

## A VARIANT OF CHEBYSHEV INEQUALITY WITH APPLICATIONS

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*Abstract.* A variant of Chebyshev inequality is established and it is applied to obtain some inequalities for expectation, variance and cumulative distribution functions as well as to provide new proofs for some classical inequalities.

### 1. Introduction

The following Chebyshev inequality (see for example [1, p. 297]) is well known:

**THEOREM 1.** *Let  $f, g : [a, b] \rightarrow \mathbf{R}$  be two absolutely continuous mappings on  $[a, b]$  whose derivatives  $f', g' : [a, b] \rightarrow \mathbf{R}$  belong to the Lebesgue space  $L_\infty[a, b]$ . Then*

$$\left| \frac{1}{b-a} \int_a^b f(t)g(t) dt - \frac{1}{b-a} \int_a^b f(t) dt \cdot \frac{1}{b-a} \int_a^b g(t) dt \right| \leq \frac{1}{12} (b-a)^2 \|f'\|_\infty \|g'\|_\infty. \quad (1)$$

The constant  $\frac{1}{12}$  is the best possible.

In [2], Matić, Pečarić and Ujević prove the following refinement of (1) which is called the “pre-Chebyshev” inequality in [3]

$$\left| \frac{1}{b-a} \int_a^b f(t)g(t) dt - \frac{1}{b-a} \int_a^b f(t) dt \cdot \frac{1}{b-a} \int_a^b g(t) dt \right| \leq \frac{1}{2\sqrt{3}} (b-a) \|f'\|_\infty \left[ \frac{1}{b-a} \int_a^b g^2(t) dt - \left( \frac{1}{b-a} \int_a^b g(t) dt \right)^2 \right]^{\frac{1}{2}}, \quad (2)$$

provided that  $f$  is as in Theorem 1 and all the integrals in (2) exist and are finite.

In [3], Barnett and Dragomir have applied the pre-Chebyshev inequality (2) to obtain the following three inequalities for the expectation, variance and cumulative distribution function of a random variable having the probability density function which is assumed to be absolutely continuous and whose derivative is essentially bounded.

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**THEOREM 2.** *Let  $X$  be a random variable whose probability density function  $f : [a, b] \rightarrow \mathbf{R}$  is absolutely continuous on  $[a, b]$  and  $f' \in L_\infty[a, b]$ . Then*

$$\left| E(X) - \frac{a+b}{2} \right| \leq \frac{1}{12}(b-a)^3 \|f'\|_\infty, \tag{3}$$

where  $E(X)$  is the expectation of the random variable  $X$ .

**THEOREM 3.** *Let the assumptions of Theorem 2 hold. If*

$$\sigma_\mu(X) := \left[ \int_a^b (t - \mu)^2 f(t) dt \right]^{\frac{1}{2}}, \quad \mu \in [a, b],$$

then,

$$\begin{aligned} & \left| \sigma_\mu^2(X) - \left( \mu - \frac{a+b}{2} \right)^2 - \frac{1}{12}(b-a)^2 \right| \\ & \leq \frac{1}{2\sqrt{3}}(b-a)^3 \left[ \frac{1}{3} \left( \mu - \frac{a+b}{2} \right)^2 + \frac{1}{180}(b-a)^2 \right]^{\frac{1}{2}} \|f'\|_\infty \\ & \leq \frac{1}{3\sqrt{15}}(b-a)^4 \|f'\|_\infty \end{aligned} \tag{4}$$

for all  $\mu \in [a, b]$ .

**THEOREM 4.** *Let the assumptions of Theorem 2 hold. Then*

$$\left| E(X) + (b-a)F(x) - x - \frac{b-a}{2} \right| \leq \frac{1}{12}(b-a)^3 \|f'\|_\infty \tag{5}$$

for all  $x \in [a, b]$ , where  $F(x) := \int_a^x f(t) dt$  is the cumulative distribution function of the random variable  $X$  having probability density function  $f : [a, b] \rightarrow \mathbf{R}$ .

In this paper, we will establish a variant of Chebyshev inequality which gives a refinement of the pre-Chebyshev inequality (2) and it is applied to provide a new proof of inequality (3) and improve the inequalities (4) and (5). It is very interesting that we can also use the variant of Chebyshev inequality to provide new and simpler proofs of some well-known classical inequalities.

### 2. A variant of Chebyshev inequality

**THEOREM 5.** *Let  $f, g : [a, b] \rightarrow \mathbf{R}$  be two absolutely continuous mappings on  $[a, b]$  and  $f' \in L_\infty[a, b]$ . Then for any  $x \in [a, b]$  we have*

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(t)g(t) dt - \frac{1}{b-a} \int_a^b f(t) dt \cdot \frac{1}{b-a} \int_a^b g(t) dt \right| \\ & \leq \frac{\|f'\|_\infty}{b-a} \int_a^b |t-x| \left| g(t) - \frac{1}{b-a} \int_a^b g(u) du \right| dt \end{aligned} \tag{6}$$

$$\leq \|f'\|_\infty \left[ \left( x - \frac{a+b}{2} \right)^2 + \frac{(b-a)^2}{12} \right]^{\frac{1}{2}} \left[ \frac{1}{b-a} \int_a^b g^2(t) dt - \left( \frac{1}{b-a} \int_a^b g(t) dt \right)^2 \right]^{\frac{1}{2}}.$$

*Proof.* It is clear that for any  $x \in [a, b]$  we have the identity

$$\begin{aligned} & \frac{1}{b-a} \int_a^b f(t)g(t) dt - \frac{1}{b-a} \int_a^b f(t) dt \cdot \frac{1}{b-a} \int_a^b g(t) dt \\ &= \frac{1}{b-a} \int_a^b [f(t) - f(x)] \left[ g(t) - \frac{1}{b-a} \int_a^b g(u) du \right] dt. \end{aligned}$$

Then the first inequality in (6) follows immediately, and by the Cauchy-Schwarz inequality we get

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(t)g(t) dt - \frac{1}{b-a} \int_a^b f(t) dt \cdot \frac{1}{b-a} \int_a^b g(t) dt \right| \\ & \leq \frac{\|f'\|_\infty}{b-a} \int_a^b |t-x| \left| g(t) - \frac{1}{b-a} \int_a^b g(u) du \right| dt \\ & \leq \frac{\|f'\|_\infty}{b-a} \left[ \int_a^b |t-x|^2 dt \right]^{\frac{1}{2}} \left[ \int_a^b \left| g(t) - \frac{1}{b-a} \int_a^b g(u) du \right|^2 dt \right]^{\frac{1}{2}}. \end{aligned}$$

However,

$$\int_a^b |t-x|^2 dt = (b-a) \left[ \left( x - \frac{a+b}{2} \right)^2 + \frac{(b-a)^2}{12} \right],$$

and

$$\int_a^b \left| g(t) - \frac{1}{b-a} \int_a^b g(u) du \right|^2 dt = \int_a^b g^2(t) dt - \frac{1}{b-a} \left( \int_a^b g(t) dt \right)^2.$$

Consequently, the second inequality in (6) follows.  $\square$

**COROLLARY 1.** *Let the assumptions of Theorem 5 hold. Then we get a variant of Chebyshev inequality as*

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(t)g(t) dt - \frac{1}{b-a} \int_a^b f(t) dt \cdot \frac{1}{b-a} \int_a^b g(t) dt \right| \\ & \leq \frac{\|f'\|_\infty}{b-a} \int_a^b \left| t - \frac{a+b}{2} \right| \left| g(t) - \frac{1}{b-a} \int_a^b g(u) du \right| dt. \end{aligned} \tag{7}$$

*The inequality (7) is sharp.*

*Proof.* The inequality (7) holds clearly by taking  $x = \frac{a+b}{2}$  in the first inequality of (6), and we may choose the function  $f = g, f : [a, b] \rightarrow \mathbf{R}, f(t) = t, t \in [a, b]$  to attain the equality in (7).  $\square$

**REMARK 1.** From (6), we see that the inequality (7) is exactly a refinement of the pre-Chebyshev inequality (2).

### 3. Some inequalities for expectation and variance

We start by using the variant of Chebyshev inequality (7) to give a new proof of inequality (3). Notice that  $\int_a^b f(t) dt = 1$  and so

$$E(X) - \frac{a+b}{2} = \int_a^b t f(t) dt - \frac{1}{b-a} \int_a^b f(t) dt \cdot \int_a^b t dt.$$

However, by (7) we get

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b t f(t) dt - \frac{1}{b-a} \int_a^b f(t) dt \cdot \frac{1}{b-a} \int_a^b t dt \right| \\ & \leq \frac{\|f'\|_\infty}{b-a} \int_a^b \left| t - \frac{a+b}{2} \right|^2 dt = \frac{1}{12} (b-a)^2 \|f'\|_\infty, \end{aligned}$$

and then (3) follows.

**THEOREM 6.** *Let  $X$  be a random variable whose probability density function  $f : [a, b] \rightarrow \mathbf{R}$  is absolutely continuous on  $[a, b]$  and  $f' \in L_\infty[a, b]$ . Then we have:*

$$\begin{aligned} & \left| \sigma_\mu^2(X) - \left( \mu - \frac{a+b}{2} \right)^2 - \frac{1}{12} (b-a)^2 \right| \\ & \leq \|f'\|_\infty \times \begin{cases} \frac{4}{3} \left( \mu - \frac{a+b}{2} \right)^4 + \frac{(b-a)^2}{6} \left( \mu - \frac{a+b}{2} \right)^2 - \frac{(b-a)^3}{6} \left( \mu - \frac{a+b}{2} \right) + \frac{(b-a)^4}{288} \\ \quad + \frac{4}{3} \left( \mu - \frac{a+b}{2} \right) \left[ \left( \mu - \frac{a+b}{2} \right)^2 + \frac{(b-a)^2}{12} \right]^{\frac{3}{2}}, & a \leq \mu \leq \frac{2a+b}{3}, \\ \frac{8}{3} \left( \mu - \frac{a+b}{2} \right)^4 + \frac{(b-a)^2}{3} \left( \mu - \frac{a+b}{2} \right)^2 + \frac{5(b-a)^4}{288}, & \frac{2a+b}{3} < \mu < \frac{a+2b}{3}, \\ \frac{4}{3} \left( \mu - \frac{a+b}{2} \right)^4 + \frac{(b-a)^2}{6} \left( \mu - \frac{a+b}{2} \right)^2 + \frac{(b-a)^3}{6} \left( \mu - \frac{a+b}{2} \right) + \frac{(b-a)^4}{288} \\ \quad - \frac{4}{3} \left( \mu - \frac{a+b}{2} \right) \left[ \left( \mu - \frac{a+b}{2} \right)^2 + \frac{(b-a)^2}{12} \right]^{\frac{3}{2}}, & \frac{a+2b}{3} \leq \mu \leq b. \end{cases} \end{aligned} \tag{8}$$

*Proof.* Observe that

$$\frac{1}{b-a} \int_a^b (t-\mu)^2 dt = \left( \mu - \frac{a+b}{2} \right)^2 + \frac{1}{12} (b-a)^2, \quad \int_a^b f(t) dt = 1,$$

and put  $g(t) = (t-\mu)^2$  in (7), we get

$$\begin{aligned} & \left| \sigma_\mu^2(X) - \left( \mu - \frac{a+b}{2} \right)^2 - \frac{(b-a)^2}{12} \right| \\ & = \left| \int_a^b (t-\mu)^2 f(t) dt - \frac{1}{b-a} \int_a^b (t-\mu)^2 dt \cdot \int_a^b f(t) dt \right| \\ & \leq \|f'\|_\infty \int_a^b \left| t - \frac{a+b}{2} \right| \left| (t-\mu)^2 - \left( \mu - \frac{a+b}{2} \right)^2 - \frac{(b-a)^2}{12} \right| dt. \end{aligned} \tag{9}$$

The last integral can be calculated as follows:

For brevity, we put

$$p(t) := (t - \mu)^2 - \left(\mu - \frac{a+b}{2}\right)^2 - \frac{(b-a)^2}{12}, \quad t \in [a, b]$$

and denote  $t_1 = \mu - [(\mu - \frac{a+b}{2})^2 + \frac{(b-a)^2}{12}]^{\frac{1}{2}}$ ,  $t_2 = \mu + [(\mu - \frac{a+b}{2})^2 + \frac{(b-a)^2}{12}]^{\frac{1}{2}}$ .

Clearly,  $p(\mu) = -(\mu - \frac{a+b}{2})^2 - \frac{(b-a)^2}{12} < 0$ ,  $p(\frac{a+b}{2}) = -\frac{(b-a)^2}{12} < 0$  and

$$p'(t) = 2(t - \mu),$$

which implies that  $p(\mu) \leq p(\frac{a+b}{2})$  and  $p(t)$  is strictly decreasing on  $[a, \mu]$  as well as strictly increasing on  $[\mu, b]$ .

Moreover, we have

$$p(a) = (b - a) \left(\mu - \frac{2a+b}{3}\right)$$

and

$$p(b) = (b - a) \left(\frac{a+2b}{3} - \mu\right)$$

which imply that  $p(a) \leq 0$  and  $p(b) > 0$  in case  $a \leq \mu \leq \frac{2a+b}{3}$ ,  $p(a) > 0$  and  $p(b) > 0$  in case  $\frac{2a+b}{3} < \mu < \frac{a+2b}{3}$  as well as  $p(a) > 0$  and  $p(b) \leq 0$  in case  $\frac{a+2b}{3} \leq \mu \leq b$ .

So, there are three possible cases to be determined.

(i) In case  $a \leq \mu \leq \frac{2a+b}{3}$ ,  $t_2 \in (\frac{a+b}{2}, b) \subset (\mu, b)$  is the unique zero of  $p(t)$  such that  $p(t) \leq 0$  for  $t \in [a, t_2]$  and  $p(t) > 0$  for  $t \in (t_2, b]$ . We have

$$\begin{aligned} & \int_a^b \left| t - \frac{a+b}{2} \right| \left| (t - \mu)^2 - \left(\mu - \frac{a+b}{2}\right)^2 - \frac{(b-a)^2}{12} \right| dt \\ &= \int_a^{\frac{a+b}{2}} \left(\frac{a+b}{2} - t\right) \left[ \left(\mu - \frac{a+b}{2}\right)^2 + \frac{(b-a)^2}{12} - (t - \mu)^2 \right] dt \\ & \quad + \int_{\frac{a+b}{2}}^{t_2} \left(t - \frac{a+b}{2}\right) \left[ \left(\mu - \frac{a+b}{2}\right)^2 + \frac{(b-a)^2}{12} - (t - \mu)^2 \right] dt \\ & \quad + \int_{t_2}^b \left(t - \frac{a+b}{2}\right) \left[ (t - \mu)^2 - \left(\mu - \frac{a+b}{2}\right)^2 - \frac{(b-a)^2}{12} \right] dt \\ &= \frac{4}{3} \left(\mu - \frac{a+b}{2}\right)^4 + \frac{(b-a)^2}{6} \left(\mu - \frac{a+b}{2}\right)^2 - \frac{(b-a)^3}{6} \left(\mu - \frac{a+b}{2}\right) \\ & \quad + \frac{(b-a)^4}{288} + \frac{4}{3} \left(\mu - \frac{a+b}{2}\right) \left[ \left(\mu - \frac{a+b}{2}\right)^2 + \frac{(b-a)^2}{12} \right]^{\frac{3}{2}}. \end{aligned} \tag{10}$$

(ii) In case  $\frac{2a+b}{3} < \mu < \frac{a+2b}{3}$ ,  $t_1 \in (a, \mu)$  and  $t_2 \in (\mu, b)$  are two zeros of  $p(t)$  such that  $p(t) > 0$  for  $t \in [a, t_1] \cup (t_2, b]$  and  $p(t) < 0$  for  $t \in (t_1, t_2)$ . We have

$$\begin{aligned}
 & \int_a^b \left| t - \frac{a+b}{2} \right| \left| (t-\mu)^2 - \left( \mu - \frac{a+b}{2} \right)^2 - \frac{(b-a)^2}{12} \right| dt \\
 &= \int_a^{t_1} \left( \frac{a+b}{2} - t \right) \left[ (t-\mu)^2 - \left( \mu - \frac{a+b}{2} \right)^2 - \frac{(b-a)^2}{12} \right] dt \\
 & \quad + \int_{t_1}^{\frac{a+b}{2}} \left( \frac{a+b}{2} - t \right) \left[ \left( \mu - \frac{a+b}{2} \right)^2 + \frac{(b-a)^2}{12} - (t-\mu)^2 \right] dt \\
 & \quad + \int_{\frac{a+b}{2}}^{t_2} \left( t - \frac{a+b}{2} \right) \left[ \left( \mu - \frac{a+b}{2} \right)^2 + \frac{(b-a)^2}{12} - (t-\mu)^2 \right] dt \\
 & \quad + \int_{t_2}^b \left( t - \frac{a+b}{2} \right) \left[ (t-\mu)^2 - \left( \mu - \frac{a+b}{2} \right)^2 - \frac{(b-a)^2}{12} \right] dt \\
 &= \frac{8}{3} \left( \mu - \frac{a+b}{2} \right)^4 + \frac{(b-a)^2}{3} \left( \mu - \frac{a+b}{2} \right)^2 + \frac{5(b-a)^4}{288}.
 \end{aligned} \tag{11}$$

(iii) In case  $\frac{2a+b}{3} \leq \mu \leq b$ ,  $t_1 \in (a, \frac{a+b}{2}) \subset (a, \mu)$  is the unique zero of  $p(t)$  such that  $p(t) > 0$  for  $t \in [a, t_1)$  and  $p(t) \leq 0$  for  $t \in (t_1, b]$ . We have

$$\begin{aligned}
 & \int_a^b \left| t - \frac{a+b}{2} \right| \left| (t-\mu)^2 - \left( \mu - \frac{a+b}{2} \right)^2 - \frac{(b-a)^2}{12} \right| dt \\
 &= \int_a^{t_1} \left( \frac{a+b}{2} - t \right) \left[ (t-\mu)^2 - \left( \mu - \frac{a+b}{2} \right)^2 - \frac{(b-a)^2}{12} \right] dt \\
 & \quad + \int_{t_1}^{\frac{a+b}{2}} \left( \frac{a+b}{2} - t \right) \left[ \left( \mu - \frac{a+b}{2} \right)^2 + \frac{(b-a)^2}{12} - (t-\mu)^2 \right] dt \\
 & \quad + \int_{\frac{a+b}{2}}^b \left( t - \frac{a+b}{2} \right) \left[ \left( \mu - \frac{a+b}{2} \right)^2 + \frac{(b-a)^2}{12} - (t-\mu)^2 \right] dt \\
 &= \frac{4}{3} \left( \mu - \frac{a+b}{2} \right)^4 + \frac{(b-a)^2}{6} \left( \mu - \frac{a+b}{2} \right)^2 + \frac{(b-a)^3}{6} \left( \mu - \frac{a+b}{2} \right) \\
 & \quad + \frac{(b-a)^4}{288} - \frac{4}{3} \left( \mu - \frac{a+b}{2} \right) \left[ \left( \mu - \frac{a+b}{2} \right)^2 + \frac{(b-a)^2}{12} \right]^{\frac{3}{2}}.
 \end{aligned} \tag{12}$$

Consequently, the inequalities (8) follow from (9) (10), (11) and (12).

The proof is completed.  $\square$

REMARK 2. It is clear that inequality (8) provides a refinement and improvement of inequality (4) for all  $\mu \in [a, b]$ .

The best inequality we can obtain from (8) is that for which  $\mu = \frac{a+b}{2}$ , giving the following corollary.

COROLLARY 2. *Let the assumptions of Theorem 6 hold. Then we have the inequality*

$$\left| \sigma_0^2(X) - \frac{(b-a)^2}{12} \right| \leq \frac{5(b-a)^4}{288} \|f'\|_\infty, \tag{13}$$

where  $\sigma_0(X) := \sigma_{\frac{a+b}{2}}(X)$ .

REMARK 3. It should be noticed that inequality (13) improves the inequality (2.5) in [3] and inequality (6.7) in [5].

#### 4. Some inequalities for cumulative distribution functions

The following theorem provides an inequality that connects the expectation  $E(X)$  and the cumulative distribution function  $F(x) := \int_a^x f(t) dt$  of a random variable  $X$  having the probability density function  $f : [a, b] \rightarrow \mathbf{R}$ .

THEOREM 7. *Let  $X$  be a random variable whose probability density function  $f : [a, b] \rightarrow \mathbf{R}$  is absolutely continuous on  $[a, b]$  and  $f' \in L_\infty[a, b]$ . Then*

$$\left| E(X) + (b-a)F(x) - x - \frac{b-a}{2} \right| \leq \|f'\|_\infty \times \begin{cases} \frac{x-a}{2}(x - \frac{a+b}{2})^2 + \frac{(x-a)^3}{6} + \frac{(b-a)^2}{8}(b-x) - \frac{(b-a)^3}{24}, & a \leq x < \frac{a+b}{2}, \\ \frac{b-x}{2}(x - \frac{a+b}{2})^2 + \frac{(b-x)^3}{6} + \frac{(b-a)^2}{8}(x-a) - \frac{(b-a)^3}{24}, & \frac{a+b}{2} \leq x \leq b. \end{cases} \tag{14}$$

*Proof.* In [4], Barnett and Dragomir established the following identity

$$(b-a)F(x) + E(X) - b = \int_a^b p(x,t) dF(t) = \int_a^b p(x,t)f(t) dt \tag{15}$$

where

$$p(x,t) := \begin{cases} t-a, & a \leq t \leq x \leq b, \\ t-b, & a \leq x < t \leq b. \end{cases}$$

Observe that

$$\frac{1}{b-a} \int_a^b p(x,t) dt = x - \frac{a+b}{2}, \quad \int_a^b f(t) dt = 1,$$

then applying the identity (15) and putting  $g(t) = p(x,t)$  in (7), we get

$$\begin{aligned} & \left| E(X) + (b-a)F(x) - x - \frac{b-a}{2} \right| \\ &= \left| \int_a^b p(x,t)f(t) dt - \frac{1}{b-a} \int_a^b p(x,t) dt \cdot \int_a^b f(t) dt \right| \\ &\leq \|f'\|_\infty \int_a^b \left| t - \frac{a+b}{2} \right| \left| p(x,t) - \frac{1}{b-a} \int_a^b p(x,s) ds \right| dt \\ &= \|f'\|_\infty \int_a^b \left| t - \frac{a+b}{2} \right| \left| p(x,t) - \left( x - \frac{a+b}{2} \right) \right| dt. \end{aligned} \tag{16}$$

If  $a \leq x < \frac{a+b}{2}$ , then  $\frac{a+b}{2} \leq x + \frac{b-a}{2} < b$  and so

$$\begin{aligned}
 & \int_a^b \left| t - \frac{a+b}{2} \right| \left| p(x,t) - \left( x - \frac{a+b}{2} \right) \right| dt \\
 &= \int_a^x \left( \frac{a+b}{2} - t \right) \left| t - a - \left( x - \frac{a+b}{2} \right) \right| dt + \int_x^{\frac{a+b}{2}} \left( \frac{a+b}{2} - t \right) \left| t - b - \left( x - \frac{a+b}{2} \right) \right| dt \\
 &\quad + \int_{\frac{a+b}{2}}^b \left( t - \frac{a+b}{2} \right) \left| t - b - \left( x - \frac{a+b}{2} \right) \right| dt \\
 &= \int_a^x \left( \frac{a+b}{2} - t \right) \left( t - x + \frac{b-a}{2} \right) dt + \int_x^{\frac{a+b}{2}} \left( \frac{a+b}{2} - t \right) \left( x + \frac{b-a}{2} - t \right) dt \\
 &\quad + \int_{\frac{a+b}{2}}^b \left( t - \frac{a+b}{2} \right) \left( x + \frac{b-a}{2} - t \right) dt + \int_{x+\frac{b-a}{2}}^b \left( t - \frac{a+b}{2} \right) \left( t - x - \frac{b-a}{2} \right) dt \\
 &= \frac{x-a}{2} \left( x - \frac{a+b}{2} \right)^2 + \frac{(x-a)^3}{6} + \frac{(b-a)^2}{8} (b-x) - \frac{(b-a)^3}{24},
 \end{aligned} \tag{17}$$

and if  $\frac{a+b}{2} \leq x \leq b$ , then  $a \leq x - \frac{b-a}{2} \leq \frac{a+b}{2}$  and so

$$\begin{aligned}
 & \int_a^b \left| t - \frac{a+b}{2} \right| \left| p(x,t) - \left( x - \frac{a+b}{2} \right) \right| dt \\
 &= \int_a^{\frac{a+b}{2}} \left( \frac{a+b}{2} - t \right) \left| t - a - \left( x - \frac{a+b}{2} \right) \right| dt + \int_{\frac{a+b}{2}}^x \left( t - \frac{a+b}{2} \right) \left| t - a - \left( x - \frac{a+b}{2} \right) \right| dt \\
 &\quad + \int_x^b \left( t - \frac{a+b}{2} \right) \left| t - b - \left( x - \frac{a+b}{2} \right) \right| dt \\
 &= \int_a^{x-\frac{b-a}{2}} \left( \frac{a+b}{2} - t \right) \left( x - \frac{b-a}{2} - t \right) dt + \int_{x-\frac{b-a}{2}}^{\frac{a+b}{2}} \left( \frac{a+b}{2} - t \right) \left( t - x + \frac{b-a}{2} \right) dt \\
 &\quad + \int_{\frac{a+b}{2}}^x \left( t - \frac{a+b}{2} \right) \left( t - x + \frac{b-a}{2} \right) dt + \int_x^b \left( t - \frac{a+b}{2} \right) \left( x + \frac{b-a}{2} - t \right) dt \\
 &= \frac{b-x}{2} \left( x - \frac{a+b}{2} \right)^2 + \frac{(b-x)^3}{6} + \frac{(b-a)^2}{8} (x-a) - \frac{(b-a)^3}{24}.
 \end{aligned} \tag{18}$$

Consequently, the inequalities (14) follow from (16), (17) and (18).

The proof is completed.  $\square$

REMARK 4. It is clear that the inequality (14) provides a refinement and improvement of inequality (5) for all  $x \in [a, b]$ .

REMARK 5. If in (14) either  $x = a$  or  $x = b$ , the inequality (3) is recaptured.

REMARK 6. Observed that  $F(x) = Pr(X \leq x) = \int_a^x f(t) dt$ , if in (14),  $x = \frac{a+b}{2}$ ,

then the best inequality that can be obtained is:

$$|E(X) + (b - a)Pr\left(X \leq \frac{a + b}{2}\right) - b| \leq \frac{1}{24}(b - a)^3 \|f'\|_\infty. \tag{19}$$

It should be noticed that the inequality (19) is an improvement of inequality (2.9) in [3] and inequality (6.11) in [5].

### 5. New proofs for some classical inequalities

Now we would like to apply the variant of Chebyshev inequality (7) to provide new and simpler proofs of the well-known classical trapezoid, midpoint and Simpson inequalities in a more general setup.

**THEOREM 8.** *Let  $f : [a, b] \rightarrow \mathbf{R}$  be such that  $f'$  is absolutely continuous on  $[a, b]$  and  $f'' \in L_\infty[a, b]$ . Then*

$$\left| \int_a^b f(x) dx - \frac{b - a}{2} [f(a) + f(b)] \right| \leq \frac{(b - a)^3}{12} \|f''\|_\infty. \tag{20}$$

*Proof.* It is clear that

$$\int_a^b f(t) dt - \frac{b - a}{2} [f(a) + f(b)] = - \int_a^b \left( t - \frac{a + b}{2} \right) f'(t) dt$$

and

$$\int_a^b \left( t - \frac{a + b}{2} \right) dt = 0.$$

Then by (7), we get

$$\begin{aligned} & \left| \int_a^b f(t) dt - \frac{b - a}{2} [f(a) + f(b)] \right| \\ &= \left| \int_a^b \left( t - \frac{a + b}{2} \right) f'(t) dt - \frac{1}{b - a} \int_a^b \left( t - \frac{a + b}{2} \right) dt \cdot \int_a^b f'(t) dt \right| \\ &\leq \|f''\|_\infty \int_a^b \left| t - \frac{a + b}{2} \right|^2 dt = \frac{(b - a)^3}{12} \|f''\|_\infty. \end{aligned}$$

i.e., the inequality (20) holds.  $\square$

**THEOREM 9.** *Let the assumptions of Theorem 8 hold. Then*

$$\left| \int_a^b f(x) dx - (b - a)f\left(\frac{a + b}{2}\right) \right| \leq \frac{(b - a)^3}{24} \|f''\|_\infty. \tag{21}$$

*Proof.* It is clear that

$$\int_a^b f(t) dt - (b - a)f\left(\frac{a + b}{2}\right) = - \int_a^b M(t) f'(t) dt$$

where

$$M(t) := \begin{cases} t - a, & a \leq t < \frac{a+b}{2}, \\ t - b, & \frac{a+b}{2} \leq t \leq b, \end{cases}$$

and

$$\int_a^b M(t) dt = 0.$$

Then by (7), we get

$$\begin{aligned} & \left| \int_a^b f(t) dt - (b-a)f\left(\frac{a+b}{2}\right) \right| \\ &= \left| \int_a^b M(t)f'(t) dt - \frac{1}{b-a} \int_a^b M(t) dt \cdot \int_a^b f'(t) dt \right| \\ &\leq \|f''\|_\infty \int_a^b \left| t - \frac{a+b}{2} \right| |M(t)| dt \\ &= \|f''\|_\infty \left[ \int_a^{\frac{a+b}{2}} \left( \frac{a+b}{2} - t \right) (t-a) dt + \int_{\frac{a+b}{2}}^b \left( t - \frac{a+b}{2} \right) (b-t) dt \right] \\ &= \frac{(b-a)^3}{24} \|f''\|_\infty. \end{aligned}$$

i.e., the inequality (21) holds.  $\square$

**THEOREM 10.** *Let the assumptions of Theorem 8 hold. Then*

$$\left| \int_a^b f(x) dx - \frac{b-a}{6} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] \right| \leq \frac{(b-a)^5}{2880} \|f^{(4)}\|_\infty. \tag{22}$$

*Proof.* It is clear that

$$\int_a^b f(t) dt - \frac{b-a}{6} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] = - \int_a^b S(t)f'''(t) dt$$

where

$$S(t) := \begin{cases} \frac{(t-a)^2}{6} \left( t - \frac{a+b}{2} \right), & a \leq t < \frac{a+b}{2}, \\ \frac{(t-b)^2}{6} \left( t - \frac{a+b}{2} \right), & \frac{a+b}{2} \leq t \leq b \end{cases}$$

and

$$\int_a^b S(t) dt = 0.$$

Then by (7), we get

$$\begin{aligned} & \left| \int_a^b f(t) dt - \frac{b-a}{6} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] \right| \\ &= \left| \int_a^b S(t)f'''(t) dt - \frac{1}{b-a} \int_a^b S(t) dt \cdot \int_a^b f'''(t) dt \right| \end{aligned}$$

$$\begin{aligned}
&\leq \|f^{(4)}\|_\infty \int_a^b \left| t - \frac{a+b}{2} \right| |S(t)| dt \\
&= \|f^{(4)}\|_\infty \left[ \int_a^{\frac{a+b}{2}} \left( \frac{a+b}{2} - t \right) |S(t)| dt + \int_{\frac{a+b}{2}}^b \left( t - \frac{a+b}{2} \right) |S(t)| dt \right] \\
&= \frac{\|f^{(4)}\|_\infty}{6} \left[ \int_a^{\frac{a+b}{2}} \left( t - \frac{a+b}{2} \right)^2 (t-a)^2 dt + \int_{\frac{a+b}{2}}^b \left( t - \frac{a+b}{2} \right)^2 (t-b)^2 dt \right] \\
&= \frac{(b-a)^5}{2880} \|f^{(4)}\|_\infty.
\end{aligned}$$

i.e., the inequality (22) holds.  $\square$

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