

PRECISE LOWER BOUND OF $f(A) - f(B)$ FOR $A > B > 0$ AND NON-CONSTANT OPERATOR MONOTONE FUNCTION f ON $[0, \infty)$

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Abstract. A and B be strictly positive operators on a Hilbert space H such that $A - B \geq m > 0$. Then the following inequalities hold for any non-constant operator monotone function f on $[0, \infty)$:

$$\begin{aligned} f(A) - f(B) &\geq f(\|B\| + m) - f(\|B\|) \\ &\geq f(\|A\|) - f(\|A\| - m) > 0. \end{aligned}$$

In particular, let $A > B$. Then

$$\begin{aligned} f(A) - f(B) &\geq f\left(\|B\| + \frac{1}{\|(A-B)^{-1}\|}\right) - f(\|B\|) \\ &\geq f(\|A\|) - f\left(\|A\| - \frac{1}{\|(A-B)^{-1}\|}\right) > 0. \end{aligned}$$

We shall state the typical concrete example of these operator inequalities.

Let $A > B$. Then the following inequalities hold as an extension of celebrated Löwner-Heinz inequality

$$\begin{aligned} A^r - B^r &\geq \left(\|B\| + \frac{1}{\|(A-B)^{-1}\|}\right)^r - \|B\|^r \\ &\geq \|A\|^r - \left(\|A\| - \frac{1}{\|(A-B)^{-1}\|}\right)^r > 0 \quad \text{for } 0 < r \leq 1 \end{aligned}$$

and also the following inequalities hold

$$\begin{aligned} \log A - \log B &\geq \log\left(\|B\| + \frac{1}{\|(A-B)^{-1}\|}\right) - \log\|B\| \\ &\geq \log\|A\| - \log\left(\|A\| - \frac{1}{\|(A-B)^{-1}\|}\right) > 0. \end{aligned}$$

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