

ON SOME INEQUALITIES FOR UNITARILY INVARIANT NORMS

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Abstract. In this paper, we present several inequalities for unitarily invariant norms by using the convexity of the function $g(r) = \|A^r X B^{2-r} + A^{2-r} X B^r\|$ on the interval $[0, 2]$. Our results are refinements of some existing inequalities.

1. Introduction

Let $M_n(\mathbb{C})$ be the space of $n \times n$ complex matrices. Let $\|\cdot\|$ denote any unitarily invariant norm on $M_n(\mathbb{C})$. So, $\|UAV\| = \|A\|$ for all $A \in M_n(\mathbb{C})$ and for all unitary matrices $U, V \in M_n(\mathbb{C})$. For $A \in M_n(\mathbb{C})$, let $s_1(A) \geq \dots \geq s_n(A)$ be the singular values of A , i.e., the eigenvalues of the positive semidefinite matrix $|A| = (AA^*)^{\frac{1}{2}}$. The Ky Fan k -norm $\|\cdot\|_{(k)}$ is defined as

$$\|A\|_{(k)} = \sum_{j=1}^k s_j(A), \quad k = 1, \dots, n,$$

and the Schatten p -norm $\|\cdot\|_p$ is defined as

$$\|A\|_p = \left(\sum_{j=1}^n s_j^p(A) \right)^{\frac{1}{p}} = (\operatorname{tr} |A|^p)^{\frac{1}{p}}, \quad 1 \leq p \leq \infty.$$

It is known that the Ky Fan k -norm $\|\cdot\|_{(k)}$ and the Schatten p -norm $\|\cdot\|_p$ are unitarily invariant [1].

Let A, B, X be $n \times n$ complex matrices such that A, B are positive semidefinite and suppose that

$$\varphi(v) = \|A^v X B^{1-v} + A^{1-v} X B^v\|, \quad 0 \leq v \leq 1. \quad (1.1)$$

The function φ is a continuous convex function on interval $[0, 1]$ and attains its minimum at $v = \frac{1}{2}$ and its maximum at $v = 0$ and $v = 1$ [2].

Replacing A, B by A^2, B^2 in (1.1) and then putting $r = 2v$, we define

$$g(r) = \|A^r X B^{2-r} + A^{2-r} X B^r\|, \quad 0 \leq r \leq 2.$$

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The function g is clearly convex on $[0, 2]$ and attains its minimum at $r = 1$. Consequently $g(1) \leq g(r)$, which implies that

$$2 \|AXB\| \leq \|A^r XB^{2-r} + A^{2-r} XB^r\|, \quad 0 \leq r \leq 2. \quad (1.2)$$

Zhan proved in [3] that if A, B, X are $n \times n$ complex matrices such that A and B are positive semidefinite, then

$$\|A^r XB^{2-r} + A^{2-r} XB^r\| \leq \frac{2}{t+2} \|A^2X + tAXB + XB^2\|, \quad (1.3)$$

for $r \in [\frac{1}{2}, \frac{3}{2}]$ and $t \in (-2, 2]$. So it follows from (1.2) and (1.3) that

$$2 \|AXB\| \leq \|A^r XB^{2-r} + A^{2-r} XB^r\| \leq \frac{2}{t+2} \|A^2X + tAXB + XB^2\|. \quad (1.4)$$

Bhatia and Kittaneh in [4] proved that if A and B are positive semidefinite, then

$$\|AB\| \leq \frac{1}{4} \|(A+B)^2\|. \quad (1.5)$$

In this paper, we will give several refinements of the inequalities (1.4) and a refinement of (1.5).

2. Main results

We begin this section with a refinement of the following inequality in (1.4)

$$2 \|AXB\| \leq \frac{2}{t+2} \|A^2X + tAXB + XB^2\|.$$

THEOREM 2.1. *Let A, B, X be $n \times n$ complex matrices such that A and B are positive semidefinite. Then*

$$\begin{aligned} 2 \|AXB\| + 2 \left(\int_{\frac{1}{2}}^{\frac{3}{2}} \|A^r XB^{2-r} + A^{2-r} XB^r\| dr - 2 \|AXB\| \right) \\ \leq \frac{2}{t+2} \|A^2X + tAXB + XB^2\|, \end{aligned} \quad (2.1)$$

where $\frac{1}{2} \leq r \leq \frac{3}{2}$, $-2 < t \leq 2$.

Proof. We present the proof in two cases:

a) For $\frac{1}{2} \leq r \leq 1$, by the convexity of the function g , Wang et al. [5] have proved that

$$g(r) \leq (2r-1)g(1) + (2-2r)g\left(\frac{1}{2}\right).$$

By integrating both sides of the inequality above, we have

$$\int_{\frac{1}{2}}^1 g(r) dr \leq g(1) \int_{\frac{1}{2}}^1 (2r-1) dr + g\left(\frac{1}{2}\right) \int_{\frac{1}{2}}^1 (2-2r) dr,$$

which implies

$$\int_{\frac{1}{2}}^1 g(r) dr \leq \frac{1}{4}g(1) + \frac{1}{4}g\left(\frac{1}{2}\right). \quad (2.2)$$

b) For $1 \leq r \leq \frac{3}{2}$, by the convexity of the function g , Wang et al. [5] have proved that

$$g(r) \leq (2r-2)g\left(\frac{3}{2}\right) + (3-2r)g(1).$$

By integrating both sides of the inequality above, we have

$$\int_1^{\frac{3}{2}} g(r) dr \leq g\left(\frac{3}{2}\right) \int_1^{\frac{3}{2}} (2r-2) dr + g(1) \int_1^{\frac{3}{2}} (3-2r) dr,$$

which implies

$$\int_1^{\frac{3}{2}} g(r) dr \leq \frac{1}{4}g(1) + \frac{1}{4}g\left(\frac{3}{2}\right). \quad (2.3)$$

It follows from (2.2), (2.3) and $g\left(\frac{3}{2}\right) = g\left(\frac{1}{2}\right)$ that

$$2 \int_{\frac{1}{2}}^{\frac{3}{2}} g(r) dr \leq g(1) + g\left(\frac{1}{2}\right),$$

which is equivalent to

$$g(1) + 2 \left(\int_{\frac{1}{2}}^{\frac{3}{2}} g(r) dr - g(1) \right) \leq g\left(\frac{1}{2}\right).$$

The last inequality is

$$\begin{aligned} 2 \|AXB\| + 2 \left(\int_{\frac{1}{2}}^{\frac{3}{2}} \|A^r XB^{2-r} + A^{2-r} XB^r\| dr - 2 \|AXB\| \right) \\ \leq \left\| A^{\frac{1}{2}} XB^{\frac{3}{2}} + A^{\frac{3}{2}} XB^{\frac{1}{2}} \right\|. \end{aligned}$$

By (1.3), we obtain

$$\begin{aligned} 2 \|AXB\| + 2 \left(\int_{\frac{1}{2}}^{\frac{3}{2}} \|A^r XB^{2-r} + A^{2-r} XB^r\| dr - 2 \|AXB\| \right) \\ \leq \frac{2}{t+2} \|A^2X + tAXB + XB^2\|. \end{aligned}$$

The proof is completed. \square

REMARK 1. Since the function g attains its minimum at $r = 1$, we have

$$\int_{\frac{1}{2}}^{\frac{3}{2}} g(r) dr - g(1) = \int_{\frac{1}{2}}^{\frac{3}{2}} (g(r) - g(1)) dr \geq 0,$$

which implies

$$\int_{\frac{1}{2}}^{\frac{3}{2}} \|A^r XB^{2-r} + A^{2-r} XB^r\| dr - 2 \|AXB\| \geq 0,$$

and so the inequality (2.1) is a refinement of the inequality

$$2 \|AXB\| \leq \frac{2}{t+2} \|A^2X + tAXB + XB^2\|.$$

REMARK 2. By a simple substitution and the triangle inequality, we know that the inequality (2.1) is equivalent to

$$\begin{aligned} 2 \|A^{\frac{1}{2}}XB^{\frac{1}{2}}\| &+ 4 \left(\int_{\frac{1}{4}}^{\frac{3}{4}} \|A^v XB^{1-v} + A^{1-v}XB^v\| dv - \|A^{\frac{1}{2}}XB^{\frac{1}{2}}\| \right) \\ &\leq \frac{2}{t+2} \|AX + tA^{\frac{1}{2}}XB^{\frac{1}{2}} + XB\| \\ &\leq \|AX + XB\|, \quad 0 \leq t \leq 2, \end{aligned}$$

which implies that

$$\begin{aligned} 2 \|A^{\frac{1}{2}}XB^{\frac{1}{2}}\| &+ 4 \left(\int_{\frac{1}{4}}^{\frac{3}{4}} \|A^v XB^{1-v} + A^{1-v}XB^v\| dv - \|A^{\frac{1}{2}}XB^{\frac{1}{2}}\| \right) \\ &\leq \|AX + XB\|. \end{aligned} \tag{2.4}$$

Zou and He in [6] obtained the following inequality

$$\begin{aligned} 2 \|A^{\frac{1}{2}}XB^{\frac{1}{2}}\| &+ 2 \left(\int_0^1 \|A^v XB^{1-v} + A^{1-v}XB^v\| dv - 2 \|A^{\frac{1}{2}}XB^{\frac{1}{2}}\| \right) \\ &\leq \|AX + XB\|. \end{aligned} \tag{2.5}$$

The right side of the inequality (2.4) is the same as that of the inequality (2.5). Nevertheless, it is easy to see that the left side of the inequality (2.4) could not be compared with that of the inequality (2.5). Thus, neither (2.4) nor (2.5) is uniformly better than the other.

Below we will give some refinements of the first inequality in (1.4).

LEMMA 2.1. [7,8] (Hermite-Hadamard Inequality) *Let h be a real-valued function which is convex on the interval $[a, b]$. Then*

$$h\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b h(t) dt \leq \frac{h(a)+h(b)}{2}.$$

Applying Lemma 2.1 to the function $g(r) = \|A^r XB^{2-r} + A^{2-r} XB^r\|$ on the interval $[\xi, 2-\xi]$ when $\xi \in [\frac{1}{2}, 1)$ and on the interval $[2-\xi, \xi]$ when $\xi \in (1, \frac{3}{2}]$ respectively, we achieve a refinement of the first inequality in (1.4).

THEOREM 2.2. *Let A, B, X be $n \times n$ complex matrices such that A and B are positive definite. Then*

$$\begin{aligned}
 2 \|AXB\| &\leq \frac{1}{|2-2\xi|} \left| \int_{\xi}^{2-\xi} \|A^r XB^{2-r} + A^{2-r} XB^r\| dr \right| \\
 &\leq \left\| A^{\xi} XB^{2-\xi} + A^{2-\xi} XB^{\xi} \right\|,
 \end{aligned}
 \tag{2.6}$$

where $\frac{1}{2} \leq \xi \leq \frac{3}{2}$. It should be noticed here that in the inequality(2.6),

$$\lim_{\xi \rightarrow 1} \frac{1}{|2-2\xi|} \left| \int_{\xi}^{2-\xi} \|A^v XB^{2-v} + A^{2-v} XB^v\| dv \right| = 2 \|AXB\|.$$

Proof. First assume that $\frac{1}{2} \leq \xi < 1$. It follows from Lemma 2.1 that

$$g\left(\frac{\xi+2-\xi}{2}\right) \leq \frac{1}{2-2\xi} \int_{\xi}^{2-\xi} g(r) dr \leq \frac{g(\xi)+g(2-\xi)}{2},$$

that is

$$g(1) \leq \frac{1}{2-2\xi} \int_{\xi}^{2-\xi} g(r) dr \leq g(\xi),$$

where $g(r) = \|A^r XB^{2-r} + A^{2-r} XB^r\|$. Thus

$$\begin{aligned}
 2 \|AXB\| &\leq \frac{1}{2-2\xi} \int_{\xi}^{2-\xi} \|A^r XB^{2-r} + A^{2-r} XB^r\| dr \\
 &\leq \left\| A^{\xi} XB^{2-\xi} + A^{2-\xi} XB^{\xi} \right\|.
 \end{aligned}
 \tag{2.7}$$

Then assume that $1 < \xi \leq \frac{3}{2}$. The proof is similar to the case of $\frac{1}{2} \leq \xi < 1$, so we obtain

$$\begin{aligned}
 2 \|AXB\| &\leq \frac{1}{2\xi-2} \int_{2-\xi}^{\xi} \|A^r XB^{2-r} + A^{2-r} XB^r\| dr \\
 &\leq \left\| A^{\xi} XB^{2-\xi} + A^{2-\xi} XB^{\xi} \right\|,
 \end{aligned}
 \tag{2.8}$$

where $g(r) = \|A^r XB^{2-r} + A^{2-r} XB^r\|$.

The inequalities in (2.6) follow by the inequalities (2.7) and (2.8). The proof is completed. \square

Applying Lemma 2.1 to the function $g(r) = \|A^r XB^{2-r} + A^{2-r} XB^r\|$ on the interval $[\xi, 1]$ when $\xi \in [\frac{1}{2}, 1)$, and on the interval $[1, \xi]$ when $\xi \in (1, \frac{3}{2}]$ respectively, we obtain the following result.

THEOREM 2.3. *Let A, B, X be $n \times n$ complex matrices such that A and B are positive definite. Then*

$$\begin{aligned} \left\| A^{\frac{1}{2}+\xi}XB^{\frac{3}{2}-\xi} + A^{\frac{3}{2}-\xi}XB^{\frac{1}{2}+\xi} \right\| &\leq \frac{1}{|1-\xi|} \left| \int_{\xi}^1 \|A^rXB^{2-r} + A^{2-r}XB^r\| dr \right| \\ &\leq \frac{1}{2} \times \left\| A^{\xi}XB^{2-\xi} + A^{2-\xi}XB^{\xi} \right\| + \|AXB\| \quad (2.9) \\ &\leq \left\| A^{\xi}XB^{2-\xi} + A^{2-\xi}XB^{\xi} \right\|. \end{aligned}$$

where $\frac{1}{2} \leq \xi \leq \frac{3}{2}$. It should be noticed here that in the inequality (2.9),

$$\lim_{\xi \rightarrow 1} \frac{1}{|1-\xi|} \left| \int_{\xi}^1 \|A^vXB^{2-v} + A^{2-v}XB^v\| dv \right| = 2\|AXB\|.$$

Proof. First assume that $\frac{1}{2} \leq \xi < 1$. It follows from Lemma 2.1 that

$$g\left(\frac{\xi+1}{2}\right) \leq \frac{1}{1-\xi} \int_{\xi}^1 g(r) dr \leq \frac{g(\xi) + g(1)}{2},$$

where $g(r) = \|A^rXB^{2-r} + A^{2-r}XB^r\|$. Thus

$$\begin{aligned} \left\| A^{\frac{1}{2}+\xi}XB^{\frac{3}{2}-\xi} + A^{\frac{3}{2}-\xi}XB^{\frac{1}{2}+\xi} \right\| &\leq \frac{1}{1-\xi} \int_{\xi}^1 g(r) dr \\ &\leq \frac{1}{2} \times \left\| A^{\xi}XB^{2-\xi} + A^{2-\xi}XB^{\xi} \right\| + \|AXB\| \quad (2.10) \\ &\leq \left\| A^{\xi}XB^{2-\xi} + A^{2-\xi}XB^{\xi} \right\|. \quad (\text{by (1.2)}) \end{aligned}$$

Then assume that $1 < \xi \leq \frac{3}{2}$. The proof is similar to the case of $\frac{1}{2} \leq \xi < 1$, so we obtain

$$\begin{aligned} \left\| A^{\frac{1}{2}+\xi}XB^{\frac{3}{2}-\xi} + A^{\frac{3}{2}-\xi}XB^{\frac{1}{2}+\xi} \right\| &\leq \frac{1}{\xi-1} \int_1^{\xi} g(r) dr \\ &\leq \frac{1}{2} \times \left\| A^{\xi}XB^{2-\xi} + A^{2-\xi}XB^{\xi} \right\| + \|AXB\| \quad (2.11) \\ &\leq \left\| A^{\xi}XB^{2-\xi} + A^{2-\xi}XB^{\xi} \right\|, \quad (\text{by (1.2)}) \end{aligned}$$

where $g(r) = \|A^rXB^{2-r} + A^{2-r}XB^r\|$.

The inequalities in (2.9) follow by the inequalities (2.10) and (2.11). The proof is completed. \square

The inequalities (2.9) and (1.2) yield another refinement of the first inequality in (1.4).

COROLLARY 2.1. *Let A, B, X be $n \times n$ complex matrices such that A and B are positive definite. Then*

$$\begin{aligned} 2 \|AXB\| &\leq \left\| A^{\frac{1}{2} + \frac{\xi}{2}} X B^{\frac{3}{2} - \frac{\xi}{2}} + A^{\frac{3}{2} - \frac{\xi}{2}} X B^{\frac{1}{2} + \frac{\xi}{2}} \right\| \\ &\leq \frac{1}{|1 - \xi|} \left| \int_{\xi}^1 \|A^r X B^{2-r} + A^{2-r} X B^r\| dr \right| \\ &\leq \left\| A^{\xi} X B^{2-\xi} + A^{2-\xi} X B^{\xi} \right\|, \end{aligned} \tag{2.12}$$

where $\frac{1}{2} \leq \xi \leq \frac{3}{2}$. It should be noticed here that in the inequality(2.12),

$$\lim_{\xi \rightarrow 1} \frac{1}{|1 - \xi|} \left| \int_{\xi}^1 \|A^v X B^{2-v} + A^{2-v} X B^v\| dv \right| = 2 \|AXB\|.$$

In the sequel, we achieve another refinement of the second inequality in (1.4).

Applying Lemma 2.1 to the function $g(r) = \|A^r X B^{2-r} + A^{2-r} X B^r\|$ on the interval $[\frac{1}{2}, \xi]$ when $\xi \in (\frac{1}{2}, 1]$, and on the interval $[\xi, \frac{3}{2}]$ when $\xi \in [1, \frac{3}{2})$ respectively, we get the following result.

THEOREM 2.4. *Let A, B, X be $n \times n$ complex matrices such that A and B are positive semidefinite. Then*

(a) for $\frac{1}{2} \leq \xi \leq 1, -2 < t \leq 2,$

$$\begin{aligned} \left\| A^{\frac{1}{4} + \frac{\xi}{2}} X B^{\frac{7}{4} - \frac{\xi}{2}} + A^{\frac{7}{4} - \frac{\xi}{2}} X B^{\frac{\xi}{2} + \frac{1}{4}} \right\| &\leq \frac{1}{\xi - \frac{1}{2}} \int_{\frac{1}{2}}^{\xi} \|A^r X B^{2-r} + A^{2-r} X B^r\| dr \\ &\leq \frac{1}{2} \times \left\| A^{\xi} X B^{2-\xi} + A^{2-\xi} X B^{\xi} \right\| \\ &\quad + \frac{1}{2} \times \left\| A^{\frac{1}{2}} X B^{\frac{3}{2}} + A^{\frac{3}{2}} X B^{\frac{1}{2}} \right\| \\ &\leq \frac{2}{t+2} \|A^2 X + t A X B + X B^2\|. \end{aligned} \tag{2.13}$$

It should be noticed here that in the inequality(2.13),

$$\lim_{\xi \rightarrow \frac{1}{2}^+} \frac{1}{\xi - \frac{1}{2}} \int_{\frac{1}{2}}^{\xi} \|A^v X B^{2-v} + A^{2-v} X B^v\| dv = \left\| A^{\frac{1}{2}} X B^{\frac{3}{2}} + A^{\frac{3}{2}} X B^{\frac{1}{2}} \right\|.$$

(b) for $1 \leq \xi \leq \frac{3}{2}, -2 < t \leq 2,$

$$\begin{aligned} \left\| A^{\frac{3}{4} + \frac{\xi}{2}} X B^{\frac{5}{4} - \frac{\xi}{2}} + A^{\frac{5}{4} - \frac{\xi}{2}} X B^{\frac{\xi}{2} + \frac{3}{4}} \right\| &\leq \frac{1}{\frac{3}{2} - \xi} \int_{\xi}^{\frac{3}{2}} \|A^r X B^{2-r} + A^{2-r} X B^r\| dr \\ &\leq \frac{1}{2} \times \left\| A^{\xi} X B^{2-\xi} + A^{2-\xi} X B^{\xi} \right\| \\ &\quad + \frac{1}{2} \times \left\| A^{\frac{1}{2}} X B^{\frac{3}{2}} + A^{\frac{3}{2}} X B^{\frac{1}{2}} \right\| \\ &\leq \frac{2}{t+2} \|A^2 X + t A X B + X B^2\|. \end{aligned} \tag{2.14}$$

It should be noticed here that in the inequality(2.14),

$$\lim_{\xi \rightarrow \frac{3}{2}^-} \frac{1}{\frac{3}{2} - \xi} \int_{\frac{1}{2}}^{\frac{3}{2}} \|A^v XB^{2-v} + A^{2-v}XB^v\| dv = \left\| A^{\frac{1}{2}}XB^{\frac{3}{2}} + A^{\frac{3}{2}}XB^{\frac{1}{2}} \right\|.$$

Proof. (a) First assume that $\frac{1}{2} < \xi \leq 1$. It follows from Lemma 2.1 that

$$g\left(\frac{\xi + \frac{1}{2}}{2}\right) \leq \frac{1}{\xi - \frac{1}{2}} \int_{\frac{1}{2}}^{\xi} g(r) dr \leq \frac{g(\xi) + g(\frac{1}{2})}{2},$$

where $g(r) = \|A^rXB^{2-r} + A^{2-r}XB^r\|$. Thus

$$\begin{aligned} \left\| A^{\frac{1}{4} + \frac{\xi}{2}}XB^{\frac{7}{4} - \frac{\xi}{2}} + A^{\frac{7}{4} - \frac{\xi}{2}}XB^{\frac{\xi}{2} + \frac{1}{4}} \right\| &\leq \frac{1}{\xi - \frac{1}{2}} \int_{\frac{1}{2}}^{\xi} \|A^rXB^{2-r} + A^{2-r}XB^r\| dr \\ &\leq \frac{1}{2} \times \left\| A^{\xi}XB^{2-\xi} + A^{2-\xi}XB^{\xi} \right\| \\ &\quad + \frac{1}{2} \times \left\| A^{\frac{1}{2}}XB^{\frac{3}{2}} + A^{\frac{3}{2}}XB^{\frac{1}{2}} \right\| \\ &\leq \frac{2}{t+2} \|A^2X + tAXB + XB^2\| \quad ((\text{by}(1.3)), \end{aligned} \tag{2.15}$$

(b) Assume that $1 \leq \xi < \frac{3}{2}$. The proof is similar to the case of $\frac{1}{2} < \xi \leq 1$, so we obtain

$$\begin{aligned} \left\| A^{\frac{3}{4} + \frac{\xi}{2}}XB^{\frac{5}{4} - \frac{\xi}{2}} + A^{\frac{5}{4} - \frac{\xi}{2}}XB^{\frac{\xi}{2} + \frac{3}{4}} \right\| &\leq \frac{1}{\frac{3}{2} - \xi} \int_{\xi}^{\frac{3}{2}} \|A^rXB^{2-r} + A^{2-r}XB^r\| dr \\ &\leq \frac{1}{2} \times \left\| A^{\xi}XB^{2-\xi} + A^{2-\xi}XB^{\xi} \right\| \\ &\quad + \frac{1}{2} \times \left\| A^{\frac{1}{2}}XB^{\frac{3}{2}} + A^{\frac{3}{2}}XB^{\frac{1}{2}} \right\| \\ &\leq \frac{2}{t+2} \|A^2X + tAXB + XB^2\| \quad ((\text{by}(1.3)), \end{aligned} \tag{2.16}$$

where $g(r) = \|A^rXB^{2-r} + A^{2-r}XB^r\|$.

The inequalities in (2.13) and (2.14) follow by the inequalities (2.15) and (2.16), respectively. The proof is completed. \square

In view of the fact that the function $g(r) = \|A^rXB^{2-r} + A^{2-r}XB^r\|$ is decreasing on the interval $[\frac{1}{2}, 1]$ and increasing on the interval $[1, \frac{3}{2}]$, by Theorem 2.4 we have the following result, which is a refinement of the second inequality in (1.4).

COROLLARY 2.2. *Let A, B, X be $n \times n$ complex matrices such that A and B are positive semidefinite. Then*

(a) for $\frac{1}{2} \leq \xi \leq 1, -2 < t \leq 2,$

$$\begin{aligned} \left\| A^\xi XB^{2-\xi} + A^{2-\xi}XB^\xi \right\| &\leq \left\| A^{\frac{1}{4}+\frac{\xi}{2}}XB^{\frac{7}{4}-\frac{\xi}{2}} + A^{\frac{7}{4}-\frac{\xi}{2}}XB^{\frac{\xi}{2}+\frac{1}{4}} \right\| \\ &\leq \frac{1}{\xi - \frac{1}{2}} \int_{\frac{1}{2}}^{\xi} \|A^rXB^{2-r} + A^{2-r}XB^r\| dr \\ &\leq \frac{1}{2} \times \left\| A^\xi XB^{2-\xi} + A^{2-\xi}XB^\xi \right\| \\ &\quad + \frac{1}{2} \times \left\| A^{\frac{1}{2}}XB^{\frac{3}{2}} + A^{\frac{3}{2}}XB^{\frac{1}{2}} \right\| \\ &\leq \frac{2}{t+2} \|A^2X + tAXB + XB^2\|. \end{aligned} \tag{2.17}$$

It should be noticed here that in the inequality(2.17),

$$\lim_{\xi \rightarrow \frac{1}{2}^+} \frac{1}{\xi - \frac{1}{2}} \int_{\frac{1}{2}}^{\xi} \|A^vXB^{2-v} + A^{2-v}XB^v\| dv = \left\| A^{\frac{1}{2}}XB^{\frac{3}{2}} + A^{\frac{3}{2}}XB^{\frac{1}{2}} \right\|.$$

(b) for $1 \leq \xi \leq \frac{3}{2}, -2 < t \leq 2,$

$$\begin{aligned} \left\| A^\xi XB^{2-\xi} + A^{2-\xi}XB^\xi \right\| &\leq \left\| A^{\frac{3}{4}+\frac{\xi}{2}}XB^{\frac{5}{4}-\frac{\xi}{2}} + A^{\frac{5}{4}-\frac{\xi}{2}}XB^{\frac{\xi}{2}+\frac{3}{4}} \right\| \\ &\leq \frac{1}{\frac{3}{2} - \xi} \int_{\xi}^{\frac{3}{2}} \|A^rXB^{2-r} + A^{2-r}XB^r\| dr \\ &\leq \frac{1}{2} \times \left\| A^\xi XB^{2-\xi} + A^{2-\xi}XB^\xi \right\| \\ &\quad + \frac{1}{2} \times \left\| A^{\frac{1}{2}}XB^{\frac{3}{2}} + A^{\frac{3}{2}}XB^{\frac{1}{2}} \right\| \\ &\leq \frac{2}{t+2} \|A^2X + tAXB + XB^2\|. \end{aligned} \tag{2.18}$$

It should be noticed here that in the inequality (2.18),

$$\lim_{\xi \rightarrow \frac{3}{2}^-} \frac{1}{\frac{3}{2} - \xi} \int_{\xi}^{\frac{3}{2}} \|A^vXB^{2-v} + A^{2-v}XB^v\| dv = \left\| A^{\frac{1}{2}}XB^{\frac{3}{2}} + A^{\frac{3}{2}}XB^{\frac{1}{2}} \right\|.$$

In the end, we give a refinement of inequality (1.5), by means of the inequality (2.4).

THEOREM 2.5. *Let A, B, X be n × n complex matrices such that A and B are positive semidefnite. Then*

$$\begin{aligned} 2\|AB\| + 4 \left(\int_{\frac{1}{4}}^{\frac{3}{4}} f(v) dv - \|AB\| \right) \\ \leq \frac{1}{2} \|(A+B)^2\|, \end{aligned} \tag{2.19}$$

where

$$f(v) = \left\| A^{\frac{1}{2}+v} B^{\frac{3}{2}-v} + A^{\frac{3}{2}-v} B^{\frac{1}{2}+v} \right\|.$$

Proof. Let

$$X = A^{\frac{1}{2}} B^{\frac{1}{2}}.$$

Then by (2.4) we have

$$2\|AB\| + 4 \left(\int_{\frac{1}{4}}^{\frac{3}{4}} f(v) dv - \|AB\| \right) \leq \left\| A^{\frac{1}{2}} B^{\frac{3}{2}} + A^{\frac{3}{2}} B^{\frac{1}{2}} \right\|. \quad (2.20)$$

By the following inequality (see [4])

$$\left\| A^{\frac{1}{2}} B^{\frac{3}{2}} + A^{\frac{3}{2}} B^{\frac{1}{2}} \right\| \leq \frac{1}{2} \|(A+B)^2\|,$$

it easily follows from (2.20) that

$$2\|AB\| + 4 \left(\int_{\frac{1}{4}}^{\frac{3}{4}} f(v) dv - \|AB\| \right) \leq \frac{1}{2} \|(A+B)^2\|.$$

The proof is completed. \square

REMARK 3. Obviously,

$$\int_{\frac{1}{4}}^{\frac{3}{4}} f(v) dv - \|AB\| \geq 0,$$

so the inequality (2.19) is a refinement of the inequality

$$\|AB\| \leq \frac{1}{4} \|(A+B)^2\|.$$

REMARK 4. It is easy to know that (2.19) is equivalent to

$$\|AB\| + 2 \left(\int_{\frac{1}{4}}^{\frac{3}{4}} \left\| A^{\frac{1}{2}+v} B^{\frac{3}{2}-v} + A^{\frac{3}{2}-v} B^{\frac{1}{2}+v} \right\| dv - \|AB\| \right) \leq \frac{1}{4} \|(A+B)^2\|. \quad (2.21)$$

Zou and He [6] obtained the inequality

$$\|AB\| + \left(\int_0^1 \left\| A^{\frac{1}{2}+v} B^{\frac{3}{2}-v} + A^{\frac{3}{2}-v} B^{\frac{1}{2}+v} \right\| dv - 2\|AB\| \right) \leq \frac{1}{4} \|(A+B)^2\|. \quad (2.22)$$

The right side of the inequality (2.21) is the same as that of the inequality (2.22). Nevertheless, it is easy to see that the left side of the inequality (2.21) could not be compared with that of the inequality (2.22). Thus, neither (2.21) nor (2.22) is uniformly better than the other.

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