

THE WEIGHTED ESTIMATE FOR THE COMMUTATOR OF THE GENERALIZED FRACTIONAL INTEGRAL

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Abstract. Let L be the infinitesimal generator of an analytic semigroup on $L^2(\mathbb{R}^n)$ with Gaussian kernel bound, and let $L^{-\alpha/2}$ be the fractional integral of L for $0 < \alpha < n$. Suppose that b is a locally integral function, then the commutator generated by b and $L^{-\alpha/2}$ is defined by $[b, L^{-\alpha/2}](f) = bL^{-\alpha/2}(f) - L^{-\alpha/2}(bf)$. When b belongs to weighted Lipschitz function space, the boundedness of $[b, L^{-\alpha/2}]$ from $L^p(\omega, \mathbb{R}^n)$ to $L^q(\omega^{1-(1-\alpha/n)q}, \mathbb{R}^n)$ is established, where $1 < p < \infty$, $0 < \beta < 1$ and $1/q = 1/p - (\alpha + \beta)/n$ with $1/p > (\alpha + \beta)/n$.

1. Introduction

In this paper, we investigate the weighted estimates for the commutator of fractional integral operator $L^{-\alpha/2}$, where L is a linear operator on $L^2(\mathbb{R}^n)$ which generates an analytic semigroup e^{-tL} with a kernel $p_t(x, y)$ satisfying a Gaussian upper bound, that is,

$$|p_t(x, y)| \leq \frac{C}{t^{n/2}} e^{-c \frac{|x-y|^2}{t}}, \quad (1.1)$$

for $x, y \in \mathbb{R}^n$ and all $t > 0$.

As Deng, Duong, Sikora and Yan point out in [1,2], the Gaussian upper bound condition (1.1) is satisfied by a large class of differential operators. Here we give two examples:

(1) Let $V \in L^1_{loc}(\mathbb{R}^n)$ be a nonnegative function on \mathbb{R}^n ($n \geq 3$). The Schrödinger operator with potential V is defined by

$$L = -\Delta + V(x) \text{ on } \mathbb{R}^n.$$

From the Feynman-Kac formula, it is known that the kernels $p_t(x, y)$ of the semigroup e^{-tL} satisfy the estimate (1.1);

(2) For the $n \times n$ order matrix $A = (a_{ij}(x))_{1 \leq i, j \leq n}$ with complex entries $a_{ij} \in L^\infty(\mathbb{R}^n)$. Assume that

$$\lambda |\xi|^2 \leq \operatorname{Re} \sum a_{ij} \xi_i \bar{\xi}_j$$

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for all $x \in \mathbb{R}^n$, $\xi = (\xi_1, \xi_2, \dots, \xi_n) \in \mathbb{C}^n$ and for some $\lambda > 0$. Let L be the divergence form operator

$$Lf = -\operatorname{div}(A\nabla f),$$

in the usual weak sense via a sesquilinear form. It turns out that Gaussian bound (1.1) on the heat kernel e^{-tL} holds in the case of real entries, or in the case of complex entries when $n = 1, 2$.

For our purpose, it is convenient to introduce some notation. For $0 < \alpha < n$, the fractional power $L^{-\alpha/2}$ of the operator L is defined by

$$L^{-\alpha/2}(f)(x) = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty e^{-tL}(f)(x) \frac{dt}{t^{-\alpha/2+1}}. \tag{1.2}$$

Note that if $L = -\Delta$ is the Laplacian on \mathbb{R}^n , then $L^{-\alpha/2}$ is the Riesz potential I_α , that is

$$I_\alpha(f)(x) = 2^{-\alpha} \pi^{-\frac{n}{2}} \frac{\Gamma(\frac{n-\alpha}{2})}{\Gamma(\frac{\alpha}{2})} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy.$$

Let b be a locally integrable function on \mathbb{R}^n , the commutator of b and $L^{-\alpha/2}$ is defined by

$$[b, L^{-\alpha/2}](f)(x) = b(x)L^{-\alpha/2}(f)(x) - L^{-\alpha/2}(bf)(x).$$

The area of fractional integral operators has been under intensive research recently. When $b \in BMO(\mathbb{R}^n)$, Chanillo [3] proved that the commutator $[b, I_\alpha]$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for $0 < \alpha < n$, $1 < p < n/\alpha$ and $1/q = 1/p - \alpha/n$. Paluszyński [4] showed that $b \in Lip_\beta(\mathbb{R}^n)$ (homogeneous Lipschitz space) if and only if $[b, I_\alpha]$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ where $0 < \beta < 1$, $1 < p < n/(\alpha + \beta)$ and $1/q = 1/p - (\alpha + \beta)/n$. When b belongs to weighted Lipschitz spaces $Lip_\beta(\omega, \mathbb{R}^n)$, Hu and Gu [5] proved that $[b, I_\alpha]$ is bounded from $L^p(\omega, \mathbb{R}^n)$ to $L^q(\omega^{1-(1-\alpha/n)q}, \mathbb{R}^n)$ for $1/q = 1/p - (\alpha + \beta)/n$ with $1 < p < (\alpha + \beta)/n$.

Duong and Yan [1] extended the result in [3] from the Laplacian $-\Delta$ to the more general operator $L^{-\alpha/2}$ defined by (1.2). They proved that for all $0 < \alpha < n$ and $b \in BMO(\mathbb{R}^n)$, both the operator $L^{-\alpha/2}$ and the commutator $[b, L^{-\alpha/2}]$ are bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ with $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$. Mo and Lu [6] obtained some boundedness properties of multiplier commutators of $L^{-\alpha/2}$ for $0 < \alpha < 1$ when b belongs to $BMO(\mathbb{R}^n)$ or the homogenous Lipschitz space. Auscher and Martell [7] were concerned with the weighted estimate of $L^{-\alpha/2}$ and its commutator. They showed that if $\omega \in A_{p,q}(\mathbb{R}^n)$ and $b \in BMO(\mathbb{R}^n)$, then both the operator $L^{-\alpha/2}$ and the commutator $[b, L^{-\alpha/2}]$ are bounded from $L^p(\omega^p, \mathbb{R}^n)$ to $L^q(\omega^q, \mathbb{R}^n)$ for $0 < \alpha < n$ and for $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$.

The purpose of this paper is to establish the weighted norm estimate of $[b, L^{-\alpha/2}]$ when b belongs to weighted Lipschitz space. Our result is the following:

THEOREM 1.1. *Assume that condition (1.1) holds and let $b \in Lip_\beta(\omega, \mathbb{R}^n)$ with $0 < \beta < 1$, $\omega \in A_{1,q}(\mathbb{R}^n)$. Then for $0 < \alpha < n$, the commutator $[b, L^{-\alpha/2}]$ satisfies*

$$\left\| [b, L^{-\alpha/2}](f) \right\|_{L^q(\omega^{1-(1-\alpha/n)q}, \mathbb{R}^n)} \leq C \|b\|_{Lip_\beta(\omega, \mathbb{R}^n)} \|f\|_{L^p(\omega, \mathbb{R}^n)}$$

for $1/q = 1/p - (\alpha + \beta)/n$ and $1 < p < n/(\alpha + \beta)$.

The paper is organized as follows. In Section 2, we will introduce some notation and definitions, and recall some preliminary results and give the proofs of some lemmas. In Section 3, we will concentrate on the proof of the theorem.

2. Some preliminaries and notation

A non-negative function ω defined on \mathbb{R}^n is called weight if it is locally integrable. A weight ω is said to belong to the Muckenhoupt class $A_p(\mathbb{R}^n)$ for $1 < p < \infty$, if there exists a constant C such that

$$\frac{1}{|B|} \int_B \omega(x) dx \left(\frac{1}{|B|} \int_B \omega(x)^{-\frac{1}{p-1}} dx \right)^{p-1} \leq C \tag{2.1}$$

for every ball $B \subset \mathbb{R}^n$. The class $A_1(\mathbb{R}^n)$ is defined by replacing the above inequality with

$$\frac{1}{|B|} \int_B \omega(y) dy \leq C \inf_{x \in B} \omega(x) \tag{2.2}$$

for every ball $B \subset \mathbb{R}^n$.

The classical $A_p(\mathbb{R}^n)$ weight theory was first introduced by Muckenhoupt in the study of weighted L^p -boundedness of Hardy-Littlewood maximal function in [8]. We also need another weight class $A_{p,q}(\mathbb{R}^n)$ introduced by Muckenhoupt and Wheeden in [9]. Given $1 \leq p \leq q < \infty$. Let s' be the dual of s such that $1/s + 1/s' = 1$. We say that $\omega \in A_{p,q}(\mathbb{R}^n)$ if there exists a constant C such that for every ball $B \subset \mathbb{R}^n$, the inequality

$$\left(\frac{1}{|B|} \int_B \omega(y)^{-p'} dy \right)^{1/p'} \left(\frac{1}{|B|} \int_B \omega(y)^q dy \right)^{1/q} \leq C, \tag{2.3}$$

holds when $1 < p < \infty$, and the inequality

$$\left(\frac{1}{|B|} \int_B \omega(y)^q dy \right)^{1/q} \leq C \inf_{x \in B} \omega(x) \tag{2.4}$$

holds when $p = 1$.

From the Hölder inequality and (2.4), we have

$$A_{1,q}(\mathbb{R}^n) \subset A_1(\mathbb{R}^n). \tag{2.5}$$

LEMMA 2.1. ([7]) *Let $0 < \alpha < n$, $1/q = 1/p - \alpha/n$ and $\omega \in A_{p,q}(\mathbb{R}^n)$. Then $L^{-\alpha/n}$ is bounded from $L^p(\omega^p, \mathbb{R}^n)$ to $L^q(\omega^q, \mathbb{R}^n)$.*

LEMMA 2.2. *Let $t \geq s_0 > 1$ and $\omega \in A_1(\mathbb{R}^n)$. Then $v = \omega^{-1+1/s_0} \in A_{s_0,t}(\mathbb{R}^n)$.*

Proof. Since

$$\left(\frac{1}{|B|} \int_B v(x)^{-s'_0} dx \right)^{1/s'_0} = \left(\frac{1}{|B|} \int_B \omega(x) dx \right)^{1/s'_0} \leq C \left(\inf_{x \in B} \omega(x) \right)^{1/s'_0},$$

and from the fact that $(\omega(x))^{-1} \leq (\inf_{x \in B} \omega(x))^{-1}$ for a.e. $x \in B \subset \mathbb{R}^n$ we have

$$\left(\frac{1}{|B|} \int_B v(x)^t dx \right)^{1/t} = \left(\frac{1}{|B|} \int_B \omega(x)^{(1/s_0-1)t} dx \right)^{1/t} \leq \left(\inf_{x \in B} \omega(x) \right)^{-1+1/s_0}.$$

Then

$$\left(\frac{1}{|B|} \int_B v(x)^{-s'_0} dx \right)^{1/s'_0} \left(\frac{1}{|B|} \int_B v(x)^t dx \right)^{1/t} \leq C.$$

This means $v = \omega^{-1+1/s_0} \in A_{s_0,t}(\mathbb{R}^n)$. \square

Let us recall the definition of weighted Lipschitz function space. Following [10], we say that a locally integrable function b belongs to the weighted Lipschitz function space $Lip_{\beta,p}(\omega, \mathbb{R}^n)$ for $1 \leq p < \infty$ and $\omega \in A_\infty(\mathbb{R}^n) = \cup_{1 \leq r < \infty} A_r(\mathbb{R}^n)$, if

$$\sup_B \frac{1}{\omega(B)^{\beta/n}} \left[\frac{1}{\omega(B)} \int_B |b(x) - b_B|^p \omega(x)^{1-p} dx \right]^{1/p} \leq C < \infty, \tag{2.6}$$

where $b_B = |B|^{-1} \int_B b(y) dy$, $\omega(B) = \int_B \omega(y) dy$ and the supremum is taken over all balls $B \subset \mathbb{R}^n$.

The Banach space of such functions modulo constants is denoted by $Lip_{\beta,p}(\omega, \mathbb{R}^n)$. The smallest bound C satisfying conditions above is then taken to be the norm of b denoted by $\|b\|_{Lip_{\beta,p}(\omega, \mathbb{R}^n)}$. Put $Lip_\beta(\omega, \mathbb{R}^n) = Lip_{\beta,1}(\omega, \mathbb{R}^n)$. Obviously, for the case $\omega = 1$, the $Lip_{\beta,p}(\omega, \mathbb{R}^n)$ space is the classical $Lip_\beta(\mathbb{R}^n)$ space. Let $\omega \in A_1(\mathbb{R}^n)$. Garca-Cuerva in [11] proved that the spaces $Lip_{\beta,p}(\omega, \mathbb{R}^n)$ coincide, and the norms $\|b\|_{Lip_{\beta,p}(\omega, \mathbb{R}^n)}$ are equivalent with respect to different values of p provided that $1 \leq p < \infty$. Since we always discuss under the assumption $\omega \in A_1(\mathbb{R}^n)$ in the following, then we denote the norm of $Lip_{\beta,p}(\omega, \mathbb{R}^n)$ by $\|\cdot\|_{Lip_\beta(\omega, \mathbb{R}^n)}$ for $1 \leq p < \infty$. It is obvious that $Lip_{0,p}(\omega, \mathbb{R}^n) = BMO(\omega, \mathbb{R}^n)$ for $1 \leq p < \infty, \omega \in A_1(\mathbb{R}^n)$.

Associated with analytic semigroup $\{e^{-tL} : t > 0\}$, Martell [12] introduced the Sharp maximal function as follows:

$$M_L^\sharp(f)(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y) - e^{-t_B L} f(y)| dy,$$

where $t_B = r_B^2$ and r_B is the radius of the ball B . Moreover, by means of the generalized good- λ inequality, Martell [12] also gave an analogue of the classical Fefferman-Stein inequality for the Sharp maximal function M_L^\sharp , that is

LEMMA 2.3. *Let $0 < p < \infty$, $\omega \in \cup_{q \geq 1} A_q(\mathbb{R}^n)$ and let $L_0^1(\mathbb{R}^n)$ be the set of functions in $L^p(\mathbb{R}^n)$ with compact support. For every $f \in L_0^1(\omega, \mathbb{R}^n)$ with $M(f) \in L^p(\omega, \mathbb{R}^n)$, we have*

$$\|f\|_{L^p(\omega, \mathbb{R}^n)} \leq C \|M(f)\|_{L^p(\omega, \mathbb{R}^n)} \leq C \|M_L^\sharp(f)\|_{L^p(\omega, \mathbb{R}^n)}.$$

LEMMA 2.4. ([2]) Assume that the semigroup e^{-tL} has a kernel $p_t(x, y)$ which satisfies the upper bound (1.1). Then for $0 < \alpha < n$, the difference operator $(I - e^{-tL})L^{-\alpha/2}$ has an associated kernel $K_{\alpha,t}(x, y)$ which satisfies

$$|K_{\alpha,t}(x, y)| \leq \frac{C}{|x-y|^{n-\alpha}} \frac{t}{|x-y|^2}.$$

LEMMA 2.5. Assume that the semigroup e^{-tL} has a kernel $p_t(x, y)$ satisfying (1.1). Let $0 < \alpha < n$, $1/q = 1/s - \beta/n$, $1/s = 1/p - \alpha/n$, $b \in \text{Lip}_\beta(\omega, \mathbb{R}^n)$ ($0 < \beta < 1$) and $\omega \in A_{1,q}(\mathbb{R}^n)$. Then for every $f \in L^p(\omega, \mathbb{R}^n)$ with $p > 1$ and $s > 1$ we have

$$\begin{aligned} & \frac{1}{|B|} \int_B |e^{-tbL}(b - b_B)L^{-\alpha/2}(f)(y)| dy \\ & \leq C \|b\|_{\text{Lip}_\beta(\omega, \mathbb{R}^n)} \omega(x)^{1-s\alpha\beta/n^2} M_{\beta,\mu,s}(L^{-\alpha/2}(f))(x) \end{aligned}$$

for a.e. $x \in B \subset \mathbb{R}^n$, where $\mu = \omega^{1+\alpha s/n}$ and

$$M_{\beta,\mu,s}(f)(x) = \sup_{x \in B} \left(\frac{1}{\mu(B)^{1-s\beta/n}} \int_B |f(y)|^s \mu(y) dy \right)^{1/s}.$$

Proof. Since $\omega \in A_{1,q}(\mathbb{R}^n) \subset A_1(\mathbb{R}^n)$, then by (2.2), we have

$$\begin{aligned} \frac{1}{|2B|} \int_{2B} |L^{-\alpha/2}f(y)|^s dy &= \frac{1}{|2B| \inf_{y \in 2B} \omega(y)} \int_{2B} |L^{-\alpha/2}f(y)|^s \inf_{y \in 2B} \omega(y) dy \\ &\leq \frac{C}{\omega(2B)} \int_{2B} |L^{-\alpha/2}f(y)|^s \omega(y) dy. \end{aligned} \quad (2.9)$$

From $1/s + \alpha/n = 1/p < 1 + \beta q/n = q/s$, one has $1 + \alpha s/n < q$. Hence, by the Hölder inequality and $\omega \in A_{1,q}(\mathbb{R}^n)$ we can deduce

$$\begin{aligned} & \left(\frac{1}{|2B|} \int_{2B} \omega(z)^{1+\alpha s/n} dz \right)^{1/(1+\alpha s/n)} \\ & \leq \left(\frac{1}{|2B|} \int_{2B} \omega(z)^q dz \right)^{1/q} \leq \frac{C}{|2B|} \int_{2B} \omega(z) dz. \end{aligned}$$

Then

$$\begin{aligned} & \frac{1}{|2B|} \int_{2B} \mu(x) dx = \frac{1}{|2B|} \int_{2B} \omega(x)^{1+\alpha s/n} dx \\ & \leq C \left(\frac{1}{|2B|} \int_{2B} \omega(x) dx \right)^{1+\alpha s/n} \leq \frac{C}{|2B|} \int_{2B} \omega(x) dx \left(\inf_{x \in 2B} \omega(x) \right)^{\alpha s/n}. \end{aligned}$$

This means that

$$\frac{1}{\left(\inf_{x \in 2B} \omega(x) \right)^{\alpha s/n} \omega(2B)} \leq \frac{C}{|2B| \left(\frac{1}{|2B|} \int_{2B} \omega(x) dx \right)^{1+\alpha s/n}} \leq \frac{C}{\mu(2B)}.$$

Therefore,

$$\left(\frac{1}{|2B|} \int_{2B} |L^{-\alpha/2} f(y)|^s dy\right)^{\frac{1}{s}} \leq C \left(\frac{1}{\mu(2B)} \int_{2B} |L^{-\alpha/2} f(z)|^s \mu(z) dz\right)^{1/s}. \tag{2.8}$$

Fix $x \in B$ for some ball B . Let $B_k = 2^k B$. Then

$$\begin{aligned} & \frac{1}{|B|} \int_B |e^{-t_B L}((b - b_B)L^{-\alpha/2} f(y))| dy \\ & \leq \frac{1}{|B|} \int_B \int_{\mathbb{R}^n} |p_{t_B}(y, z)(b(z) - b_B)L^{-\alpha/2} f(z)| dz dy \\ & \leq \frac{1}{|B|} \int_B \int_{2B} |p_{t_B}(y, z)(b(z) - b_B)L^{-\alpha/2} f(z)| dz dy \\ & \quad + \sum_{k=1}^{\infty} \frac{1}{|B|} \int_B \int_{B_{k+1} \setminus B_k} |p_{t_B}(y, z)(b(z) - b_B)L^{-\alpha/2} f(z)| dz dy \\ & = M + N. \end{aligned}$$

By (1.1), it holds that

$$|p_{t_B}(y, z)| \leq C|2B|^{-1}$$

for any $y \in B$ and $z \in 2B$.

Then by means of the Hölder inequality, (2.6) and (2.8), we have

$$\begin{aligned} M & \leq \frac{C}{|2B|} \int_{2B} |(b(z) - b_B)L^{-\alpha/2} f(z)| dz \\ & \leq C \left(\frac{1}{|2B|} \int_{2B} |b(z) - b_{2B}|^{s'} \omega(z)^{1-s'} dz\right)^{1/s'} \left(\frac{1}{|2B|} \int_{2B} |L^{-\alpha/2} f(z)|^s \omega(z) dz\right)^{1/s} \\ & \quad + \frac{C}{|2B|} \int_{2B} |b(z) - b_{2B}| dz \left(\frac{1}{|2B|} \int_{2B} |L^{-\alpha/2} f(z)|^s dz\right)^{1/s} \\ & \leq C \|b\|_{Lip_{\beta}(\omega, \mathbb{R}^n)} \frac{(\omega(2B))^{1+\beta/n}}{|B|} \left(\frac{1}{\mu(2B)} \int_{2B} |L^{-\alpha/2} f(z)|^s \mu(z) dz\right)^{1/s}. \end{aligned}$$

From (2.2) we get

$$\begin{aligned} \omega(2B)^{\beta/n} & = \left(\inf_{x \in 2B} \omega(x)\right)^{-\alpha\beta s/n^2} \left(\int_{2B} \omega(x) \left(\inf_{x \in 2B} \omega(x)\right)^{\alpha s/n} dx\right)^{\beta/n} \\ & \leq C \left(\frac{\omega(2B)}{|2B|}\right)^{-\alpha\beta s/n^2} \mu(2B)^{\beta/n}. \end{aligned} \tag{2.9}$$

Since $1/q = 1/s - \beta/n > 0$, we have $1 - \alpha\beta s/n^2 > 0$. Then

$$\left(\frac{\omega(2B)}{|2B|}\right)^{1-\alpha\beta s/n^2} \leq C(\omega(x))^{1-s\alpha\beta/n^2} \tag{2.10}$$

for a.e. $x \in 2B$. Thus

$$M \leq C(\omega(x))^{1-s\alpha\beta/n^2} \|b\|_{Lip_\beta(\omega, \mathbb{R}^n)} M_{\beta, \mu, s}(L^{-\alpha/2} f)(x).$$

Moreover, for any $z \in B$ and $y \in B_{k+1} \setminus B_k$, we have $|z - y| \geq 2^{k-1}r_B$ and

$$|p_{t_B}(y, z)| \leq C \frac{e^{-c2^{2(k-1)}2^{n(k+1)}}}{|B_{k+1}|}$$

from (1.1). Since $|b_B - b_{B_k}| \leq Ck \frac{\omega(2B)}{|2B|} \omega(B_k)^{\beta/n}$, similarly the estimate of M we have

$$\begin{aligned} N &\leq C \sum_{k=1}^{\infty} \frac{e^{-c2^{2(k-1)}2^{n(k+1)}}}{|B_{k+1}|} \int_{B_{k+1}} |b(z) - b_{B_{k+1}}| |L^{-\alpha/2} f(z)| dz \\ &\quad + C \sum_{k=1}^{\infty} \frac{e^{-c2^{2(k-1)}2^{n(k+1)}}}{|B_{k+1}|} |b_B - b_{B_{k+1}}| \int_{B_{k+1}} |L^{-\alpha/2} f(z)| dz \\ &\leq C \|b\|_{Lip_\beta(\omega, \mathbb{R}^n)} \omega(x)^{1-s\alpha\beta/n^2} M_{\beta, \mu, s}(L^{-\alpha/2} f)(x). \quad \square \end{aligned}$$

3. Proof of the main result

Let $1/q = 1/s - \beta/n$, $1/s = 1/p - \alpha/n$ and $\mu = \omega^{1+\alpha s/n} = \omega^{s/p}$. Choose two real numbers s_1 and s_0 such that $s_1 > s_0 > 1$. We will prove that there exists a constant C such that for all $x \in \mathbb{R}^n$ and for all $B \ni x$, it holds that

$$\begin{aligned} &\frac{1}{|B|} \int_B |[b, L^{-\alpha/2}]f(y) - e^{-t_B L}([b, L^{-\alpha/2}]f)(y)| dy \\ &\leq C \|b\|_{Lip_\beta(\omega, \mathbb{R}^n)} \left(\omega(x)^{1-s\alpha\beta/n^2} M_{\beta, \mu, s}(L^{-\alpha/2} f)(x) \right. \\ &\quad \left. + \omega(x)^{1-\alpha/n} (M_{\alpha+\beta, \omega, s_0}(f)(x) + M_{\alpha+\beta, \omega, s_1}(f)(x)) \right). \end{aligned} \tag{3.1}$$

Since $\omega \in A_{1,q}(\mathbb{R}^n)$ and $s/p = 1 + \alpha s/n < q$, then $\mu = \omega^{s/p} \in A_1(\mathbb{R}^n)$ and $\omega^{1/p} \in A_{s,q}(\mathbb{R}^n)$ (see [9]). From (3.1), Theorem 1.1 will be proved by Lemma 2.1, Lemma 2.2, Lemma 2.3 and the continuity of the maximal function $M_{\beta, \mu, s} f$, that is to say

$$\begin{aligned} &\|[b, L^{-\alpha/2}]f\|_{L^q(\omega^{1-(1-\alpha/n)q}, \mathbb{R}^n)} \\ &\leq C \|M_L^\sharp [b, L^{-\alpha/2}]f\|_{L^q(\omega^{1-(1-\alpha/n)q}, \mathbb{R}^n)} \\ &\leq C \|b\|_{Lip_\beta(\omega, \mathbb{R}^n)} \left(\|M_{\beta, \mu, s}(L^{-\alpha/2} f)\|_{L^q(\mu, \mathbb{R}^n)} \right. \\ &\quad \left. + \|M_{\alpha+\beta, \omega, s_0}(f)\|_{L^q(\omega, \mathbb{R}^n)} + \|M_{\alpha+\beta, \omega, s_1}(f)\|_{L^q(\omega, \mathbb{R}^n)} \right) \\ &\leq C \|b\|_{Lip_\beta(\omega, \mathbb{R}^n)} \left(\|L^{-\alpha/2} f\|_{L^s(\omega^{s/p}, \mathbb{R}^n)} + \|f\|_{L^p(\omega, \mathbb{R}^n)} \right) \\ &\leq C \|b\|_{Lip_\beta(\omega, \mathbb{R}^n)} \|f\|_{L^p(\omega, \mathbb{R}^n)}. \end{aligned}$$

Now let us prove (3.1). For an arbitrary fixed $x \in \mathbb{R}^n$, choose a ball B which contains x . Let $f_1 = f\chi_{2B}$ and $f_2 = f - f_1$. We write

$$[b, L^{-\alpha/2}]f = (b - b_B)L^{-\alpha/2}f - L^{-\alpha/2}(b - b_B)f_1 - L^{-\alpha/2}(b - b_B)f_2,$$

and

$$\begin{aligned} e^{-t_B L}([b, L^{-\alpha/2}]f) &= e^{-t_B L}((b - b_B)L^{-\alpha/2}f) - e^{-t_B L}(L^{-\alpha/2}(b - b_B)f_1) \\ &\quad - e^{-t_B L}(L^{-\alpha/2}(b - b_B)f_2). \end{aligned}$$

Then

$$\begin{aligned} &\frac{1}{|B|} \int_B \left| [b, L^{-\alpha/2}]f(y) - e^{-t_B L}([b, L^{-\alpha/2}]f)(y) \right| dy \\ &\leq \frac{1}{|B|} \int_B \left| (b - b_B)L^{-\alpha/2}f(y) \right| dy + \frac{1}{|B|} \int_B \left| L^{-\alpha/2}((b - b_B)f_1)(y) \right| dy \\ &\quad + \frac{1}{|B|} \int_B \left| e^{-t_B L}((b - b_B)L^{-\alpha/2}f)(y) \right| dy \\ &\quad + \frac{1}{|B|} \int_B \left| e^{-t_B L}(L^{-\alpha/2}(b - b_B)f_1)(y) \right| dy \\ &\quad + \frac{1}{|B|} \int_B \left| (L^{-\alpha/2} - e^{-t_B L}L^{-\alpha/2})((b - b_B)f_2)(y) \right| dy \\ &= I + II + III + IV + V. \end{aligned}$$

By the Hölder inequality, (2.6), (2.8) and (2.10), we have

$$\begin{aligned} I &\leq \frac{C}{|B|} \int_B \left| (b(y) - b_B)L^{-\alpha/2}(f)(y) \right| dy \\ &\leq C \left(\frac{1}{|B|} \int_B |b(y) - b_B|^{s'} \omega(y)^{1-s'} dy \right)^{1/s'} \left(\frac{1}{|B|} \int_B \left| L^{-\alpha/2}(f)(y) \right|^s \omega(y) dy \right)^{1/s} \\ &\leq C \|b\|_{Lip_\beta(\omega, \mathbb{R}^n)} \frac{(\omega(B))^{1+\beta/n}}{|B|} \left(\frac{1}{\inf_B \omega(y)^{\alpha s/n} \omega(B)} \int_B \left| L^{-\alpha/2}f(y) \right|^s \mu(y) dy \right)^{1/s} \\ &\leq C \|b\|_{Lip_\beta(\omega, \mathbb{R}^n)} (\omega(x))^{1-s\alpha\beta/n^2} M_{\beta, \mu, s}(L^{-\alpha/2}f)(x). \end{aligned}$$

Let $1/t = 1/s_0 - \alpha/n$ and $v = \omega^{-1+1/s_0}$. Then from (2.2) we have

$$\begin{aligned} &\left(\frac{1}{|B|} \int_B v(y)^{-t'} dy \right)^{1/t'} \leq \left(\frac{1}{|B|} \int_B \omega(y)^{1-\alpha t'/n} dy \right)^{1/t'} \\ &\leq C \left(\inf_{x \in B} \omega(x) \right)^{-\alpha/n} \left(\frac{1}{|B|} \int_B \omega(y) dy \right)^{1/t'} \leq C \left(\inf_{x \in B} \omega(x) \right)^{1/t' - \alpha/n}. \end{aligned} \tag{2.11}$$

Suppose $s_2 = s_1 s_0 / (s_1 - s_0)$. We have

$$\begin{aligned} & \left(\int_{2B} |(b(y) - b_{2B})f(y)|^{s_0} \omega(y)^{1-s_0} dy \right)^{1/s_0} \\ & \leq \left(\int_{2B} |b(y) - b_{2B}|^{s_2} \omega(y)^{1-s_2} dy \right)^{1/s_2} \left(\int_{2B} |f(y)|^{s_1} \omega(y) dy \right)^{1/s_1} \\ & \leq C \|b\|_{Lip_\beta(\omega, \mathbb{R}^n)} \omega(2B)^{1/s_0 - \alpha/n} \left(\frac{1}{\omega(2B)^{1-s_1(\alpha+\beta)/n}} \int_{2B} |f(y)|^{s_1} \omega(y) dy \right)^{1/s_1} \\ & \leq C \|b\|_{Lip_\beta(\omega, \mathbb{R}^n)} \omega(2B)^{1/t} M_{\alpha+\beta, \omega, s_1}(f)(x). \end{aligned} \quad (2.12)$$

Notice that

$$\begin{aligned} & |b_B - b_{2B}| \left(\int_{2B} |f(y)|^{s_0} \omega(y)^{1-s_0} dy \right)^{1/s_0} \\ & \leq C \|b\|_{Lip_\beta(\omega, \mathbb{R}^n)} \left(\inf_{x \in 2B} \omega(x) \right)^{-1} \frac{\omega(2B)^{1+\beta/n}}{|2B|} \omega(2B)^{1/s_0 - (\alpha+\beta)/n} M_{\alpha+\beta, \omega, s}(f)(x) \\ & \leq C \|b\|_{Lip_\beta(\omega, \mathbb{R}^n)} \omega(2B)^{1/t} M_{\alpha+\beta, \omega, s_0}(f)(x). \end{aligned} \quad (2.13)$$

Then by Lemma 2.1 and (2.11)–(2.13), we obtain

$$\begin{aligned} II &= \frac{1}{|B|} \int_B \left| L^{-\alpha/2}((b - b_B)f\chi_{2B})(y) \right| v(y)v(y)^{-1} dy \\ & \leq \left(\frac{1}{|B|} \int_B \left| L^{-\alpha/2}((b - b_B)f\chi_{2B})(y) \right|^t v(y)^t dy \right)^{1/t} \left(\frac{1}{|B|} \int_B v(y)^{-t} dy \right)^{1/t'} \\ & \leq C \left(\inf_{x \in B} \omega(x) \right)^{1/t' - \alpha/n} |B|^{-1/t} \left(\int_{2B} |(b(y) - b_B)f(y)|^{s_0} \omega(y)^{1-s_0} dy \right)^{1/s_0} \\ & \leq C \left(\inf_{x \in B} \omega(x) \right)^{1/t' - \alpha/n} |B|^{-1/t} \left\{ \left(\int_{2B} |(b(y) - b_{2B})f(y)|^{s_0} \omega(y)^{1-s_0} dy \right)^{1/s_0} \right. \\ & \quad \left. + |b_B - b_{2B}| \left(\int_{2B} |f(y)|^{s_0} \omega(y)^{1-s_0} dy \right)^{1/s_0} \right\} \\ & \leq C \|b\|_{Lip_\beta(\omega, \mathbb{R}^n)} \left(\inf_{x \in B} \omega(x) \right)^{1/t' - \alpha/n} \left(\frac{\omega(2B)}{|2B|} \right)^{1/t} \\ & \quad \left\{ M_{\alpha+\beta, \omega, s_0}(f)(x) + M_{\alpha+\beta, \omega, s}(f)(x) \right\} \\ & \leq C \|b\|_{Lip_\beta(\omega, \mathbb{R}^n)} \omega(x)^{1-\alpha/n} \left\{ M_{\alpha+\beta, \omega, s_0}(f)(x) + M_{\alpha+\beta, \omega, s}(f)(x) \right\}. \end{aligned}$$

Similar to the estimate of Lemma 2.6, we have

$$\begin{aligned} III + IV & \leq C \|b\|_{Lip_\beta(\omega, \mathbb{R}^n)} \left(\omega(x)^{1-s\alpha\beta/n^2} M_{\beta, \mu, s}(L^{-\alpha/2}(f))(x) \right. \\ & \quad \left. + \omega(x)^{1-\alpha/n} (M_{\alpha+\beta, \omega, s_0}(f)(x) + M_{\alpha+\beta, \omega, s_1}(f)(x)) \right). \end{aligned}$$

Let us see what happens with the term V. Using Lemma 2.4, we have

$$\begin{aligned} V &\leq \frac{1}{|B|} \int_B \int_{(2B)^c} |K_{\alpha, \beta}(y, z)| |(b(z) - b_B)f(z)| dz dy \\ &\leq C \sum_{k=1}^{\infty} \int_{B_{k+1} \setminus B_k} \frac{1}{|x_0 - z|^{n-\alpha}} \frac{r_B}{|x_0 - z|} |(b(z) - b_B)f(z)| dz \\ &\leq C \sum_{k=1}^{\infty} 2^{-k} \frac{1}{|B_{k+1}|^{1-\alpha/n}} \int_{B_{k+1}} |(b(z) - b_B)f(z)| dz \\ &\quad + C \sum_{k=1}^{\infty} 2^{-k} |b_{B_{k+1}} - b_B| \frac{1}{|B_{k+1}|^{1-\alpha/n}} \int_{B_{k+1}} |f(z)| dz. \end{aligned}$$

Similar to the estimate of II, we get

$$V \leq C \omega(x)^{1-\alpha/n} \|b\|_{\text{Lip}_\beta(\omega, \mathbb{R}^n)} M_{\alpha+\beta, \omega, s_0}(x).$$

Combining the above estimates I, II, III, IV and V, we obtain (3.1). Then the proof of the Theorem 1.1 is completed.

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