geqslant 0 \end{aligned}$$

which means that $F^{\Delta}(x) \geqslant 0$. So $F(x) \geqslant F(a) = 0$. The proof is complete. □

THEOREM 3.3. *Let $a, b \in \mathbb{T}$ and $p \geqslant 1$. Assume $f, \sigma : \mathbb{T} \to \mathbb{R}$ be delta differentiable at $t \in \mathbb{T}^{\kappa}$. If f is a non-negative and increasing function on $[a,b]_{\mathbb{T}}$ and satisfies*

$$f^p(x) f^{\Delta}(x) \geqslant \frac{p}{(b-a)^{p-1}} \left[f^{\sigma^2}(x)\right]^p [\sigma^2(x) - a]^{p-1} \sigma^{\Delta}(x) \tag{3.8}$$

and

$$f(a) \geqslant \frac{p+1}{(b-a)^{p-1}} \mu^p(a), \tag{3.9}$$

then

$$\int_a^b f^{p+2}(x)\Delta x \geqslant \frac{1}{(b-a)^{p-1}} \left[\int_a^b f(x)\Delta x\right]^{p+1}. \tag{3.10}$$

Proof. Define

$$F(x) = \int_a^x f^{p+2}(t)\Delta t - \frac{1}{(b-a)^{p-1}} \left[\int_a^x f(t)\Delta t\right]^{p+1}$$

and $g(x) = \int_a^x f(t)\Delta t$. Using Lemma 2.5, it follows that

$$\begin{aligned} F^{\Delta}(x) &= f^{p+2}(x) - \frac{1}{(b-a)^{p-1}} [g^{p+1}(x)]^{\Delta} \\ &\geqslant f^{p+2}(x) - \frac{p+1}{(b-a)^{p-1}} [g^{\sigma}(x)]^p g^{\Delta}(x) \\ &\geqslant f(x)\left\{ f^{p+1}(x) - \frac{p+1}{(b-a)^{p-1}} [g^{\sigma}(x)]^p \right\} \\ &= f(x) G(x), \end{aligned}$$

where

$$G(x) = f^{p+1}(x) - \frac{p+1}{(b-a)^{p-1}} [g^{\sigma}(x)]^p.$$

Using Lemma 2.5 and (3.4) again, we have

$$\begin{aligned} G^{\Delta}(x) &\geqslant (p+1) f^p(x) f^{\Delta}(x) - \frac{p(p+1)}{(b-a)^{p-1}} \left[g^{\sigma^2}(x)\right]^{p-1} f^{\sigma^2}(x) \sigma^{\Delta}(x) \\ &\geqslant (p+1)\left\{ f^p(x) f^{\Delta}(x) - \frac{p}{(b-a)^{p-1}} \left[f^{\sigma^2}(x)\right]^p [\sigma^2(x) - a]^{p-1} \sigma^{\Delta}(x) \right\}. \end{aligned}$$

By Lemma 2.1, we conclude that $G(x)$ is an increasing function. Hence, $G(x) \geqslant G(a) \geqslant 0$ which means that $F^{\Delta}(x) \geqslant 0$. So $F(x) \geqslant F(a) = 0$. The proof is complete. $\square$

REMARK 3.2. If $\mathbb{T} = h\mathbb{Z}$ in Theorem 3.3, we deduce Theorem 4 in [4].

THEOREM 3.4. *Let $a, b \in \mathbb{T}$ and $p \geqslant 3$. Assume $f, \sigma : \mathbb{T} \to \mathbb{R}$ be delta differentiable at $t \in \mathbb{T}^{\kappa}$. If f is a non-negative and increasing function on $[a,b]_{\mathbb{T}}$ and satisfies*

$$[f^{\sigma}(x)]^{\Delta} \geqslant (p-2)[\sigma^2(x) - a]^{p-3}\sigma^{\Delta}(x) \tag{3.11}$$

and

$$f^{\sigma}(a) \geqslant (p-1)\mu^{p-2}(a), \tag{3.12}$$

then

$$\int_a^b [f^{\sigma}(x)]^p \Delta x \geqslant \left[\int_a^b f^{\rho}(x)\Delta x\right]^{p-1}. \tag{3.13}$$

Proof. Define

$$F(x) = \int_a^x [f^{\sigma}(t)]^p \Delta t - \left[\int_a^x f^{\rho}(t)\Delta t\right]^{p-1}$$

and $g(x) = \int_a^x f^{\rho}(t)\Delta t$. Using Lemma 2.5, it follows that

$$\begin{aligned} F^{\Delta}(x) &= [f^{\sigma}(x)]^p - [g^{p-1}(x)]^{\Delta} \\ &\geqslant [f^{\sigma}(x)]^p - (p-1)[g^{\sigma}(x)]^{p-2} g^{\Delta}(x) \\ &= f(x)G(x), \end{aligned}$$

where

$$G(x) = [f^{\sigma}(x)]^{p-1} - (p-1)[g^{\sigma}(x)]^{p-2}.$$

Using Lemma 2.5 and (3.4) again, we have

$$\begin{aligned} G^{\Delta}(x) &\geqslant (p-1)\left\{[f^{\sigma}(x)]^{p-2}[f^{\sigma}(x)]^{\Delta} - (p-1)(p-2)\left[g^{\sigma^2}(x)\right]^{p-3}[g^{\sigma}(x)]^{\Delta}\right\} \\ &\geqslant (p-1)[f^{\sigma}(x)]^{p-2}\left\{[f^{\sigma}(x)]^{\Delta} - (p-2)[\sigma^2(x) - a]^{p-3}\sigma^{\Delta}(x)\right\} \\ &\geqslant 0. \end{aligned}$$

By Lemma 2.1, we conclude that $G(x)$ is an increasing function. Hence,

$$\begin{aligned} G(x) &\geqslant G(a) \\ &= [f^{\sigma}(a)]^{p-1} - (p-1)[g^{\sigma}(a)]^{p-2} \\ &\geqslant [f^{\sigma}(a)]^{p-2}[f^{\sigma}(a) - (p-1)\mu^{p-2}(a)] \\ &\geqslant 0 \end{aligned}$$

which means that $F^{\Delta}(x) \geqslant 0$. So $F(x) \geqslant F(a) = 0$. The proof is complete. $\square$

THEOREM 3.5. *Let $a,b \in \mathbb{T}$ and $p \geqslant 1$. Assume $f,\sigma : \mathbb{T} \to \mathbb{R}$ be delta differentiable at $t \in \mathbb{T}^{\kappa}$. If f is a non-negative and increasing function on $[a,b]_{\mathbb{T}}$ and satisfies*

$$[f^{\sigma}(x)]^{\Delta} \geqslant p * \sigma^{\Delta}(x) \tag{3.14}$$

and

$$f^{\sigma}(a) \geqslant \frac{p+1}{(b-a)^{p-1}}\mu^{p}(a), \tag{3.15}$$

then

$$\int_a^b [f^{\sigma}(x)]^{p+2}\Delta x \geqslant \frac{1}{(b-a)^{p-1}}\left[\int_a^b f^{\rho}(x)\Delta x\right]^{p+1}. \tag{3.16}$$

Proof. Define

$$F(x) = \int_a^x [f^{\sigma}(t)]^{p+2}\Delta t - \frac{1}{(b-a)^{p-1}}\left(\int_a^x f^{\rho}(t)\Delta t\right)^{p+1}$$

and $g(x) = \int_a^x f^{\rho}(t)\Delta t$. Using Lemma 2.5, it follows that

$$\begin{aligned} F^{\Delta}(x) &\geqslant [f^{\sigma}(x)]^{p+2} - \frac{p+1}{(b-a)^{p-1}}[g^{\sigma}(x)]^{p} g^{\Delta}(x) \\ &\geqslant f^{\sigma}(x)\left\{[f^{\sigma}(x)]^{p+1} - \frac{p+1}{(b-a)^{p-1}}[g^{\sigma}(x)]^{p}\right\}, \end{aligned}$$

where

$$G(x) = [f^{\sigma}(x)]^{p+1} - \frac{p+1}{(b-a)^{p-1}}[g^{\sigma}(x)]^{p}.$$

Using Lemma 2.5 again, we have

$$G^{\Delta}(x) \geqslant (p+1)\left\{[f^{\sigma}(x)]^{p}[f^{\sigma}(x)]^{\Delta} - \frac{p}{(b-a)^{p-1}}\left[g^{\sigma^2}(x)\right]^{p-1} f(x)(\sigma(x))^{\Delta}\right\}.$$

Since f is a non-negative and increasing function, then

$$g^{\sigma^2}(x) = \int_a^{\sigma^2(x)} f^{\rho}(t)\Delta t \leqslant f^{\rho\sigma^2}(x)[\sigma^2(x) - a] \leqslant f^{\sigma}(x)(b-a). \tag{3.17}$$

Hence,

$$G^{\Delta}(x) \geqslant (p+1)[f^{\sigma}(x)]^{p}\left\{[f^{\sigma}(x)]^{\Delta} - p[\sigma(x)]^{\Delta}\right\}.$$

By Lemma 2.1, we conclude that $G(x)$ is an increasing function. Hence,

$$\begin{aligned} G(x) &\geqslant G(a) \\ &= [f^{\sigma}(a)]^{p+1} - \frac{p+1}{(b-a)^{p-1}}[g^{\sigma}(a)]^{p} \\ &= [f^{\sigma}(a)]^{p+1} - \frac{p+1}{(b-a)^{p-1}}[f^{\rho}(a)\mu(a)]^{p} \\ &\geqslant [f^{\sigma}(a)]^{p}\left[f^{\sigma}(a) - \frac{p+1}{(b-a)^{p-1}}[\mu(a)]^{p}\right] \\ &\geqslant 0 \end{aligned}$$

which means that $F^{\Delta}(x) \geqslant 0$. So $F(x) \geqslant F(a) = 0$. The proof is complete. $\square$

REMARK 3.3. If $\mathbb{T} = \mathbb{R}$ in Theorem 3.5, we deduce Theorem 1.1.

REMARK 3.4. Our main results give some sufficient conditions for Qi type inequalities from different directions. On the other hand, we also unify the previous conclusions obtained by other authors.

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