

## MULTI-WEIGHTED BOUNDEDNESS FOR MULTILINEAR ROUGH FRACTIONAL INTEGRALS AND MAXIMAL OPERATORS

XIANGXING TAO AND YANLONG SHI

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*Abstract.* In this article, several sufficient conditions on the weights  $(\vec{v}, u)$  are given such that the multilinear rough fractional integrals  $I_{\Omega, \alpha}^{(m)}$  and the rough multi-sublinear fractional maximal operators  $M_{\Omega, \alpha}^{(m)}$  are bounded from the product spaces  $L_{v_1}^{p_1}(\mathbb{R}^n) \times L_{v_2}^{p_2}(\mathbb{R}^n) \times \cdots \times L_{v_m}^{p_m}(\mathbb{R}^n)$  to the space  $L_u^q(\mathbb{R}^n)$ . The weak multi-weighted boundedness has also been derived. These results will extend the early and recent works in this direction.

### 1. Introduction

The fractional type operators and their weighted boundedness theory play important roles in harmonic analysis and other fields, and the multilinear operators arise in numerous situations involving product-like operations, see [5], [6], [7], [8], [9] for instance. In this paper we will study the weighted boundedness for the rough multi(sub)linear fractional operators  $M_{\Omega, \alpha}^{(m)}$  and  $I_{\Omega, \alpha}^{(m)}$ , which are the more generalizations of the classical setting. Let  $m$  and  $n$  be the nonnegative integers with  $n \geq 2$  and  $m \geq 1$ , and let  $\mathbb{S}^{mn-1}$  denote the unit sphere of  $\mathbb{R}^{mn}$ , and suppose that  $\Omega$  is a homogeneous function of degree zero on  $\mathbb{R}^{mn}$  only with the condition  $\Omega \in L^s(\mathbb{S}^{mn-1})$  for some  $s > 1$ . We consider the rough multi-sublinear fractional maximal operator  $M_{\Omega, \alpha}^{(m)}$  defined by

$$M_{\Omega, \alpha}^{(m)}(\vec{f})(x) = \sup_{r>0} \frac{1}{r^{mn-\alpha}} \int_{|\vec{y}|<r} |\Omega(\vec{y})| \prod_{i=1}^m |f_i(x - y_i)| d\vec{y}, \quad (1.1)$$

and the rough multilinear fractional integral  $I_{\Omega, \alpha}^{(m)}$  defined by

$$I_{\Omega, \alpha}^{(m)}(\vec{f})(x) = \int_{(\mathbb{R}^n)^m} \frac{\Omega(\vec{y})}{|\vec{y}|^{mn-\alpha}} \prod_{i=1}^m f_i(x - y_i) d\vec{y}, \quad (1.2)$$

where  $\vec{y} = (y_1, y_2, \dots, y_m)$  and  $x, y_1, y_2, \dots, y_m \in \mathbb{R}^n$ ,  $d\vec{y} = dy_1 dy_2 \cdots dy_m$ , and  $\vec{f}$  denotes the  $m$ -tuple  $(f_1, f_2, \dots, f_m)$ .

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In case  $m = 1$ , we denote  $M_{\Omega,\alpha}^{(1)}$  and  $I_{\Omega,\alpha}^{(1)}$  by  $M_{\Omega,\alpha}$  and  $I_{\Omega,\alpha}$ , which are the rough fractional maximal operator and the rough fractional integral respectively. In case  $\Omega = 1$ , we denote  $M_{1,\alpha}^{(m)}$  and  $I_{1,\alpha}^{(m)}$  by  $M_\alpha^{(m)}$  and  $I_\alpha^{(m)}$ , which are the multi-sublinear fractional maximal operator and the multilinear fractional integral respectively. If  $\Omega = 1$  and  $m = 1$ , then they become the classical fractional maximal operator  $M_\alpha$  and the classical fractional integral  $I_\alpha$ .

Historically, in 1974 Muckenhoupt and Wheeden [11] proved that the fractional maximal operator  $M_\alpha$  and the fractional integral operator  $I_\alpha$  were of weak type  $(L^1(\omega), L^{\frac{n}{n-\alpha}}(\omega^{\frac{n}{n-\alpha}}))$  and of strong type  $(L^p(\omega^p), L^q(\omega^q))$  for  $1/q = 1/p - \alpha/n$ ,  $0 < \alpha < n$  and  $1 < p < n/\alpha$ , if the positive weight  $\omega \in A_{p,q}$ , which means that

$$\sup_{Q \subset \mathbb{R}^n} \left( \frac{1}{|Q|} \int_Q \omega(x)^q dx \right)^{1/q} \left( \frac{1}{|Q|} \int_Q \omega(x)^{-p' dx} \right)^{1/p'} < \infty, \tag{1.3}$$

where  $Q$  denotes the cube in  $\mathbb{R}^n$  with the sides parallel to the coordinate axes and the supremum is taken over all cubes, and as usual,  $p'$  is the exponent conjugate to  $p$  satisfying  $1/p + 1/p' = 1$ . In 1998, Ding and Lu [3] extended Muckenhoupt-Wheeden's result to get that, when  $s/(s-1) = s' < p < n/\alpha$  and  $1/q = 1/p - \alpha/n$ , and if  $\Omega \in L^s(\mathbb{S}^{n-1})$  and  $\omega(x)^{s'} \in A_{p/s', q/s'}$ , then the two operators  $M_{\Omega,\alpha}$  and  $I_{\Omega,\alpha}$  are all bounded from  $L_{\omega^p}^p(\mathbb{R}^n)$  to  $L_{\omega^q}^q(\mathbb{R}^n)$ .

In 1992, Grafakos [5] first studied the multilinear type maximal function and multilinear type fractional integral. It is well known that the study of multilinear integral operators has recently received increasing attentions, see, for example, [6], [7], [9], [14] and [15]. Recently, following the work of [8], the  $A_{(\vec{p},q)}$  was introduced in [2] and [10] by

$$\sup_{Q \subset \mathbb{R}^n} \left( \frac{1}{|Q|} \int_Q v_{\vec{\omega}}(x)^q dx \right)^{\frac{1}{q}} \prod_{i=1}^m \left( \frac{1}{|Q|} \int_Q \omega_i(x)^{-p'_i dx} \right)^{\frac{1}{p'_i}} < \infty$$

where  $v_{\vec{\omega}} = \prod_{i=1}^m \omega_i$ . They proved that, if  $0 < \alpha < mn$ ,  $1 < p_1, \dots, p_m < \infty$  and  $1/q = 1/p_1 + \dots + 1/p_m - \alpha/n$ , one has  $\vec{\omega} \in A_{(\vec{p},q)}$  if and only if the multilinear operators  $M_\alpha^{(m)}$  and  $I_\alpha^{(m)}$  are bounded from  $L^{p_1}(\omega_1^{p_1}) \times \dots \times L^{p_m}(\omega_m^{p_m})$  to  $L^q(v_{\vec{\omega}}^q)$ . Further, the following multi(sub)linear fractional operators with homogeneous kernels were considered in [2],

$$M_{\Pi\Omega_i,\alpha}^{(m)}(\vec{f})(x) = \sup_{r>0} \frac{1}{r^{mn-\alpha}} \int_{|\vec{y}|<r} \prod_{i=1}^m |\Omega_i(y_i)| |f_i(x-y_i)| d\vec{y},$$

$$I_{\Pi\Omega_i,\alpha}^{(m)}(\vec{f})(x) = \int_{(\mathbb{R}^n)^m} \frac{\prod_{i=1}^m \Omega_i(y_i) f_i(x-y_i)}{|\vec{y}|^{mn-\alpha}} d\vec{y},$$

and the sufficient conditions for the  $(L^{p_1}(\omega_1^{p_1}) \times \dots \times L^{p_m}(\omega_m^{p_m}), L^q(v_{\vec{\omega}}^q))$  boundedness were given, see Theorem 2.5 and 2.6 in [2] for details. But for the rough multi(sub)linear fractional operators  $M_{\Omega,\alpha}^{(m)}$  and  $I_{\Omega,\alpha}^{(m)}$  defined by (1.1) and (1.2), the similar weighted estimates remain unknown. Our first purpose is to study some weighted estimates for the

operators  $M_{\Omega,\alpha}^{(m)}$  and  $I_{\Omega,\alpha}^{(m)}$  related to the  $A_{p,q}$  weights and  $A_{(\vec{p},q)}$  weights, we obtain the following two theorems.

**THEOREM 1.1.** *Let  $0 < \alpha < mn$  and  $\Omega$  be homogeneous of degree zero on  $\mathbb{R}^{mn}$  with  $\Omega \in L^s(\mathbb{S}^{mn-1})$  for  $s > 1$ . Let  $1/q = 1/p_1 + \dots + 1/p_m - \alpha/n > 0$ . Assume that  $\vec{\omega}^{s'} \in A_{(\vec{p},q)}^{(s',q)}$ .*

(a) *If each  $p_i > s'$ , then there is a constant  $C$  independent of  $\vec{f}$  such that*

$$\left\| M_{\Omega,\alpha}^{(m)}(\vec{f}) \right\|_{L_{\vec{\omega}}^{q, \infty}(\mathbb{R}^n)} \leq C \prod_{i=1}^m \|f_i\|_{L_{\omega_i}^{p_i}(\mathbb{R}^n)}.$$

(b) *If each  $p_i \geq s'$ , then there is a constant  $C$  independent of  $\vec{f}$  such that*

$$\left\| M_{\Omega,\alpha}^{(m)}(\vec{f}) \right\|_{L_{\vec{\omega}}^{q, \infty}(\mathbb{R}^n)} \leq C \prod_{i=1}^m \|f_i\|_{L_{\omega_i}^{p_i}(\mathbb{R}^n)}.$$

Here, and henceforth,  $L^{q, \infty}(\mathbb{R}^n)$  denotes the weak  $L^q(\mathbb{R}^n)$  space.

**THEOREM 1.2.** *Let  $0 < \alpha < mn$  and  $\Omega$  be homogeneous of degree zero on  $\mathbb{R}^{mn}$  with  $\Omega \in L^s(\mathbb{S}^{mn-1})$  for  $s > 1$ , and let  $1/q = 1/p_1 + 1/p_2 + \dots + 1/p_m - \alpha/n > 0$ . Assume that the weights  $\omega_i^{s'} \in A_{p_i/s', q_i/s'}$  with some  $q_i > p_i$  satisfying  $1/q_1 + 1/q_2 + \dots + 1/q_m = 1/q$ .*

(a) *If each  $p_i > s'$ , then there is a constant  $C$  independent of  $\vec{f}$  such that*

$$\left\| M_{\Omega,\alpha}^{(m)}(\vec{f}) \right\|_{L_{\vec{\omega}}^{q, \infty}(\mathbb{R}^n)} + \left\| I_{\Omega,\alpha}^{(m)}(\vec{f}) \right\|_{L_{\vec{\omega}}^{q, \infty}(\mathbb{R}^n)} \leq C \prod_{i=1}^m \|f_i\|_{L_{\omega_i}^{p_i}(\mathbb{R}^n)};$$

(b) *If  $p_i \geq s'$ , then there is a constant  $C$  independent of  $\vec{f}$  such that*

$$\left\| M_{\Omega,\alpha}^{(m)}(\vec{f}) \right\|_{L_{\vec{\omega}}^{q, \infty}(\mathbb{R}^n)} + \left\| I_{\Omega,\alpha}^{(m)}(\vec{f}) \right\|_{L_{\vec{\omega}}^{q, \infty}(\mathbb{R}^n)} \leq C \prod_{i=1}^m \|f_i\|_{L_{\omega_i}^{p_i}(\mathbb{R}^n)}.$$

On the other hand, for the two-weighted boundedness of the classical fractional operators, Chanillo, Watson and Wheeden [1], García-Cuerva and Martell [4], Pérez [12], Sawyer and Wheeden [13] showed that, when  $0 < \alpha < n$  and  $1 < p < q < \infty$ , if  $(u, v) \in \mathcal{A}_{p,q}^{\alpha,r,1}$  for some  $r > 1$ , then

$$\left( \int_{\mathbb{R}^n} |M_\alpha f(x)|^q u(x) dx \right)^{1/q} \leq C \left( \int_{\mathbb{R}^n} |f(x)|^p v(x) dx \right)^{1/p}; \tag{1.4}$$

and if  $(u, v) \in \mathcal{A}_{p,q}^{\alpha,r,r}$  for some  $r > 1$ , then

$$\left( \int_{\mathbb{R}^n} |I_\alpha f(x)|^q u(x) dx \right)^{1/q} \leq C \left( \int_{\mathbb{R}^n} |f(x)|^p v(x) dx \right)^{1/p} \tag{1.5}$$

with the constant  $C$  independent of  $f \in L^p_v$ .

It is natural to ask whether the conclusions above can be extended to the multilinear cases with the rough kernels. This is our important objective of the paper, we will derive some sufficient conditions on weights  $(\vec{v}, u)$  and establish the multi-weighted norm inequalities for the rough multi(sub)linear operators  $M_{\Omega, \alpha}^{(m)}$  and  $I_{\Omega, \alpha}^{(m)}$ . To this end, we should introduce the notations  $\mathcal{A}_{p, q}^{\alpha, r, t}$  as follows.

DEFINITION 1.3. Let  $0 \leq \alpha, r, t < \infty$  and  $1 < p, q < \infty$ , we call that the pair of positive weights  $(u, v)$  satisfies the  $\mathcal{A}_{p, q}^{\alpha, r, t}$  condition, i.e.  $(u, v) \in \mathcal{A}_{p, q}^{\alpha, r, t}$ , if

$$\sup_{Q \subset \mathbb{R}^n} |Q|^{\frac{1}{q} + \frac{\alpha}{n} - \frac{1}{p}} \left( \frac{1}{|Q|} \int_Q u(x)^t dx \right)^{\frac{1}{tq}} \left( \frac{1}{|Q|} \int_Q v(x)^{r(1-p')} dx \right)^{\frac{1}{rp'}} < \infty.$$

One notes that, in special case  $r = t = 1$  and  $1/q = 1/p - \alpha/n$ ,  $(w^q, w^p) \in \mathcal{A}_{p, q}^{\alpha, 1, 1}$  if and only if  $w \in A_{p, q}$ . It's easy to see that  $\mathcal{A}_{p, q}^{\alpha, r, t} \subset \mathcal{A}_{p, q}^{\alpha, r, 1} \subset \mathcal{A}_{p, q}^{\alpha, 1, 1}$  and  $\mathcal{A}_{p, q}^{\alpha, r, t} \subset \mathcal{A}_{p, q}^{\alpha, 1, t} \subset \mathcal{A}_{p, q}^{\alpha, 1, 1}$  for any  $r, t > 1$ .

THEOREM 1.4. Let  $0 < \alpha' < mn$  and  $\Omega$  be homogeneous of degree zero on  $\mathbb{R}^{mn}$  with  $\Omega \in L^s(\mathbb{S}^{mn-1})$  for  $s > 1$ . If for each  $i$  we assume that  $s' < p_i < q_i < \infty$ , and the weights  $(u, v_i) \in \mathcal{A}_{p_i/s', q_i/s'}^{\alpha'/m, r_i, 1}$  for some  $r_i > 1$ . Then for any  $f_i \in L^{p_i}_{v_i}(\mathbb{R}^n)$  it follows that

$$\left\| M_{\Omega, \alpha}^{(m)}(\vec{f}) \right\|_{L^q_u(\mathbb{R}^n)} \leq C \prod_{i=1}^m \|f_i\|_{L^{p_i}_{v_i}(\mathbb{R}^n)}$$

with the constant  $C > 0$  independent of  $\vec{f}$ , where  $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} + \dots + \frac{1}{q_m}$ .

THEOREM 1.5. Let  $0 < \alpha' < mn$  and  $\Omega$  be homogeneous of degree zero on  $\mathbb{R}^{mn}$  with  $\Omega \in L^s(\mathbb{S}^{mn-1})$  for  $s > 1$ . If for each  $i$  we assume that  $s' < p_i < q_i < \infty$ , and the weights  $(u, v_i) \in \mathcal{A}_{p_i/s', q_i/s'}^{\alpha'/m, 1, 1}$ . Then for any  $f_i \in L^{p_i}_{v_i}(\mathbb{R}^n)$  it follows that

$$\left\| M_{\Omega, \alpha}^{(m)}(\vec{f}) \right\|_{L^{q, \infty}_u(\mathbb{R}^n)} \leq C \prod_{i=1}^m \|f_i\|_{L^{p_i}_{v_i}(\mathbb{R}^n)}$$

with the constant  $C > 0$  independent of  $\vec{f}$ , where  $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} + \dots + \frac{1}{q_m}$ .

THEOREM 1.6. Let  $0 < \alpha' < mn$  and  $\Omega$  be homogeneous of degree zero on  $\mathbb{R}^{mn}$  with  $\Omega \in L^s(\mathbb{S}^{mn-1})$  for  $s > 1$ . If for each  $i$  we assume that  $s' < p_i < q_i < \infty$ , and the weights  $(u, v_i) \in \mathcal{A}_{p_i/s', q_i/s'}^{\alpha'/m, r_i, t_i}$  for some  $r_i > 1$  and  $t_i > 1$ . Then for any  $f_i \in L^{p_i}_{v_i}(\mathbb{R}^n)$  it follows that

$$\left\| I_{\Omega, \alpha}^{(m)}(\vec{f}) \right\|_{L^q(\mathbb{R}^n)} \leq C \prod_{i=1}^m \|f_i\|_{L^{p_i}_{v_i}(\mathbb{R}^n)}$$

with the constant  $C > 0$  independent of  $\vec{f}$ , where  $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} + \dots + \frac{1}{q_m}$ .

**THEOREM 1.7.** *Let  $0 < \alpha' < mn$  and  $\Omega$  be homogeneous of degree zero on  $\mathbb{R}^{mn}$  with  $\Omega \in L^s(\mathbb{S}^{mn-1})$  for  $s > 1$ . If for each  $i$  we assume that  $s' < p_i < q_i < \infty$ , and the weights  $(u, v_i) \in \mathcal{A}_{p_i/s', q_i/s'}^{\alpha'/m, 1, t_i}$  for some  $t_i > 1$ . Then for any  $f_i \in L_{v_i}^{p_i}(\mathbb{R}^n)$  it follows that*

$$\left\| I_{\Omega, \alpha}^{(m)}(\vec{f}) \right\|_{L_{u'}^{q, \infty}(\mathbb{R}^n)} \leq C \prod_{i=1}^m \|f_i\|_{L_{v_i}^{p_i}(\mathbb{R}^n)}$$

with the constant  $C > 0$  independent of  $\vec{f}$ , where  $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} + \dots + \frac{1}{q_m}$ .

This paper is organized as follows. In section 2, we will give some basic estimates and the the unweighted boundedness for  $M_{\Omega, \alpha}^{(m)}$  and  $I_{\Omega, \alpha}^{(m)}$ , i.e. Theorem 2.4 and 2.5. In section 3, we will prove Theorem 1.1 and 1.2. Finally in section 4, we will derive the slight stronger results on the multi-weighted boundedness for  $M_{\Omega, \alpha}^{(m)}$  and  $I_{\Omega, \alpha}^{(m)}$ , see Theorem 4.1 and 4.2, and Theorem 4.5 and 4.6, which will imply Theorem 1.4–1.7 above.

Throughout this paper, the letter  $C$  always remains to denote a positive constant that may varies at each occurrence but is independent of the essential variable.

### 2. Elemental estimates for $M_{\Omega, \alpha}^{(m)}$ and $I_{\Omega, \alpha}^{(m)}$

In this section, we give some elemental estimates and prove the unweighted bound- edness for  $M_{\Omega, \alpha}^{(m)}$  and  $I_{\Omega, \alpha}^{(m)}$ , i.e., Theorem 2.4 and 2.5.

**LEMMA 2.1.** [9] *Let  $0 < \alpha < mn$  and  $1/q = 1/p_1 + 1/p_2 + \dots + 1/p_m - \alpha/n > 0$ .*

(a) *If each  $p_i > 1$ , then*

$$\left\| I_{\alpha}^{(m)}(\vec{f}) \right\|_{L^q(\mathbb{R}^n)} \leq C \prod_{i=1}^m \|f_i\|_{L^{p_i}(\mathbb{R}^n)}.$$

(b) *If  $p_i \geq 1$ , then*

$$\left\| I_{\alpha}^{(m)}(\vec{f}) \right\|_{L^{q, \infty}(\mathbb{R}^n)} \leq C \prod_{i=1}^m \|f_i\|_{L^{p_i}(\mathbb{R}^n)}.$$

**LEMMA 2.2.** *Let  $\Omega$  be homogeneous of degree zero on  $\mathbb{R}^{mn}$  with  $\Omega \in L^s(\mathbb{S}^{mn-1})$  for some  $s > 1$ , and let  $1/s + 1/s' = 1$ . Then there exists a constant  $C > 0$  such that, for any  $x \in \mathbb{R}^n$  and  $f_i \in L_{loc}(\mathbb{R}^n)$  ( $i = 1, 2, \dots, m$ ),*

$$M_{\Omega, \alpha}^{(m)}(\vec{f})(x) \leq C \left[ M_{\alpha, s'}^{(m)}(|f_1|^{s'}, |f_2|^{s'}, \dots, |f_m|^{s'})(x) \right]^{1/s'}.$$

*Proof.* By the Hölder inequality, we have

$$\begin{aligned} & \frac{1}{r^{mn-\alpha}} \int_{|\vec{y}|<r} |\Omega(\vec{y})| \prod_{i=1}^m |f_i(x-y_i)| d\vec{y} \\ & \leq \frac{1}{r^{mn-\alpha}} \left( \int_{|\vec{y}|<r} \prod_{i=1}^m |f_i(x-y_i)|^{s'} d\vec{y} \right)^{1/s'} \left( \int_{|\vec{y}|<r} |\Omega(\vec{y})|^s d\vec{y} \right)^{1/s} \\ & \leq C \sup_{r>0} \left( \frac{1}{r^{mn-\alpha s'}} \int_{|\vec{y}|<r} \prod_{i=1}^m |f_i(x-y_i)|^{s'} d\vec{y} \right)^{1/s'} \\ & \leq C \left[ M_{\alpha s'}^{(m)} (|f_1|^{s'}, |f_2|^{s'}, \dots, |f_m|^{s'}) (x) \right]^{1/s'}. \end{aligned}$$

This completes the proof of the Lemma.  $\square$

LEMMA 2.3. *Let  $0 < \alpha < mn$  and  $0 < \varepsilon < \min\{\alpha, mn - \alpha\}$ , then there is a constant  $C > 0$  such that, for any  $x \in \mathbb{R}^n$  and  $f_i \in L_{loc}(\mathbb{R}^n)$  ( $i = 1, 2, \dots, m$ ),*

$$\left| I_{\Omega, \alpha}^{(m)}(\vec{f})(x) \right| \leq C \left[ M_{\Omega, \alpha + \varepsilon}^{(m)}(\vec{f})(x) \right]^{\frac{1}{2}} \left[ M_{\Omega, \alpha - \varepsilon}^{(m)}(\vec{f})(x) \right]^{\frac{1}{2}}.$$

*Proof.* This lemma can be seen by standard method. Fix  $x \in \mathbb{R}^n$  and  $0 < \varepsilon < \min\{\alpha, mn - \alpha\}$ , for any  $\delta > 0$ , we decompose as follows,

$$\left| I_{\Omega, \alpha}^{(m)}(\vec{f})(x) \right| \leq \left( \int_{|\vec{y}| \leq \delta} + \int_{|\vec{y}| > \delta} \right) \frac{|\Omega(\vec{y})|}{|\vec{y}|^{mn-\alpha}} \prod_{i=1}^m |f_i(x-y_i)| d\vec{y} := I_1 + I_2.$$

It's easy to see that

$$\begin{aligned} I_1 &= \sum_{j=1}^{\infty} \int_{2^{-j}\delta < |\vec{y}| \leq 2^{-j+1}\delta} \frac{|\Omega(\vec{y})|}{|\vec{y}|^{mn-\alpha}} \prod_{i=1}^m |f_i(x-y_i)| d\vec{y} \\ &\leq C \sum_{j=0}^{\infty} \frac{1}{(2^{-j}\delta)^{mn-\alpha}} \int_{|\vec{y}| \leq 2^{-j}\delta} |\Omega(\vec{y})| \prod_{i=1}^m |f_i(x-y_i)| d\vec{y} \\ &\leq C \sum_{j=0}^{\infty} (2^{-j}\delta)^{\varepsilon} M_{\Omega, \alpha - \varepsilon}^{(m)}(\vec{f})(x) \leq C \delta^{\varepsilon} M_{\Omega, \alpha - \varepsilon}^{(m)}(\vec{f})(x), \end{aligned}$$

and

$$\begin{aligned} I_2 &= \sum_{j=0}^{\infty} \int_{2^j\delta < |\vec{y}| \leq 2^{j+1}\delta} \frac{|\Omega(\vec{y})|}{|\vec{y}|^{mn-\alpha}} \prod_{i=1}^m |f_i(x-y_i)| d\vec{y} \\ &\leq \sum_{j=0}^{\infty} \frac{1}{(2^j\delta)^{mn-\alpha}} \int_{|\vec{y}| \leq 2^{j+1}\delta} |\Omega(\vec{y})| \prod_{i=1}^m |f_i(x-y_i)| d\vec{y} \\ &\leq C \sum_{j=0}^{\infty} (2^j\delta)^{-\varepsilon} M_{\Omega, \alpha + \varepsilon}^{(m)}(\vec{f})(x) \leq C \delta^{-\varepsilon} M_{\Omega, \alpha + \varepsilon}^{(m)}(\vec{f})(x). \end{aligned}$$

Thus we get

$$|I_{\Omega,\alpha}^{(m)}(\vec{f})(x)| \leq C\delta^\varepsilon M_{\Omega,\alpha-\varepsilon}^{(m)}(\vec{f})(x) + C\delta^{-\varepsilon} M_{\Omega,\alpha+\varepsilon}^{(m)}(\vec{f})(x).$$

Now we take  $\delta > 0$  such that

$$\delta^\varepsilon M_{\Omega,\alpha-\varepsilon}^{(m)}(\vec{f})(x) = \delta^{-\varepsilon} M_{\Omega,\alpha+\varepsilon}^{(m)}(\vec{f})(x),$$

which implies the Lemma.  $\square$

**THEOREM 2.4.** *Let  $0 < \alpha < mn$  and  $\Omega$  be homogeneous of degree zero on  $\mathbb{R}^{mn}$  with  $\Omega \in L^s(\mathbb{S}^{mn-1})$  for  $s > 1$ . Let  $1/q = 1/p_1 + 1/p_2 + \dots + 1/p_m - \alpha/n > 0$ .*

(a) *If each  $p_i > s'$ , then there is a constant  $C$  independent of  $\vec{f}$  such that*

$$\left\| M_{\Omega,\alpha}^{(m)}(\vec{f}) \right\|_{L^q(\mathbb{R}^n)} \leq C \prod_{i=1}^m \|f_i\|_{L^{p_i}(\mathbb{R}^n)}.$$

(b) *If  $p_i \geq s'$ , then there is a constant  $C$  independent of  $\vec{f}$  such that*

$$\left\| M_{\Omega,\alpha}^{(m)}(\vec{f}) \right\|_{L^{q,\infty}(\mathbb{R}^n)} \leq C \prod_{i=1}^m \|f_i\|_{L^{p_i}(\mathbb{R}^n)}.$$

*Proof.* Fix  $x \in \mathbb{R}^n$ ,  $0 < \alpha < mn$ , we have

$$\frac{1}{r^{mn-\alpha}} \int_{|\vec{y}| < r} \prod_{i=1}^m |f_i(x - y_i)| d\vec{y} \leq \int_{|\vec{y}| < r} \frac{1}{|\vec{y}|^{mn-\alpha}} \prod_{i=1}^m |f_i(x - y_i)| d\vec{y}.$$

Taking the supremum for  $r > 0$  on both sides of the inequality above, we have

$$M_{\Omega,\alpha}^{(m)}(\vec{f})(x) \leq I_{\Omega,\alpha}^{(m)}(|f_1|, |f_2|, \dots, |f_m|)(x).$$

Then, applying Lemma 2.1, we immediately obtain that

$$\left\| M_{\Omega,\alpha}^{(m)}(\vec{f}) \right\|_{L^q(\mathbb{R}^n)} \leq C \prod_{i=1}^m \|f_i\|_{L^{p_i}(\mathbb{R}^n)} \tag{2.1}$$

if each  $p_i > 1$ , and

$$\left\| M_{\Omega,\alpha}^{(m)}(\vec{f}) \right\|_{L^{q,\infty}(\mathbb{R}^n)} \leq C \prod_{i=1}^m \|f_i\|_{L^{p_i}(\mathbb{R}^n)} \tag{2.2}$$

if some  $p_i = 1$ .

Here we point out that, if each  $p_i \geq s'$ , one can see from the assumption  $1/p_1 + 1/p_2 + \dots + 1/p_m - \alpha/n > 0$  that  $\alpha < nm/s'$ , and so  $0 < \alpha s' < mn$ .

Now if each  $p_i > s'$ , we use Lemma 2.2 and the inequality (2.1) to get that

$$\left\| M_{\Omega,\alpha}^{(m)}(\vec{f}) \right\|_{L^q(\mathbb{R}^n)} \leq C \prod_{i=1}^m \left\| |f_i|^{s'} \right\|_{L^{p_i/s'}(\mathbb{R}^n)}^{1/s'} \leq C \prod_{i=1}^m \|f_i\|_{L^{p_i}(\mathbb{R}^n)}.$$

If  $p_i \geq s'$ , we use Lemma 2.2 and (2.2) to get for any  $\lambda > 0$  that

$$\begin{aligned} & \left| \left\{ x : |M_{\Omega, \alpha}^{(m)}(\vec{f})(x)| > \lambda \right\} \right| \\ & \leq C \left| \left\{ x : |[M_{\alpha s'}^{(m)}(|f_1|^{s'}, |f_2|^{s'}, \dots, |f_m|^{s'})(x)]|^{\frac{1}{s'}} > \lambda \right\} \right| \\ & \leq C \left( \frac{1}{\lambda^{s'}} \prod_{i=1}^m \| |f_i|^{s'} \|_{L^{p_i/s'}(\mathbb{R}^n)} \right)^{q/s'} \leq C \left( \frac{1}{\lambda} \prod_{i=1}^m \| f_i \|_{L^{p_i}(\mathbb{R}^n)} \right)^q. \end{aligned}$$

Thus we complete the proof of Theorem 2.4.  $\square$

**THEOREM 2.5.** *Suppose the same conditions and notations as that in Theorem 2.4,*

(a) *if each  $p_i > s'$ , then there is a constant  $C$  independent of  $\vec{f}$  such that*

$$\left\| I_{\Omega, \alpha}^{(m)}(\vec{f}) \right\|_{L^q(\mathbb{R}^n)} \leq C \prod_{i=1}^m \| f_i \|_{L^{p_i}(\mathbb{R}^n)};$$

(b) *if  $p_i \geq s'$ , then there is a constant  $C$  independent of  $\vec{f}$  such that*

$$\left\| I_{\Omega, \alpha}^{(m)}(\vec{f}) \right\|_{L^{q, \infty}(\mathbb{R}^n)} \leq C \prod_{i=1}^m \| f_i \|_{L^{p_i}(\mathbb{R}^n)}.$$

*Proof.* Taking a small positive number  $\varepsilon$  with  $0 < \varepsilon < \min\{\alpha, mn - \alpha, n/q\}$ , one can then see that

$$\frac{1}{q_1} := \frac{1}{p_1} + \frac{1}{p_2} + \dots + \frac{1}{p_m} - \frac{\alpha + \varepsilon}{n} = \frac{1}{q} - \frac{\varepsilon}{n} > 0,$$

$$\frac{1}{q_2} := \frac{1}{p_1} + \frac{1}{p_2} + \dots + \frac{1}{p_m} - \frac{\alpha - \varepsilon}{n} = \frac{1}{q} + \frac{\varepsilon}{n} > 0.$$

Now if each  $p_i > s'$ , using Lemma 2.3, the Hölder inequality for  $\frac{1}{q_1} + \frac{1}{q_2} = \frac{2}{q}$ , and applying Theorem 2.4, we obtain that

$$\begin{aligned} \left\| I_{\Omega, \alpha}^{(m)}(\vec{f}) \right\|_{L^q(\mathbb{R}^n)}^2 & \leq C \left( \int_{\mathbb{R}^n} \left[ M_{\Omega, \alpha + \varepsilon}^{(m)}(\vec{f})(x) \right]^{\frac{q}{2}} \left[ M_{\Omega, \alpha - \varepsilon}^{(m)}(\vec{f})(x) \right]^{\frac{q}{2}} dx \right)^{2/q} \\ & \leq C \left\| M_{\Omega, \alpha + \varepsilon}^{(m)}(\vec{f}) \right\|_{L^{q_1}(\mathbb{R}^n)} \left\| M_{\Omega, \alpha - \varepsilon}^{(m)}(\vec{f}) \right\|_{L^{q_2}(\mathbb{R}^n)} \\ & \leq C \prod_{i=1}^m \| f_i \|_{L^{p_i}(\mathbb{R}^n)}^2. \end{aligned}$$

Similarly, if  $p_i = s'$  for some  $i$ , we take  $A^2 = \lambda^{\frac{2q_1}{q_1+q_2}} \left( \prod_{i=1}^m \|f_i\|_{L^{p_i}} \right)^{\frac{q_2-q_1}{q_2+q_1}}$  for any  $\lambda > 0$ , then applying Lemma 2.3 and Theorem 2.4, we give that

$$\begin{aligned} & \left| \left\{ x : |I_{\Omega,\alpha}^{(m)}(\vec{f})(x)| > \lambda \right\} \right| \\ & \leq \left| \left\{ x : |M_{\Omega,\alpha-\varepsilon}^{(m)}(\vec{f})(x)| > A^2/C^2 \right\} \right| + \left| \left\{ x : |M_{\Omega,\alpha+\varepsilon}^{(m)}(\vec{f})(x)| > \lambda^2/A^2 \right\} \right| \\ & \leq C \left( \frac{1}{A^2} \prod_{i=1}^m \|f_i\|_{L^{p_i}(\mathbb{R}^n)} \right)^{q_2} + C \left( \frac{A^2}{\lambda^2} \prod_{i=1}^m \|f_i\|_{L^{p_i}(\mathbb{R}^n)} \right)^{q_1} \\ & \leq C \left( \frac{1}{\lambda} \prod_{i=1}^m \|f_i\|_{L^{p_i}(\mathbb{R}^n)} \right)^q. \end{aligned}$$

This yields the desired inequality of Theorem 2.5. Thus we complete the proof of Theorem 2.5.  $\square$

### 3. Weighted estimates for operators $I_{\Omega,\alpha}^{(m)}$ and $M_{\Omega,\alpha}^{(m)}$

In this section, we will prove the weighted norm inequalities for rough multi-(sub)linear operators  $M_{\Omega,\alpha}^{(m)}$  and  $I_{\Omega,\alpha}^{(m)}$ , i.e., Theorem 1.1 and 1.2.

A locally integrable nonnegative function  $\omega$  on  $\mathbb{R}^n$  is said to belong to  $A_p$  ( $1 < p < \infty$ ) if there exists  $C$  such that

$$\sup_{Q \subset \mathbb{R}^n} \left( \frac{1}{|Q|} \int_Q \omega(x) dx \right) \left( \frac{1}{|Q|} \int_Q \omega(x)^{-\frac{1}{p-1}} dx \right)^{p-1} \leq C < \infty,$$

where  $Q$  denotes any cube in  $\mathbb{R}^n$ . Recall (1.3), the definition of  $A_{p,q}$  weight defined in the first section, one can see that  $\omega \in A_{p,p}$  if and only if  $\omega^p \in A_p$  for  $1 < p < \infty$ . Moreover, if  $1/q = 1/p - \alpha/n$  with  $1 < p < n/\alpha$  and  $0 < \alpha < n$ , then it's easy to deduce that

$$\omega(x) \in A_{p,q} \Leftrightarrow \omega(x)^q \in A_{q(n-\alpha)/n} \Leftrightarrow \omega(x)^q \in A_{1+q/p'}.$$

From the characterization of  $A_p$  weights, one can shows that

LEMMA 3.1. [3] *Let  $0 < \alpha < n$ ,  $1 \leq s' < p < n/\alpha$ ,  $1/q = 1/p - \alpha/n$  and  $\omega(x)^{s'} \in A_{p/s',q/s'}$ . Then there exists a small positive number  $\varepsilon$  with  $0 < \varepsilon < \min\{\alpha, n/p - \alpha, n/q'\}$  such that  $\omega(x)^{s'} \in A_{p/s',q_\varepsilon/s'}$  and  $\omega(x)^{s'} \in A_{p/s',\tilde{q}_\varepsilon/s'}$ , where  $1/q_\varepsilon = 1/p - (\alpha + \varepsilon)/n$  and  $1/\tilde{q}_\varepsilon = 1/p - (\alpha - \varepsilon)/n$ .*

LEMMA 3.2. [11] *Let  $0 \leq \alpha < n$ ,  $1 < p < n/\alpha$ ,  $1/q = 1/p - \alpha/n$  and  $\omega \in A_{p,q}$ . Then there exists a constant  $C$  independent of  $f$  such that*

$$\left( \int_{\mathbb{R}^n} |M_\alpha f(x) \omega(x)|^q dx \right)^{1/q} \leq C \left( \int_{\mathbb{R}^n} |f(x) \omega(x)|^p dx \right)^{1/p}.$$

LEMMA 3.3. [10] Let  $0 < \alpha < mn$ ,  $\frac{1}{q} = \frac{1}{p_1} + \dots + \frac{1}{p_m} - \frac{\alpha}{n} > 0$ , and  $\vec{\omega} \in A_{\vec{p},q}$ .  
 (i) if  $1 < p_1, \dots, p_m < \infty$ , then

$$\left\| M_{\alpha}^{(m)}(\vec{f}) \right\|_{L_{v_{\vec{\omega}}}^q(\mathbb{R}^n)} + \left\| I_{\alpha}^{(m)}(\vec{f}) \right\|_{L_{v_{\vec{\omega}}}^q(\mathbb{R}^n)} \leq C \prod_{i=1}^m \|f_i\|_{L_{\omega_i^{p_i}}^{p_i}(\mathbb{R}^n)};$$

(ii) if  $1 \leq p_1 \dots, p_m < \infty$ , then

$$\left\| M_{\alpha}^{(m)}(\vec{f}) \right\|_{L_{v_{\vec{\omega}}}^{q,\infty}(\mathbb{R}^n)} + \left\| I_{\alpha}^{(m)}(\vec{f}) \right\|_{L_{v_{\vec{\omega}}}^{q,\infty}(\mathbb{R}^n)} \leq C \prod_{i=1}^m \|f_i\|_{L_{\omega_i^{p_i}}^{p_i}(\mathbb{R}^n)};$$

with the absolute constant  $C$  independent of  $f_i$ , where  $v_{\vec{\omega}} = \prod_{i=1}^m \omega_i$ .

The proof of Theorem 1.1. By Lemma 2.2 we have

$$\left\| M_{\Omega,\alpha}^{(m)}(\vec{f}) \right\|_{L_{v_{\vec{\omega}}}^q(\mathbb{R}^n)} \leq C \left\| M_{\alpha s'}^{(m)}(|f_1|^{s'}, |f_2|^{s'}, \dots, |f_m|^{s'}) \right\|_{L_{v_{\vec{\omega}}}^{q/s'}(\mathbb{R}^n)}^{1/s'}. \tag{3.1}$$

This, together with Lemma 3.3 under the assumption  $\vec{\omega}^{s'} \in A_{(\frac{\vec{p}}{s'}, \frac{q}{s'})}$ , yields the theorem.  $\square$

The proof of Theorem 1.2. Noting the assumptions  $\omega_i^{s'} \in A_{p_i/s', q_i/s'}$  imply that  $\vec{\omega}^{s'} \in A_{(\frac{\vec{p}}{s'}, \frac{q}{s'})}$ , so the boundedness for  $M_{\Omega,\alpha}^{(m)}$  follows from Theorem 1.1, it's left to show the boundedness for  $I_{\Omega,\alpha}^{(m)}$ .

We can choose  $\alpha_i$  satisfying  $\frac{1}{q_i} = \frac{1}{p_i} - \frac{\alpha_i}{n}$  and  $0 < \alpha_i s' < n$ . Since  $\omega_i^{s'} \in A_{p_i/s', q_i/s'}$ , we get from Lemma 3.1 that there exists a small positive number  $\varepsilon$  such that

$$\omega_i^{s'} \in A_{p_i/s', \gamma_i/s'} \quad \text{where} \quad \frac{1}{\gamma_i} = \frac{1}{p_i} - \frac{\alpha_i^+}{n} \quad \text{and} \quad \alpha_i^+ = \alpha_i + \frac{\varepsilon}{m} < \frac{n}{s'},$$

and

$$\omega_i^{s'} \in A_{p_i/s', \xi_i/s'} \quad \text{where} \quad \frac{1}{\xi_i} = \frac{1}{p_i} - \frac{\alpha_i^-}{n} \quad \text{and} \quad \alpha_i^- = \alpha_i - \frac{\varepsilon}{m} > 0.$$

Denote by

$$\frac{1}{\beta_1} := \frac{1}{p_1} + \frac{1}{p_2} + \dots + \frac{1}{p_m} - \frac{\alpha + \varepsilon}{n} = \frac{1}{q} - \frac{\varepsilon}{n} > 0,$$

$$\frac{1}{\beta_2} := \frac{1}{p_1} + \frac{1}{p_2} + \dots + \frac{1}{p_m} - \frac{\alpha - \varepsilon}{n} = \frac{1}{q} + \frac{\varepsilon}{n} > 0,$$

we have that

$$\frac{1}{\beta_1} = \frac{1}{\gamma_1} + \frac{1}{\gamma_2} + \dots + \frac{1}{\gamma_m}, \quad \alpha_1^+ + \alpha_2^+ + \dots + \alpha_m^+ = \alpha + \varepsilon,$$

and

$$\frac{1}{\beta_2} = \frac{1}{\xi_1} + \frac{1}{\xi_2} + \cdots + \frac{1}{\xi_m}, \quad \alpha_1^- + \alpha_2^- + \cdots + \alpha_m^- = \alpha - \varepsilon.$$

Then, by Lemma 2.3 and the Hölder inequality, and by Theorem 1.1 we get

$$\begin{aligned} & \left\| I_{\Omega, \alpha}^{(m)}(\vec{f}) \right\|_{L_{v_{\vec{\omega}}}^q(\mathbb{R}^n)} = \left( \int_{\mathbb{R}^n} \left| I_{\Omega, \alpha}^{(m)}(\vec{f})(x) v_{\vec{\omega}}(x) \right|^q dx \right)^{1/q} \\ & \leq C \left( \int_{\mathbb{R}^n} \left[ M_{\Omega, \alpha + \varepsilon}^{(m)}(\vec{f})(x) v_{\vec{\omega}}(x) \right]^{\frac{q}{2}} \left[ M_{\Omega, \alpha - \varepsilon}^{(m)}(\vec{f})(x) v_{\vec{\omega}}(x) \right]^{\frac{q}{2}} dx \right)^{1/q} \\ & \leq \left\| M_{\Omega, \alpha + \varepsilon}^{(m)}(\vec{f}) \right\|_{L_{v_{\vec{\omega}}}^{\beta_1}(\mathbb{R}^n)}^{1/2} \left\| M_{\Omega, \alpha - \varepsilon}^{(m)}(\vec{f}) \right\|_{L_{v_{\vec{\omega}}}^{\beta_2}(\mathbb{R}^n)}^{1/2} \\ & \leq C \prod_{i=1}^m \|f_i\|_{L_{\omega_i}^{p_i}(\mathbb{R}^n)}, \end{aligned}$$

which is the strong type estimate.

Similarly, if  $p_i \geq s'$ , we take  $A^2 = \lambda^{\frac{2\beta_1}{\beta_1 + \beta_2}} \left( \prod_{i=1}^m \|f_i\|_{L_{\omega_i}^{p_i}} \right)^{\frac{\beta_2 - \beta_1}{\beta_2 + \beta_1}}$  for any  $\lambda > 0$ , then applying Lemma 2.3 and Theorem 1.1, we give that

$$\begin{aligned} & \left| v_{\vec{\omega}} \left\{ x : \left| I_{\Omega, \alpha}^{(m)}(\vec{f})(x) \right| > \lambda \right\} \right| \\ & \leq \left| v_{\vec{\omega}} \left\{ x : \left| M_{\Omega, \alpha - \varepsilon}^{(m)}(\vec{f})(x) \right| > A^2/C^2 \right\} \right| + \left| v_{\vec{\omega}} \left\{ x : \left| M_{\Omega, \alpha + \varepsilon}^{(m)}(\vec{f})(x) \right| > \lambda^2/A^2 \right\} \right| \\ & \leq C \left( \frac{1}{A^2} \prod_{i=1}^m \|f_i\|_{L_{\omega_i}^{p_i}(\mathbb{R}^n)} \right)^{\beta_2} + C \left( \frac{A^2}{\lambda^2} \prod_{i=1}^m \|f_i\|_{L_{\omega_i}^{p_i}(\mathbb{R}^n)} \right)^{\beta_1} \\ & \leq C \left( \frac{1}{\lambda} \prod_{i=1}^m \|f_i\|_{L_{\omega_i}^{p_i}(\mathbb{R}^n)} \right)^q. \end{aligned}$$

This finishes the proof of Theorem 1.2.  $\square$

#### 4. Multi-weighted estimates for $M_{\Omega, \alpha}^{(m)}$ and $I_{\Omega, \alpha}^{(m)}$

In this section, we will obtain the multi-weighted norm estimates for the fractional multi-sublinear maximal operator  $M_{\Omega, \alpha}^{(m)}$  and the multilinear fractional integral operator  $I_{\Omega, \alpha}^{(m)}$ . The slight stronger theorems will be proved here than stated in the Section one, which will imply Theorem 1.4–1.7 respectively.

**THEOREM 4.1.** *Let  $\Omega$  be homogeneous of degree zero on  $\mathbb{R}^{mn}$  with  $\Omega \in L^s(\mathbb{S}^{mn-1})$  for  $s > 1$ . Assume, for each  $i = 1, 2, \dots, m$ , that  $0 < \alpha_i s' < n$ ,  $s' < p_i < q_i < \infty$  and*

the weights  $(u, v_i) \in \mathcal{A}_{p_i/s', q_i/s'}^{\alpha_i s', r_i, 1}$  with some  $r_i > 1$ . Let  $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} + \dots + \frac{1}{q_m}$  and  $\alpha = \alpha_1 + \alpha_2 + \dots + \alpha_m$ . Then for any  $f_i \in L_{v_i}^{p_i}(\mathbb{R}^n)$  it follows that

$$\left\| M_{\Omega, \alpha}^{(m)}(\vec{f}) \right\|_{L_u^q(\mathbb{R}^n)} \leq C \prod_{i=1}^m \|f_i\|_{L_{v_i}^{p_i}(\mathbb{R}^n)} \tag{4.1}$$

with the constant  $C > 0$  independent of  $\vec{f}$ .

*Proof.* Using Lemma 2.2, the condition  $\alpha s' = \alpha_1 s' + \dots + \alpha_m s'$ , and the Hölder inequality for integral with the index  $\frac{1}{q/s'} = \frac{1}{q_1/s'} + \dots + \frac{1}{q_m/s'}$ , we have

$$\begin{aligned} & \left\| M_{\Omega, \alpha}^{(m)}(\vec{f}) \right\|_{L_u^q(\mathbb{R}^n)} \\ & \leq C \left( \int_{\mathbb{R}^n} \left| M_{\alpha s'}^{(m)}(|f_1|^{s'}, |f_2|^{s'}, \dots, |f_m|^{s'})(x) \right|^{\frac{q}{s'}} u(x) dx \right)^{1/q} \\ & \leq C \left( \int_{\mathbb{R}^n} \left| \prod_{i=1}^m M_{\alpha_i s'}(|f_i|^{s'}) \right|^{\frac{q}{s'}} u(x) dx \right)^{1/q} \\ & \leq C \prod_{i=1}^m \left\| M_{\alpha_i s'}(|f_i|^{s'}) \right\|_{L_u^{q_i/s'}(\mathbb{R}^n)}^{1/s'} \end{aligned}$$

Since  $(u, v_i) \in \mathcal{A}_{p_i/s', q_i/s'}^{\alpha_i s', r_i, 1}$  for each  $i = 1, 2, \dots, m$ , we apply the two-weighted boundedness for operators  $M_{\alpha_i s'}$ , the inequality (1.4), we obtain that, for  $|f_i|^{s'} \in L_{v_i}^{p_i/s'}(\mathbb{R}^n)$ ,

$$\begin{aligned} \left\| M_{\Omega, \alpha}^{(m)}(\vec{f}) \right\|_{L_u^q(\mathbb{R}^n)} & \leq C \prod_{i=1}^m \left\| M_{\alpha_i s'}(|f_i|^{s'}) \right\|_{L_u^{q_i/s'}(\mathbb{R}^n)}^{1/s'} \\ & \leq C \prod_{i=1}^m \left\| |f_i|^{s'} \right\|_{L_{v_i}^{p_i/s'}(\mathbb{R}^n)}^{1/s'} \leq C \prod_{i=1}^m \|f_i\|_{L_{v_i}^{p_i}(\mathbb{R}^n)}. \end{aligned}$$

This proves the inequality (4.1).  $\square$

**THEOREM 4.2.** Let  $\Omega$  be homogeneous of degree zero on  $\mathbb{R}^{mn}$  with  $\Omega \in L^s(\mathbb{S}^{mn-1})$  for  $s > 1$ . Assume, for each  $i = 1, 2, \dots, m$ , that  $0 < \alpha_i s' < n$ ,  $s' < p_i < q_i < \infty$  and the weights  $(u, v_i) \in \mathcal{A}_{p_i/s', q_i/s'}^{\alpha_i s', r_i, t_i}$  with some  $r_i > 1$  and  $t_i > 1$ . Let  $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} + \dots + \frac{1}{q_m}$  and  $\alpha = \alpha_1 + \alpha_2 + \dots + \alpha_m$ . Then for any  $f_i \in L_{v_i}^{p_i}(\mathbb{R}^n)$  it follows that

$$\left\| I_{\Omega, \alpha}^{(m)}(\vec{f}) \right\|_{L_u^q(\mathbb{R}^n)} \leq C \prod_{i=1}^m \|f_i\|_{L_{v_i}^{p_i}(\mathbb{R}^n)} \tag{4.2}$$

with the constant  $C > 0$  independent of  $\vec{f}$ .

*Proof.* Under the conditions of Theorem 4.2, we can choose a positive number  $\varepsilon$  such that, for each  $i = 1, 2, \dots, m$ ,

$$0 < \varepsilon < \min \left\{ \alpha, mn - \alpha, \frac{n}{q}, \frac{\alpha_i q_i}{q}, \frac{(n - \alpha_i s') q_i}{q s'}, \left( \frac{q_i}{p_i} - 1 \right) \frac{n}{q}, \frac{n}{q t'_i} \right\}.$$

Let  $\frac{1}{l_1} = \frac{1}{q} - \frac{\varepsilon}{n}$ ,  $\frac{1}{l_2} = \frac{1}{q} + \frac{\varepsilon}{n}$ , then  $\frac{1}{l_1} + \frac{1}{l_2} = \frac{2}{q}$ . Applying Lemma 2.3 and the Hölder inequality, we get

$$\begin{aligned} \left\| I_{\Omega, \alpha}^{(m)}(\vec{f}) \right\|_{L_{u_i}^q(\mathbb{R}^n)} &\leq C \left( \int_{\mathbb{R}^n} \left[ M_{\Omega, \alpha + \varepsilon}^{(m)}(\vec{f})(x) \right]^{\frac{q}{2}} \left[ M_{\Omega, \alpha - \varepsilon}^{(m)}(\vec{f})(x) \right]^{\frac{q}{2}} u(x) dx \right)^{1/q} \\ &\leq C \left\| M_{\Omega, \alpha + \varepsilon}^{(m)}(\vec{f}) \right\|_{L_{u_i^{1/q}}^{l_1}(\mathbb{R}^n)}^{1/2} \left\| M_{\Omega, \alpha - \varepsilon}^{(m)}(\vec{f}) \right\|_{L_{u_i^{2/q}}^{l_2}(\mathbb{R}^n)}^{1/2}. \end{aligned} \tag{4.3}$$

Now we choose here  $\alpha_i^+ = \alpha_i + \varepsilon q / q_i$  and  $\alpha_i^- = \alpha_i - \varepsilon q / q_i$  for each  $i$ , then  $0 < \alpha_i^- s' < \alpha_i^+ s' < n$  and

$$\alpha_1^+ + \alpha_2^+ + \dots + \alpha_m^+ = \alpha + \varepsilon, \quad \alpha_1^- + \alpha_2^- + \dots + \alpha_m^- = \alpha - \varepsilon.$$

We also take  $g_i = q_i l_1 / q$  and  $h_i = q_i l_2 / q$  for each  $i$ . Then we have

$$\frac{1}{g_i} + \frac{\alpha_i^+}{n} = \frac{1}{q_i} + \frac{\alpha_i}{n} = \frac{1}{h_i} + \frac{\alpha_i^-}{n}$$

and due to the way we choose  $\varepsilon$ , we have  $s' < p_i < g_i < \infty$ ,  $s' < p_i < h_i < \infty$ , and moreover

$$\frac{1}{l_1} = \frac{1}{g_1} + \frac{1}{g_2} + \dots + \frac{1}{g_m}, \quad \frac{1}{l_2} = \frac{1}{h_1} + \frac{1}{h_2} + \dots + \frac{1}{h_m}.$$

Observe that  $1 < l_1 / q < t_i$ , we have

$$\begin{aligned} & \left| \mathcal{Q} \right|^{\frac{s'}{g_i} + \frac{\alpha_i^+ s'}{n} - \frac{s'}{p_i}} \left( \frac{1}{|\mathcal{Q}|} \int_{\mathcal{Q}} u(x)^{\frac{l_1}{q}} dx \right)^{\frac{s'}{g_i}} \left( \frac{1}{|\mathcal{Q}|} \int_{\mathcal{Q}} v_i(x)^{r_i(1-(p_i/s')')} dx \right)^{\frac{1}{r_i(p_i/s')'}} \\ & \leq \left| \mathcal{Q} \right|^{\frac{s'}{q_i} + \frac{\alpha_i^+ s'}{n} - \frac{s'}{p_i}} \left( \frac{1}{|\mathcal{Q}|} \int_{\mathcal{Q}} u(x)^{t_i} dx \right)^{\frac{s'}{t_i q_i}} \left( \frac{1}{|\mathcal{Q}|} \int_{\mathcal{Q}} v_i(x)^{r_i(1-(p_i/s')')} dx \right)^{\frac{1}{r_i(p_i/s')'}}. \end{aligned}$$

Thus, for each  $i$ , the pair of weights  $(u_i^{1/q}, v_i)$  satisfies the  $\mathcal{A}_{p_i/s', g_i/s'}^{\alpha_i^+, r_i, 1}$ -condition whenever  $(u_i^{1/q}, v_i) \in \mathcal{A}_{p_i/s', q_i/s'}^{\alpha_i^+, r_i, t_i}$ . Then, Theorem 4.1 implies that  $M_{\Omega, \alpha + \varepsilon}^{(m)}$  is bounded from  $L_{v_1}^{p_1}(\mathbb{R}^n) \times L_{v_2}^{p_2}(\mathbb{R}^n) \times \dots \times L_{v_m}^{p_m}(\mathbb{R}^n)$  to  $L_{u_i^{1/q}}^{l_1}(\mathbb{R}^n)$ .

On the other hand, we can see  $l_2 / q < 1 < t_i$  and so

$$\begin{aligned} & \left| \mathcal{Q} \right|^{\frac{s'}{h_i} + \frac{\alpha_i^- s'}{n} - \frac{s'}{p_i}} \left( \frac{1}{|\mathcal{Q}|} \int_{\mathcal{Q}} u(x)^{\frac{l_2}{q}} dx \right)^{\frac{s'}{h_i}} \left( \frac{1}{|\mathcal{Q}|} \int_{\mathcal{Q}} v_i(x)^{r_i(1-(p_i/s')')} dx \right)^{\frac{1}{r_i(p_i/s')'}} \\ & \leq \left| \mathcal{Q} \right|^{\frac{s'}{q_i} + \frac{\alpha_i^- s'}{n} - \frac{s'}{p_i}} \left( \frac{1}{|\mathcal{Q}|} \int_{\mathcal{Q}} u(x)^{r_i} dx \right)^{\frac{s'}{t_i q_i}} \left( \frac{1}{|\mathcal{Q}|} \int_{\mathcal{Q}} v_i(x)^{r_i(1-(p_i/s')')} dx \right)^{\frac{1}{r_i(p_i/s')'}}. \end{aligned}$$

In this case, for each  $i$ , the pair of weights  $(u^{l_2/q}, v_i)$  verifies  $\mathcal{A}_{p_i/s', h_i/s'}^{\alpha_i^-, r_i, 1}$ -condition whenever  $(u^{l_2/q}, v_i) \in \mathcal{A}_{p_i/s', q_i/s'}^{\alpha_i^+, r_i, t_i}$ . Then, Theorem 4.1 says that  $M_{\Omega, \alpha - \varepsilon}^{(m)}$  is a bounded operator from  $L_{v_1}^{p_1}(\mathbb{R}^n) \times L_{v_2}^{p_2}(\mathbb{R}^n) \times \dots \times L_{v_m}^{p_m}(\mathbb{R}^n)$  to  $L_{u^{l_2/q}}^{l_1}(\mathbb{R}^n)$ .

Combing the above estimates together, we get

$$\left\| M_{\Omega, \alpha + \varepsilon}^{(m)}(\vec{f}) \right\|_{L_{u^{l_1/q}}^{l_1}(\mathbb{R}^n)}^{1/2} \left\| M_{\Omega, \alpha - \varepsilon}^{(m)}(\vec{f}) \right\|_{L_{u^{l_2/q}}^{l_2}(\mathbb{R}^n)}^{1/2} \leq C \prod_{i=1}^m \|f_i\|_{L_{v_i}^{p_i}(\mathbb{R}^n)}.$$

This and inequality (4.3) yield the desired inequality (4.2). The proof of Theorem 4.2 is complete.  $\square$

Now if we put in Theorem 4.1 and 4.2 that  $\alpha_i = \alpha/m$  for each  $i$ , then we obtain Theorem 1.4 and 1.6.

Finally, we consider the two-weighted weak-type estimate for the operators  $M_{\Omega, \alpha}^{(m)}$ . We recall the definition of the two weight  $(u, v) \in \mathcal{A}_{p, q}^{\alpha, r, t}$  in Definition 1.3, and remark that, in the special case  $p = 1$ ,  $\mathcal{A}_{1, q}^{\alpha, r, t} = \mathcal{A}_{1, q}^{\alpha, 1, t}$  for any  $0 \leq \alpha, r, t < \infty$  and  $1 \leq q < \infty$ , which means that

$$\sup_{Q \subset \mathbb{R}^n} |Q|^{\frac{1}{q} + \frac{\alpha}{n} - 1} \left( \frac{1}{|Q|} \int_Q u(x)^t dx \right)^{\frac{1}{tq}} \leq C v(x) \quad \text{for a.e. } x \in Q.$$

LEMMA 4.3. [4] *Let  $1 \leq p \leq q < \infty$  and  $0 \leq \alpha < n$ . Suppose that  $(u, v)$  be a pair of weights in  $\mathcal{A}_{p, q}^{\alpha, 1, 1}$ . Then for every  $\lambda > 0$ ,*

$$u(\{x \in \mathbb{R}^n : |M_{\alpha} f(x)| > \lambda\}) \leq \frac{C}{\lambda^q} \left( \int_{\mathbb{R}^n} |f(x)|^p v(x) dx \right)^{q/p}.$$

LEMMA 4.4. *Let  $0 < \alpha_i < n$ ,  $1 \leq p_i \leq q_i < \infty$  and  $(u, v_i) \in \mathcal{A}_{p_i, q_i}^{\alpha_i, 1, 1}$  for each  $i = 1, 2, \dots, m$ . Let  $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} + \dots + \frac{1}{q_m}$  and  $\alpha = \alpha_1 + \alpha_2 + \dots + \alpha_m$ . Then for every  $f_i \in L_{v_i}^{p_i}(\mathbb{R}^n)$ , there is a constant  $C > 0$ , independent of  $f_i$ , such that*

$$\left\| M_{\alpha}^{(m)}(\vec{f}) \right\|_{L_u^{q, \infty}(\mathbb{R}^n)} \leq C \prod_{i=1}^m \|f_i\|_{L_{v_i}^{p_i}(\mathbb{R}^n)}. \tag{4.4}$$

*Proof.* For any fixed  $\lambda > 0$ , we denote by

$$\mu_i = \lambda^{q/q_i} \|f_i\|_{L_{v_i}^{p_i}(\mathbb{R}^n)} \left( \prod_{k=1}^m \|f_k\|_{L_{v_k}^{p_k}(\mathbb{R}^n)} \right)^{-q/q_i}, \quad i = 1, 2, \dots, m-1.$$

Then we have

$$\begin{aligned} u\left(\left\{x: \left|M_{\alpha}^{(m)}(\vec{f})(x)\right| > \lambda\right\}\right) &\leq u\left(\left\{x: \left|\prod_{i=1}^m M_{\alpha_i} f_i(x)\right| > \frac{\lambda}{C}\right\}\right) \\ &\leq \sum_{i=1}^{m-1} u\left(\left\{x: |M_{\alpha_i} f_i(x)| > \mu_i\right\}\right) + u\left(\left\{x: |M_{\alpha_m} f_m(x)| > \frac{\lambda}{C} \prod_{k=1}^{m-1} \mu_k^{-1}\right\}\right) \\ &\leq \sum_{i=1}^{m-1} C\left(\mu_i^{-1} \|f_i\|_{L_{v_i}^{p_i}(\mathbb{R}^n)}\right)^{q_i} + C\left(\lambda^{-1} \left[\prod_{i=1}^{m-1} \mu_i\right] \|f_m\|_{L_{v_m}^{p_m}(\mathbb{R}^n)}\right)^{q_m} \\ &\leq C\left(\frac{1}{\lambda} \prod_{k=1}^m \|f_k\|_{L_{v_k}^{p_k}(\mathbb{R}^n)}\right)^q. \end{aligned}$$

This yields the desired inequality (4.4).  $\square$

**THEOREM 4.5.** *Let  $\Omega$  be homogeneous of degree zero on  $\mathbb{R}^{mn}$  with  $\Omega \in L^s(\mathbb{S}^{mn-1})$  for  $s > 1$ . Assume, for each  $i = 1, 2, \dots, m$ , that  $0 < \alpha_i < n$ ,  $s' < p_i < q_i < \infty$  and the weights  $(u, v_i) \in \mathcal{A}_{p_i/s', q_i/s'}^{\alpha_i s', 1, 1}$ . Let  $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} + \dots + \frac{1}{q_m}$  and  $\alpha = \alpha_1 + \alpha_2 + \dots + \alpha_m$ . Then for any  $f_i \in L_{v_i}^{p_i}(\mathbb{R}^n)$  it follows that*

$$\|M_{\Omega, \alpha}^{(m)}(\vec{f})\|_{L_u^{q, \infty}(\mathbb{R}^n)} \leq C \prod_{i=1}^m \|f_i\|_{L_{v_i}^{p_i}(\mathbb{R}^n)}.$$

**THEOREM 4.6.** *Let  $\Omega$  be homogeneous of degree zero on  $\mathbb{R}^{mn}$  with  $\Omega \in L^s(\mathbb{S}^{mn-1})$  for  $s > 1$ . Assume, for each  $i = 1, 2, \dots, m$ , that  $0 < \alpha_i < n$ ,  $s' < p_i < q_i < \infty$  and the weights  $(u, v_i) \in \mathcal{A}_{p_i/s', q_i/s'}^{\alpha_i s', 1, t_i}$  with some  $t_i > 1$ . Let  $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} + \dots + \frac{1}{q_m}$  and  $\alpha = \alpha_1 + \alpha_2 + \dots + \alpha_m$ . Then for any  $f_i \in L_{v_i}^{p_i}(\mathbb{R}^n)$  it follows that*

$$\|I_{\Omega, \alpha}^{(m)}(\vec{f})\|_{L_u^{q, \infty}(\mathbb{R}^n)} \leq C \prod_{i=1}^m \|f_i\|_{L_{v_i}^{p_i}(\mathbb{R}^n)}.$$

Applying Lemma 2.2 and Lemma 4.4, and by the method used in last sections, we can prove the two theorems above, which imply Theorem 1.5 and 1.6 by simply letting  $\alpha_i = \alpha/m$  for each  $i = 1, 2, \dots, m$ . We omit the details of the proof of Theorem 4.5 and 4.6.

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Xiangxing Tao  
 Institute of Applied Mathematics, School of Science  
 Zhejiang University of Science and Technology  
 Hangzhou, Zhejiang 310023  
 P.R. China  
 e-mail: xxtao@zust.edu.cn

Yanlong Shi  
 Department of fundamental Courses  
 Zhejiang Pharmaceutical College  
 Ningbo, 315100  
 P.R. China  
 e-mail: shiyan-long@hotmail.com