

## EMBEDDINGS BETWEEN WEIGHTED LOCAL MORREY-TYPE SPACES AND WEIGHTED LEBESGUE SPACES

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*Abstract.* In this paper, the embeddings between weighted local Morrey-type spaces and weighted Lebesgue spaces are investigated.

### 1. Introduction

Throughout the paper, we always denote by  $c$  and  $C$  a positive constant, which is independent of main parameters but it may vary from line to line. However a constant with subscript or superscript such as  $c_1$  does not change in different occurrences. By  $a \lesssim b$ , ( $b \gtrsim a$ ) we mean that  $a \leq \lambda b$ , where  $\lambda > 0$  depends on inessential parameters. If  $a \lesssim b$  and  $b \lesssim a$ , we write  $a \approx b$  and say that  $a$  and  $b$  are equivalent. We will denote by  $\mathbf{1}$  the function  $\mathbf{1}(x) = 1$ ,  $x \in \mathbb{R}^n$ . For  $x \in \mathbb{R}^n$  and  $r > 0$ , let  $B(x, r)$  be the open ball centered at  $x$  of radius  $r$  and  ${}^cB(x, r) := \mathbb{R}^n \setminus B(x, r)$ .

Let  $A, B$  be some sets and  $\varphi, \psi$  be non-negative functions defined on  $A \times B$  (It may happen that  $\varphi(\alpha, \beta) = \infty$  or  $\psi(\alpha, \beta) = \infty$  for some  $\alpha \in A$ ,  $\beta \in B$ ). We say that  $\varphi$  is dominated by  $\psi$  (or  $\psi$  dominates  $\varphi$ ) on  $A \times B$  uniformly in  $\alpha \in A$  and write

$$\varphi(\alpha, \beta) \lesssim \psi(\alpha, \beta) \quad \text{uniformly in} \quad \alpha \in A$$

or

$$\psi(\alpha, \beta) \gtrsim \varphi(\alpha, \beta) \quad \text{uniformly in} \quad \alpha \in A,$$

if for each  $\beta \in B$  there exists  $C(\beta) > 0$  such that

$$\varphi(\alpha, \beta) \leq C(\beta)\psi(\alpha, \beta)$$

for all  $\alpha \in A$ . We also say that  $\varphi$  is equivalent to  $\psi$  on  $A \times B$  uniformly in  $\alpha \in A$  and write

$$\varphi(\alpha, \beta) \approx \psi(\alpha, \beta) \quad \text{uniformly in} \quad \alpha \in A,$$

if  $\varphi$  and  $\psi$  dominate each other on  $A \times B$  uniformly in  $\alpha \in A$  (see, for instance, [3]).

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Given two quasi-normed vector spaces  $X$  and  $Y$ , we write  $X = Y$  if  $X$  and  $Y$  are equal in the algebraic and the topological sense (their quasi-norms are equivalent). The symbol  $X \hookrightarrow Y$  ( $Y \hookleftarrow X$ ) means that  $X \subset Y$  and the natural embedding  $I$  of  $X$  in  $Y$  is continuous, that is, there exist a constant  $c > 0$  such that  $\|z\|_Y \leq c\|z\|_X$  for all  $z \in X$ . The best constant of the embedding  $X \hookrightarrow Y$  is  $\|I\|_{X \rightarrow Y}$ .

Let  $A$  be any measurable subset of  $\mathbb{R}^n$ ,  $n \geq 1$ . By  $\mathfrak{M}(A)$  we denote the set of all measurable functions on  $A$ . The symbol  $\mathfrak{M}^+(A)$  stands for the collection of all  $f \in \mathfrak{M}(A)$  which are non-negative on  $A$ . The family of all weight functions (also called just weights) on  $A$ , that is, measurable, positive and finite a.e. on  $A$ , is given by  $W(A)$ .

For  $p \in (0, \infty]$  and  $w \in \mathfrak{M}^+(A)$ , we define the functional  $\|\cdot\|_{p,w,A}$  on  $\mathfrak{M}(A)$  by

$$\|f\|_{p,w,A} := \begin{cases} (\int_A |f(x)|^p w(x) dx)^{1/p} & \text{if } p < \infty, \\ \text{ess sup}_A |f(x)|w(x) & \text{if } p = \infty. \end{cases}$$

If, in addition,  $w \in W(A)$ , then the weighted Lebesgue space  $L_p(A, w)$  is given by

$$L_{p,w}(A) \equiv L_p(A, w) := \{f \in \mathfrak{M}(A) : \|f\|_{p,w,A} < \infty\}$$

and it is equipped with the quasi-norm  $\|\cdot\|_{p,w,A}$ .

When  $w \equiv 1$  on  $A$ , we write simply  $L_p(A)$  and  $\|\cdot\|_{p,A}$  instead of  $L_p(A, w)$  and  $\|\cdot\|_{p,w,A}$ , respectively.

We adopt the following usual conventions.

CONVENTION 1.1. (i) Throughout the paper we put  $0/0 = 0$ ,  $0 \cdot (\pm\infty) = 0$  and  $1/(\pm\infty) = 0$ .

(ii) We put

$$p' := \begin{cases} \frac{p}{1-p} & \text{if } 0 < p < 1, \\ +\infty & \text{if } p = 1, \\ \frac{p}{p-1} & \text{if } 1 < p < +\infty, \\ 1 & \text{if } p = +\infty. \end{cases}$$

(iii) If  $I = (a, b) \subseteq \mathbb{R}$  and  $g$  is a monotone function on  $I$ , then by  $g(a)$  and  $g(b)$  we mean the limits  $\lim_{x \rightarrow a+} g(x)$  and  $\lim_{x \rightarrow b-} g(x)$ , respectively.

Local Morrey-type spaces were widely investigated during last decades. The research includes the study of classical operators of Harmonic and Real Analysis – maximal, singular and potential operators – in these spaces (see, for instance, [18]–[20] and [1]–[11]), and the investigation of the functional-analytic properties of local Morrey-type spaces and relation of these spaces with other known function spaces (see, for instance, [12], [15] and [17]). We refer the reader to the surveys [1] and [2] for a comprehensive discussion of the history of local Morrey-type spaces.

The weighted local Morrey-type spaces and weighted complementary local Morrey-type spaces are defined as follows.

DEFINITION 1.2. Let  $0 < p, \theta \leq \infty$ ,  $\omega \in \mathfrak{M}^+(0, \infty)$  and  $v \in W(\mathbb{R}^n)$ . We denote by  $LM_{p\theta, \omega}(\mathbb{R}^n, v)$  the weighted local Morrey-type space, the set of all  $f \in L_{p, v}^{\text{loc}}(\mathbb{R}^n)$  with

$$\|f\|_{LM_{p\theta, \omega}(\mathbb{R}^n, v)} := \|\omega(r)\|f\|_{p, v, B(0, r)}\|_{\theta, (0, \infty)} < \infty.$$

DEFINITION 1.3. Let  $0 < p, \theta \leq \infty$ ,  $\omega \in \mathfrak{M}^+(0, \infty)$  and  $v \in W(\mathbb{R}^n)$ . We denote by  ${}^cLM_{p\theta, \omega}(\mathbb{R}^n, v)$  the weighted complementary local Morrey-type space, the set of all functions such that  $f \in L_{p, v}({}^cB(0, t))$  for all  $t > 0$  with

$$\|f\|_{{}^cLM_{p\theta, \omega}(\mathbb{R}^n, v)} := \|\omega(r)\|f\|_{p, v, {}^cB(0, r)}\|_{\theta, (0, \infty)} < \infty.$$

REMARK 1.4. In [5] and [7] it were proved that the spaces  $LM_{p\theta, \omega}(\mathbb{R}^n) := LM_{p\theta, \omega}(\mathbb{R}^n, \mathbf{1})$  and  ${}^cLM_{p\theta, \omega}(\mathbb{R}^n) := {}^cLM_{p\theta, \omega}(\mathbb{R}^n, \mathbf{1})$  are non-trivial, i.e. consists not only of functions equivalent to 0 on  $\mathbb{R}^n$ , if and only if

$$\|\omega\|_{\theta, (t, \infty)} < \infty, \quad \text{for some } t > 0, \quad (1.1)$$

and

$$\|\omega\|_{\theta, (0, t)} < \infty, \quad \text{for some } t > 0, \quad (1.2)$$

respectively. The same conclusion is true for  $LM_{p\theta, \omega}(\mathbb{R}^n, v)$  and  ${}^cLM_{p\theta, \omega}(\mathbb{R}^n, v)$  for any  $v \in W(\mathbb{R}^n)$ .

DEFINITION 1.5. Let  $0 < p, \theta \leq \infty$ . We denote by  $\Omega_\theta$  the set all functions  $\omega \in \mathfrak{M}^+(0, \infty)$  such that

$$0 < \|\omega\|_{\theta, (t, \infty)} < \infty, \quad t > 0,$$

and by  ${}^c\Omega_\theta$  the set all functions  $\omega \in \mathfrak{M}^+(0, \infty)$  such that

$$0 < \|\omega\|_{\theta, (0, t)} < \infty, \quad t > 0.$$

Let  $v \in W(\mathbb{R}^n)$ . It is easy to see that  $LM_{p\theta, \omega}(\mathbb{R}^n, v)$  and  ${}^cLM_{p\theta, \omega}(\mathbb{R}^n, v)$  are quasi-normed vector spaces when  $\omega \in \Omega_\theta$  and  $\omega \in {}^c\Omega_\theta$ , respectively.

Let  $f \in L_1^{\text{loc}}(\mathbb{R}^n)$ . The maximal operator  $M$  is defined for all  $x \in \mathbb{R}^n$  by

$$Mf(x) := \sup_{t>0} \frac{1}{|B(x, t)|} \int_{B(x, t)} |f(y)| dy,$$

where  $|B(x, t)|$  is the Lebesgue measure of the ball  $B(x, t)$ .

The boundedness of the maximal operator from  $LM_{p\theta_1, \omega_1}(\mathbb{R}^n)$  to  $LM_{p\theta_2, \omega_2}(\mathbb{R}^n)$  for general  $\omega_1$  and  $\omega_2$  was studied in [4]–[6], [9] and [3]. In [4]–[6] and [9], for a certain range of the parameters  $p$ ,  $\theta_1$  and  $\theta_2$ , necessary and sufficient conditions on  $\omega_1$  and  $\omega_2$  were obtained ensuring the boundedness of  $M$  from  $LM_{p\theta_1, \omega_1}(\mathbb{R}^n)$  to  $LM_{p\theta_2, \omega_2}(\mathbb{R}^n)$ , namely the following statement was proved.

THEOREM 1.6. *If  $n \in \mathbb{N}$ ,  $1 < p < \infty$ ,  $0 < \theta_1 \leq \theta_2 \leq \infty$ ,  $\omega_1 \in \Omega_{\theta_1}$ , and  $\omega_2 \in \Omega_{\theta_2}$ , then the condition*

$$\left\| \omega_2(r) \left( \frac{r}{t+r} \right)^{n/p} \right\|_{\theta_2, (0, \infty)} \lesssim \|\omega_1\|_{\theta_1, (t, \infty)} \tag{1.3}$$

*uniformly in  $t \in (0, \infty)$  is necessary and sufficient for the boundedness of  $M$  from  $LM_{p\theta_1, \omega_1}(\mathbb{R}^n)$  to  $LM_{p\theta_2, \omega_2}(\mathbb{R}^n)$ . Moreover,*

$$\|M\|_{LM_{p\theta_1, \omega_1}(\mathbb{R}^n) \rightarrow LM_{p\theta_2, \omega_2}(\mathbb{R}^n)} \approx \sup_{t \in (0, \infty)} \|\omega_1\|_{\theta_1, (t, \infty)}^{-1} \left\| \omega_2(r) \left( \frac{r}{t+r} \right)^{n/p} \right\|_{\theta_2, (0, \infty)}$$

*uniformly in  $\omega_1 \in \Omega_{\theta_1}$  and  $\omega_2 \in \Omega_{\theta_2}$ .*

Note that in [4]–[6] this was proved under the additional assumption  $\theta_1 \leq p$ .

If  $\theta_2 < \theta_1$ , then sufficient conditions on  $\omega_1$  and  $\omega_2$  for the boundedness of  $M$  from  $LM_{p\theta_1, \omega_1}(\mathbb{R}^n)$  to  $LM_{p\theta_2, \omega_2}(\mathbb{R}^n)$  are given in [9]. However, the problem of finding necessary and sufficient condition on  $\omega_1$  and  $\omega_2$  ensuring the boundedness of  $M$  from  $LM_{p\theta_1, \omega_1}(\mathbb{R}^n)$  to  $LM_{p\theta_2, \omega_2}(\mathbb{R}^n)$  for the case  $\theta_2 < \theta_1$  is still open. In [3] the solution of this problem is given for very particular case in which  $\theta_1 = \infty$  and  $\omega_1(r) \equiv 1$ . In other words, for all admissible values of the parameters  $p_1$ ,  $p_2$  and  $\theta$  authors find necessary and sufficient conditions on  $\omega$  ensuring the boundedness of the maximal operator from  $L_{p_1}(\mathbb{R}^n) = LM_{p_1, \omega_1, 1}(\mathbb{R}^n)$  to  $LM_{p_2, \theta, \omega}(\mathbb{R}^n)$ .

THEOREM 1.7. ([3], see also [1]) *Let  $n \in \mathbb{N}$ ,  $0 < p_2 \leq p_1 \leq \infty$ ,  $0 < \theta \leq \infty$ , and  $\omega \in \Omega_\theta$ .*

1. *If  $1 < p_2 = p_1$ ,  $0 < \theta \leq \infty$  or  $0 < p_2 < p_1$ ,  $1 < p_1$ ,  $\theta = \infty$ , then*

$$\|M\|_{L_{p_1}(\mathbb{R}^n) \rightarrow LM_{p_2, \theta, \omega}(\mathbb{R}^n)} \approx \left\| r^{n(1/p_2 - 1/p_1)} \omega(r) \right\|_{\theta, (0, \infty)} \tag{1.4}$$

*uniformly in  $\omega \in \Omega_\theta$ .*

*In particular, if  $1 < p \leq \infty$ ,  $0 < \theta \leq \infty$ . then*

$$\|M\|_{L_p(\mathbb{R}^n) \rightarrow LM_{p, \theta, \omega}(\mathbb{R}^n)} \approx \|\omega\|_{\theta, (0, \infty)}$$

*uniformly in  $\omega \in \Omega_\theta$ .*

2. *If  $0 < p_2 < p_1$ ,  $1 < p_1$  and  $\theta < \infty$ , then*

$$\begin{aligned} \|M\|_{L_{p_1}(\mathbb{R}^n) \rightarrow LM_{p_2, \theta, \omega}(\mathbb{R}^n)} &\approx \left\| t^{n(1/p_2 - 1/p_1) - 1/s} \|\omega\|_{\theta, (t, \infty)} \right\|_{s, (0, \infty)} \\ &\approx \left\| t^{n(1/p_2 - 1/p_1) - 1/s} \left\| \left( \frac{r}{r+t} \right)^{n/p_2} \omega(r) \right\|_{\theta, (0, \infty)} \right\|_{s, (0, \infty)} \end{aligned} \tag{1.5}$$

*uniformly in  $\omega \in \Omega_\theta$ , where*

$$s = \begin{cases} \frac{p_1 \theta}{p_1 - \theta} & \text{if } \theta < p_1, \\ \infty & \text{if } \theta \geq p_1. \end{cases} \tag{1.6}$$

The idea used in [3] is mainly based on the following theorems.

**THEOREM 1.8.** ([3], see also [1]) *Let  $n \in \mathbb{N}$ ,  $0 < p_2 \leq p_1 \leq \infty$ ,  $0 < \theta \leq \infty$ , and  $\omega \in \Omega_\theta$ .*

1. *If  $p_2 = p_1$ ,  $0 < \theta \leq \infty$  or  $0 < p_2 < p_1$ ,  $\theta = \infty$ , then*

$$\|I\|_{L_{p_1}(\mathbb{R}^n) \rightarrow LM_{p_2\theta,\omega}(\mathbb{R}^n)} \approx \left\| r^{n(1/p_2-1/p_1)} \omega(r) \right\|_{\theta,(0,\infty)} \tag{1.7}$$

*uniformly in  $\omega \in \Omega_\theta$ .*

2. *If  $0 < p_2 < p_1$  and  $\theta < \infty$ , then*

$$\|I\|_{L_{p_1}(\mathbb{R}^n) \rightarrow LM_{p_2\theta,\omega}(\mathbb{R}^n)} \approx \left\| t^{n(1/p_2-1/p_1)-1/s} \|\omega\|_{\theta,(t,\infty)} \right\|_{s,(0,\infty)} \tag{1.8}$$

*uniformly in  $\omega \in \Omega_\theta$ , where  $s$  is defined by (1.6).*

**THEOREM 1.9.** *Let  $X, Y$  be a quasi-normed vector spaces of measurable functions on  $\mathbb{R}^n$  and let  $M$  is bounded on  $X$ . Moreover, assume that  $Y$  satisfies the monotonicity property, that is,*

$$0 \leq g \leq f \quad \Rightarrow \quad \|g\|_Y \lesssim \|f\|_Y.$$

*Then  $M$  is bounded from  $X$  to  $Y$  if and only if  $X \hookrightarrow Y$ , and*

$$\|M\|_{X \rightarrow Y} \approx \|I\|_{X \rightarrow Y}.$$

Note that in [3], Theorem 1.9 was proved for  $X = L_{p_1}(\mathbb{R}^n)$  and  $Y = LM_{p_2\theta,\omega}(\mathbb{R}^n)$ . The proof of Theorem 1.9 is straightforward and we omit it.

The aim of this paper is to characterize the embeddings between weighted local Morrey-type spaces and weighted Lebesgue spaces, that is, the embeddings

$$L_{p_1}(\mathbb{R}^n, v_1) \hookrightarrow LM_{p_2\theta,\omega}(\mathbb{R}^n, v_2), \tag{1.9}$$

$$L_{p_1}(\mathbb{R}^n, v_1) \hookrightarrow {}^cLM_{p_2\theta,\omega}(\mathbb{R}^n, v_2), \tag{1.10}$$

$$L_{p_1}(\mathbb{R}^n, v_1) \hookrightarrow LM_{p_2\theta,\omega}(\mathbb{R}^n, v_2), \tag{1.11}$$

$$L_{p_1}(\mathbb{R}^n, v_1) \hookrightarrow {}^cLM_{p_2\theta,\omega}(\mathbb{R}^n, v_2). \tag{1.12}$$

The method of investigation is based on using the characterizations of the direct and reverse multidimensional Hardy inequalities.

Our main results are Theorems 3.1, 3.2, 4.1 and 4.2. Note that Theorem 3.1 is a generalization of Theorem 1.8 to the weighted case. Theorems 3.2, 4.1 and 4.2 are characterizations of embeddings (1.10), (1.11) and (1.12), respectively. Using Theorems 3.1 and 3.2 we are able to calculate the norms  $\|M\|_{L_{p_1}(\mathbb{R}^n, v_1) \rightarrow LM_{p_2\theta,\omega}(\mathbb{R}^n, v_2)}$  and  $\|M\|_{L_{p_1}(\mathbb{R}^n, v_1) \rightarrow {}^cLM_{p_2\theta,\omega}(\mathbb{R}^n, v_2)}$ , when  $v_1$  is a weight function from the Muckenhoupt class  $A_{p_1}$ ,  $1 < p_1 < \infty$  (see Corollaries 3.3 and 3.4). Theorems 4.1 and 4.2 make it possible to describe the associate spaces of  $LM_{p\theta,\omega}(\mathbb{R}^n, v)$  and  ${}^cLM_{p\theta,\omega}(\mathbb{R}^n, v)$  (see Theorems 4.4 and 4.5).

The paper is organized as follows. Section 2 contains some preliminaries along with the standard ingredients used in the proofs. In Sections 3 and 4 we give the characterizations of the embeddings (1.9), (1.10) and (1.11), (1.12), respectively.

### 2. Some Hardy-type inequalities

In [13], M. Christ and L. Grafakos showed that the  $n$ -dimensional Hardy inequality

$$\int_{\mathbb{R}^n} \left( \frac{1}{|B(0, |x|)|} \int_{B(0, |x|)} f(y) dy \right)^p dx \leq \left( \frac{p}{p-1} \right)^p \int_{\mathbb{R}^n} f(x)^p dx, \quad 1 < p < \infty,$$

holds for all  $f \in \mathfrak{M}^+(\mathbb{R}^n)$ , the constant  $(\frac{p}{p-1})^p$  being again the best possible. In [14], P. Drábek, H. P. Heining and A. Kufner extended this Hardy inequality to general  $n$ -dimensional weights  $u, w$  and to the whole range of the parameters  $p, q, 1 < p < \infty, 0 < q < \infty$ . The necessary and sufficient conditions for the validity of the inequality

$$\left( \int_{\mathbb{R}^n} \left( \int_{B(0, |x|)} f(y) dy \right)^q u(x) dx \right)^{1/q} \leq C \left( \int_{\mathbb{R}^n} f(x)^p w(x) dx \right)^{1/p} \tag{2.1}$$

for all  $f \in \mathfrak{M}^+(\mathbb{R}^n)$  are exactly the analogous of the corresponding conditions for dimension one. It is easy to see that if  $u$  is a function on  $\mathbb{R}^n$  such that  $u(x) = v(|x|)/|x|^{n-1}, x \in \mathbb{R}^n$ , for some function  $v \in \mathfrak{M}^+(0, \infty)$ , then (2.1) is equivalent to the following inequality

$$\left( \int_0^\infty \left( \int_{B(0, t)} f(y) dy \right)^q v(t) dt \right)^{1/q} \leq C \left( \int_{\mathbb{R}^n} f(x)^p w(x) dx \right)^{1/p}. \tag{2.2}$$

According to the above remark, we can formulate the following theorems.

**THEOREM 2.1.** *Let  $1 \leq p \leq \infty, 0 < q \leq \infty, v \in \mathfrak{M}^+(0, \infty)$  and  $w \in \mathfrak{M}^+(\mathbb{R}^n)$ . Denote by*

$$(Hf)(t) := \int_{B(0, t)} f(x) dx, \quad f \in \mathfrak{M}^+(\mathbb{R}^n), \quad t \geq 0.$$

*Then the inequality*

$$\|Hf\|_{q, v, (0, \infty)} \leq c \|f\|_{p, w, \mathbb{R}^n} \tag{2.3}$$

*holds for all  $f \in \mathfrak{M}^+(\mathbb{R}^n)$  if and only if  $A(p, q) < \infty$ , and the best constant in (2.3), that is,*

$$B(p, q) := \sup_{f \in \mathfrak{M}^+(\mathbb{R}^n)} \|Hf\|_{q, v, (0, \infty)} / \|f\|_{p, w, \mathbb{R}^n}$$

*satisfies  $B(p, q) \approx A(p, q)$ , where*

(a) *for  $1 < p \leq q < \infty$ ,*

$$A(p, q) := \sup_{t > 0} \left( \int_t^\infty v(s) ds \right)^{1/q} \left( \int_{B(0, t)} w(x)^{1-p'} dx \right)^{1/p'};$$

(b) *for  $1 < p < \infty, 0 < q < p$  and  $1/r = 1/q - 1/p$ ,*

$$A(p, q) := \left( \int_0^\infty \left( \int_t^\infty v(s) ds \right)^{r/p} v(t) \left( \int_{B(0, t)} w(x)^{1-p'} dx \right)^{r/p'} dt \right)^{1/r};$$

(c) for  $1 < p < \infty$ ,  $q = \infty$ ,

$$A(p, q) := \sup_{t > 0} \left( \operatorname{ess\,sup}_{t < s < \infty} v(s) \right) \left( \int_{B(0, t)} w(x)^{1-p'} dx \right)^{1/p'};$$

(d) for  $p = q = \infty$ ,

$$A(p, q) := \sup_{t > 0} \left( \operatorname{ess\,sup}_{t < s < \infty} v(s) \right) \int_{B(0, t)} \frac{dx}{w(x)};$$

(e) for  $p = \infty$ ,  $0 < q < \infty$ ,

$$A(p, q) := \left( \int_0^\infty v(t) \left( \int_{B(0, t)} \frac{dx}{w(x)} \right)^q dt \right)^{1/q};$$

(f) for  $p = 1$ ,  $1 \leq q < \infty$ ,

$$A(p, q) := \sup_{t \in (0, \infty)} \left( \int_t^\infty v(s) ds \right)^{1/q} \operatorname{ess\,sup}_{x \in B(0, t)} w(x)^{-1};$$

(g) for  $p = 1$ ,  $0 < q < 1$ ,

$$A(p, q) := \left( \int_0^\infty \left( \int_t^\infty v(s) ds \right)^{q'} v(t) \left( \operatorname{ess\,sup}_{x \in B(0, t)} w(x)^{-1} \right)^{q'} dt \right)^{1/q'};$$

(h) for  $p = 1$ ,  $q = \infty$ ,

$$A(p, q) := \sup_{t \in (0, \infty)} \left( \operatorname{ess\,sup}_{t < s < \infty} v(s) \right) \left( \operatorname{ess\,sup}_{x \in B(0, t)} w(x)^{-1} \right).$$

**THEOREM 2.2.** Let  $1 \leq p \leq \infty$ ,  $0 < q \leq \infty$ ,  $v \in \mathfrak{M}^+(0, \infty)$  and  $w \in \mathfrak{M}^+(\mathbb{R}^n)$ . Denote by

$$(H^*f)(t) := \int_{\varepsilon_B(0, t)} f(x) dx, \quad f \in \mathfrak{M}^+(\mathbb{R}^n), \quad t \geq 0.$$

Then the inequality

$$\|H^*f\|_{q, v, (0, \infty)} \leq c \|f\|_{p, w, \mathbb{R}^n} \tag{2.4}$$

holds for all  $f \in \mathfrak{M}^+(\mathbb{R}^n)$  if and only if  $A^*(p, q) < \infty$ , and the best constant in (2.4), that is,

$$B^*(p, q) := \sup_{f \in \mathfrak{M}^+(\mathbb{R}^n)} \|H^*f\|_{q, v, (0, \infty)} / \|f\|_{p, w, \mathbb{R}^n}$$

satisfies  $B^*(p, q) \approx A^*(p, q)$ . Here

(a) for  $1 < p \leq q < \infty$ ,

$$A^*(p, q) := \sup_{t>0} \left( \int_0^t v(s) ds \right)^{1/q} \left( \int_{c_{B(0,t)}} w(x)^{1-p'} dx \right)^{1/p'} ;$$

(b) for  $1 < p < \infty$ ,  $0 < q < p$  and  $1/r = 1/q - 1/p$ ,

$$A^*(p, q) := \left( \int_0^\infty \left( \int_0^t v(s) ds \right)^{r/p} v(t) \left( \int_{c_{B(0,t)}} w(x)^{1-p'} dx \right)^{r/p'} dt \right)^{1/r} ;$$

(c) for  $1 < p < \infty$ ,  $q = \infty$ ,

$$A^*(p, q) := \sup_{t>0} \left( \operatorname{ess\,sup}_{0<s<t} v(s) \right) \left( \int_{c_{B(0,t)}} w(x)^{1-p'} dx \right)^{1/p'} ;$$

(d) for  $p = q = \infty$ ,

$$A^*(p, q) := \sup_{t>0} \left( \operatorname{ess\,sup}_{0<s<t} v(s) \right) \int_{c_{B(0,t)}} \frac{dx}{w(x)} ;$$

(e) for  $p = \infty$ ,  $0 < q < \infty$ ,

$$A^*(p, q) := \left( \int_0^\infty v(t) \left( \int_{c_{B(0,t)}} \frac{dx}{w(x)} \right)^q dt \right)^{1/q} ;$$

(f) for  $p = 1$ ,  $1 \leq q < \infty$ ,

$$A^*(p, q) := \sup_{t \in (0, \infty)} \left( \int_0^t v(s) ds \right)^{1/q} \operatorname{ess\,sup}_{x \in c_{B(0,t)}} w(x)^{-1} ;$$

(g) for  $p = 1$ ,  $0 < q < 1$ ,

$$A^*(p, q) := \left( \int_0^\infty \left( \int_0^t v(s) ds \right)^{q'} v(t) \left( \operatorname{ess\,sup}_{x \in c_{B(0,t)}} w(x)^{-1} \right)^{q'} dt \right)^{1/q'} ;$$

(h) for  $p = 1$ ,  $q = \infty$ ,

$$A^*(p, q) := \sup_{t \in (0, \infty)} \left( \operatorname{ess\,sup}_{0<s<t} v(s) \right) \left( \operatorname{ess\,sup}_{x \in c_{B(0,t)}} w(x)^{-1} \right).$$

**THEOREM 2.3.** Let  $0 < q \leq \infty$ ,  $v \in \mathfrak{M}^+(0, \infty)$  and  $w \in \mathfrak{M}^+(\mathbb{R}^n)$ . Denote by

$$(Sf)(t) := \operatorname{ess\,sup}_{x \in B(0,t)} f(x), \quad f \in \mathfrak{M}^+(\mathbb{R}^n), \quad t \geq 0.$$

Then the inequality

$$\|(Sf)v\|_{q,(0,\infty)} \leq c \|fw\|_{\infty,\mathbb{R}^n}$$

holds for all  $f \in \mathfrak{M}^+(\mathbb{R}^n)$  if and only if

$$\left\| v(r) \left( \operatorname{ess\,sup}_{x \in B(0,r)} w(x)^{-1} \right) \right\|_{q,(0,\infty)} < \infty,$$

and

$$\sup_{f \in \mathfrak{M}^+(\mathbb{R}^n)} \|(Sf)v\|_{q,(0,\infty)} / \|fw\|_{\infty,\mathbb{R}^n} \approx \left\| v(r) \left( \operatorname{ess\,sup}_{x \in B(0,r)} w(x)^{-1} \right) \right\|_{q,(0,\infty)}.$$

**THEOREM 2.4.** Let  $0 < q \leq \infty$ ,  $v \in \mathfrak{M}^+(0, \infty)$  and  $w \in \mathfrak{M}^+(\mathbb{R}^n)$ . Denote by

$$(S^*f)(t) := \operatorname{ess\,sup}_{x \in {}^cB(0,t)} f(x), \quad f \in \mathfrak{M}^+(\mathbb{R}^n), \quad t \geq 0.$$

Then the inequality

$$\|(S^*f)v\|_{q,(0,\infty)} \leq c \|fw\|_{\infty,\mathbb{R}^n}$$

holds for all  $f \in \mathfrak{M}^+(\mathbb{R}^n)$  if and only if

$$\left\| v(r) \left( \operatorname{ess\,sup}_{x \in {}^cB(0,r)} w(x)^{-1} \right) \right\|_{q,(0,\infty)} < \infty,$$

and

$$\sup_{f \in \mathfrak{M}^+(\mathbb{R}^n)} \|(S^*f)v\|_{q,(0,\infty)} / \|fw\|_{\infty,\mathbb{R}^n} \approx \left\| v(r) \left( \operatorname{ess\,sup}_{x \in {}^cB(0,r)} w(x)^{-1} \right) \right\|_{q,(0,\infty)}.$$

For the convenience of the reader we repeat the relevant material from [16] without proofs, thus making our exposition self-contained.

Let  $\varphi$  be non-decreasing and finite function on the interval  $I := (a, b) \subseteq \mathbb{R}$ . We assign to  $\varphi$  the function  $\lambda$  defined on subintervals of  $I$  by

$$\begin{aligned} \lambda([y, z]) &= \varphi(z+) - \varphi(y-), \\ \lambda([y, z]) &= \varphi(z-) - \varphi(y-), \\ \lambda((y, z]) &= \varphi(z+) - \varphi(y+), \\ \lambda((y, z]) &= \varphi(z-) - \varphi(y+). \end{aligned} \tag{2.5}$$

$\lambda$  is a non-negative, additive and regular function of intervals. Thus (cf. [24], Chapter 10), it admits a unique extension to a non-negative Borel measure  $\lambda$  on  $I$ .

Note also that the associated Borel measure can be determined, e.g., only by putting

$$\lambda([y, z]) = \varphi(z+) - \varphi(y-) \quad \text{for any } [y, z] \subset I$$

(since the Borel subsets of  $I$  can be generated by subintervals  $[y, z] \subset I$ ).

If  $J \subseteq I$ , then the Lebesgue-Stieltjes integral  $\int_J f d\varphi$  is defined as  $\int_J f d\lambda$ . We shall also use the Lebesgue-Stieltjes integral  $\int_J f d\varphi$  when  $\varphi$  is a non-increasing and finite on the interval  $I$ . In such a case we put

$$\int_J f d\varphi := - \int_J f d(-\varphi).$$

We adopt the following conventions.

CONVENTION 2.5. Let  $I = (a, b) \subseteq \mathbb{R}$ ,  $f : I \rightarrow [0, \infty]$  and  $h : I \rightarrow [0, \infty]$ . Assume that  $h$  is non-decreasing and left-continuous on  $I$ . If  $h : I \rightarrow [0, \infty]$ , then the symbol  $\int_I f dh$  means the usual Lebesgue-Stieltjes integral (with the measure  $\lambda$  associated to  $h$  is given by  $\lambda([\alpha, \beta]) = h(\beta) - h(\alpha)$  if  $[\alpha, \beta] \subset (a, b)$  - cf. (2.5)). However, if  $h = \infty$  on some subinterval  $(c, b)$  with  $c \in I$ , then we define  $\int_I f dh$  only if  $f = 0$  on  $[c, b)$  and we put

$$\int_I f dh = \int_{(a,c)} f dh.$$

CONVENTION 2.6. Let  $I = (a, b) \subseteq \mathbb{R}$ ,  $f : I \rightarrow [0, +\infty]$  and  $h : I \rightarrow [-\infty, 0]$ . Assume that  $h$  is non-decreasing and right-continuous on  $I$ . If  $h : I \rightarrow (-\infty, 0]$ , then the symbol  $\int_I f dh$  means the usual Lebesgue-Stieltjes integral. However, if  $h = -\infty$  on some subinterval  $(a, c)$  with  $c \in I$ , then we define  $\int_I f dh$  only if  $f = 0$  on  $(a, c]$  and we put

$$\int_I f dh = \int_{(c,b)} f dh.$$

THEOREM 2.7. Let  $w \in \mathfrak{M}^+(\mathbb{R}^n)$  and  $u \in \mathfrak{M}^+(0, \infty)$  be such that  $\|u\|_{q,(t,\infty)} < \infty$  for all  $t \in (0, \infty)$ .

(a) Assume that  $0 < q \leq p \leq 1$ . Then

$$\|gw\|_{p,\mathbb{R}^n} \leq c \|(Hg)u\|_{q,(0,\infty)} \tag{2.6}$$

holds for all  $g \in \mathfrak{M}^+(\mathbb{R}^n)$  if and only if

$$C(p, q) := \sup_{t \in (0, \infty)} \|w\|_{p', c_{B(0,t)}} \|u\|_{q,(t,\infty)}^{-1} < \infty. \tag{2.7}$$

The best possible constant in (2.6), that is,

$$D(p, q) := \sup_{g \in \mathfrak{M}^+(\mathbb{R}^n)} \|gw\|_{p,\mathbb{R}^n} / \|(Hg)u\|_{q,(0,\infty)}$$

satisfies  $D(p, q) \approx C(p, q)$ .

(b) Let  $0 < p \leq 1$ ,  $p < q \leq \infty$  and  $1/r = 1/p - 1/q$ . Then (2.6) holds if and only if

$$C(p, q) := \left( \int_{(0,\infty)} \|w\|_{p', c_{B(0,t)}}^r d \left( \|u\|_{q,(t-\infty)}^{-r} \right) \right)^{1/r} + \frac{\|w\|_{p', \mathbb{R}^n}}{\|u\|_{q,(0,\infty)}} < \infty,$$

and  $D(p, q) \approx C(p, q)$ , where

$$\|u\|_{q,(t-\infty)} := \lim_{s \rightarrow t-} \|u\|_{q,(s,\infty)}, \quad t \in (0, \infty).$$

**THEOREM 2.8.** *Let  $w \in \mathfrak{M}^+(\mathbb{R}^n)$  and  $u \in \mathfrak{M}^+(0, \infty)$  be such that  $\|u\|_{q,(0,t)} < \infty$  for all  $t \in (0, \infty)$ .*

(a) *Assume that  $0 < q \leq p \leq 1$ . Then*

$$\|gw\|_{p,\mathbb{R}^n} \leq c \|(H^*g)u\|_{q,(0,\infty)} \tag{2.8}$$

*holds for all  $g \in \mathfrak{M}^+(\mathbb{R}^n)$  if and only if*

$$C^*(p, q) := \sup_{t \in (0, \infty)} \|w\|_{p',B(0,t)} \|u\|_{q,(0,t)}^{-1} < \infty. \tag{2.9}$$

*The best possible constant in (2.8), that is,*

$$D^*(p, q) := \sup_{g \in \mathfrak{M}^+(\mathbb{R}^n)} \|gw\|_{p,\mathbb{R}^n} / \|(H^*g)u\|_{q,(0,\infty)}$$

*satisfies  $D^*(p, q) \approx C^*(p, q)$ .*

(b) *Let  $0 < p \leq 1$ ,  $p < q \leq \infty$  and  $1/r = 1/p - 1/q$ . Then (2.8) holds if and only if*

$$C^*(p, q) := \left( \int_{(0, \infty)} \|w\|_{p',B(0,t)}^r d\left(-\|u\|_{q,(0,t+)}^{-r}\right) \right)^{1/r} + \frac{\|w\|_{p',\mathbb{R}^n}}{\|u\|_{q,(0,\infty)}} < \infty,$$

*and  $D^*(p, q) \approx C^*(p, q)$ , where*

$$\|u\|_{q,(0,t+)} := \lim_{s \rightarrow t+} \|u\|_{q,(0,s)}, \quad t \in (0, \infty).$$

**REMARK 2.9.** Let  $q < \infty$  in Theorems 2.7 and 2.8. Then

$$\|u\|_{q,(t-\infty)} = \|u\|_{q,(t,\infty)} \quad \text{and} \quad \|u\|_{q,(0,t+)} = \|u\|_{q,(0,t)} \quad \text{for all } t \in (0, \infty),$$

which implies that

$$C(p, q) = \left( \int_{(a,b)} \|w\|_{p',c_B(0,t)}^r d\left(\|u\|_{q,(t,\infty)}^{-r}\right) \right)^{1/r} + \frac{\|w\|_{p',\mathbb{R}^n}}{\|u\|_{q,(0,\infty)}},$$

and

$$C^*(p, q) = \left( \int_{(0,\infty)} \|w\|_{p',B(0,t)}^r d\left(-\|u\|_{q,(0,t)}^{-r}\right) \right)^{1/r} + \frac{\|w\|_{p',\mathbb{R}^n}}{\|u\|_{q,(0,\infty)}}.$$

### 3. Characterizations of $L_{p_1}(\mathbb{R}^n, v_1) \hookrightarrow LM_{p_2\theta,\omega}(\mathbb{R}^n, v_2)$ and $L_{p_1}(\mathbb{R}^n, v_1) \hookrightarrow {}^cLM_{p_2\theta,\omega}(\mathbb{R}^n, v_2)$

In this section we characterize (1.9) and (1.10).

**THEOREM 3.1.** *Let  $0 < p_1 \leq \infty$ ,  $0 < p_2 < \infty$ ,  $0 < \theta \leq \infty$ ,  $v_1, v_2 \in W(\mathbb{R}^n)$  and  $\omega \in \Omega_\theta$ .*

(i) If  $p_2 < p_1 \leq \theta < \infty$ , then

$$\|I\|_{L_{p_1}(\mathbb{R}^n, v_1) \rightarrow LM_{p_2, \theta, \omega}(\mathbb{R}^n, v_2)} \approx \sup_{t>0} \|\omega\|_{\theta, (t, \infty)} \left\| v_1^{-1/p_1} v_2^{1/p_2} \right\|_{\frac{p_1 p_2}{p_1 - p_2}, B(0, t)}$$

uniformly in  $\omega \in \Omega_\theta$ .

(ii) If  $p_2 < p_1 < \infty$ ,  $0 < \theta < p_1$ , then

$$\|I\|_{L_{p_1}(\mathbb{R}^n, v_1) \rightarrow LM_{p_2, \theta, \omega}(\mathbb{R}^n, v_2)} \approx \left\| \|\omega\|_{\theta, (t, \infty)}^{\theta/p_1} \left\| v_1^{-1/p_1} v_2^{1/p_2} \right\|_{\frac{p_1 p_2}{p_1 - p_2}, B(0, t)} \right\|_{\frac{\theta p_1}{p_1 - \theta}, \omega^\theta, (0, \infty)}$$

uniformly in  $\omega \in \Omega_\theta$ .

(iii) If  $p_2 < p_1 < \infty$ ,  $\theta = \infty$ , then

$$\|I\|_{L_{p_1}(\mathbb{R}^n, v_1) \rightarrow LM_{p_2, \theta, \omega}(\mathbb{R}^n, v_2)} \approx \sup_{t>0} \|\omega\|_{\theta, (t, \infty)} \left\| v_1^{-1/p_1} v_2^{1/p_2} \right\|_{\frac{p_1 p_2}{p_1 - p_2}, B(0, t)}$$

uniformly in  $\omega \in \Omega_\theta$ .

(iv) If  $p_1 = \theta = \infty$ ,  $0 < p_2 < \infty$ , then

$$\|I\|_{L_{p_1}(\mathbb{R}^n, v_1) \rightarrow LM_{p_2, \theta, \omega}(\mathbb{R}^n, v_2)} \approx \sup_{t>0} \|\omega\|_{\theta, (t, \infty)} \left\| v_1^{-1} v_2^{1/p_2} \right\|_{p_2, B(0, t)}$$

uniformly in  $\omega \in \Omega_\theta$ .

(v) If  $p_1 = \infty$ ,  $0 < p_2 < \infty$ ,  $0 < \theta < \infty$ , then

$$\|I\|_{L_{p_1}(\mathbb{R}^n, v_1) \rightarrow LM_{p_2, \theta, \omega}(\mathbb{R}^n, v_2)} \approx \left\| \|\omega(t)\| \left\| v_1^{-1} v_2^{1/p_2} \right\|_{p_2, B(0, t)} \right\|_{\theta, (0, \infty)}$$

uniformly in  $\omega \in \Omega_\theta$ .

(vi) If  $p := p_1 = p_2 \leq \theta < \infty$ , then

$$\|I\|_{L_{p_1}(\mathbb{R}^n, v_1) \rightarrow LM_{p_2, \theta, \omega}(\mathbb{R}^n, v_2)} \approx \sup_{t>0} \|\omega\|_{\theta, (t, \infty)} \|v_1^{-1/p} v_2^{1/p}\|_{\infty, B(0, t)}$$

uniformly in  $\omega \in \Omega_\theta$ .

(vii) If  $0 < p := p_1 = p_2 < \infty$ ,  $0 < \theta < p$ , then

$$\|I\|_{L_{p_1}(\mathbb{R}^n, v_1) \rightarrow LM_{p_2, \theta, \omega}(\mathbb{R}^n, v_2)} \approx \left\| \|\omega\|_{\theta, (t, \infty)}^{\theta/p} \left\| v_1^{-1/p} v_2^{1/p} \right\|_{\infty, B(0, t)} \right\|_{\frac{\theta p}{p - \theta}, \omega^\theta, (0, \infty)}$$

uniformly in  $\omega \in \Omega_\theta$ .

(viii) If  $0 < p := p_1 = p_2 < \infty$ ,  $\theta = \infty$ , then

$$\|I\|_{L_{p_1}(\mathbb{R}^n, v_1) \rightarrow LM_{p_2, \theta, \omega}(\mathbb{R}^n, v_2)} \approx \sup_{t>0} \|\omega\|_{\theta, (t, \infty)} \|v_1^{-1/p} v_2^{1/p}\|_{\infty, B(0, t)}$$

uniformly in  $\omega \in \Omega_\theta$ .

(ix) If  $p := p_1 = p_2 = \infty$ ,  $0 < \theta < \infty$ , then

$$\|I\|_{L_{p_1}(\mathbb{R}^n, v_1) \rightarrow LM_{p_2, \theta, \omega}(\mathbb{R}^n, v_2)} \approx \left\| \|\omega(t)\| \left\| v_1^{-1} v_2 \right\|_{\infty, B(0, t)} \right\|_{\theta, (0, \infty)}$$

uniformly in  $\omega \in \Omega_\theta$ .

*Proof.* (i)–(viii). Denote by

$$q_1 := p_1/p_2, \quad q_2 := \theta/p_2, \quad w_1 := v_1 v_2^{-q_1}, \quad w_2 := \omega^\theta.$$

Since

$$\begin{aligned} \|I\|_{L_{p_1}(\mathbb{R}^n, v_1) \rightarrow LM_{p_2 \theta, \omega}(\mathbb{R}^n, v_2)} &= \sup_{f \neq 0} \frac{\|\omega(r)\|f\|_{p_2, v_2, B(0, r)}\|_{\theta, (0, \infty)}}{\|f\|_{p_1, v_1, \mathbb{R}^n}} \\ &= \left( \sup_{g \neq 0} \frac{\|H(|g|)\|_{q_2, w_2, (0, \infty)}}{\|g\|_{q_1, w_1, \mathbb{R}^n}} \right)^{1/p_2}, \end{aligned}$$

it remains to apply Theorem 2.1.

(ix). Note that

$$\begin{aligned} \|I\|_{L_\infty(\mathbb{R}^n, v_1) \rightarrow LM_{\infty \theta, \omega}(\mathbb{R}^n, v_2)} &= \sup_{f \neq 0} \frac{\|\omega(r)\|f\|_{\infty, v_2, B(0, r)}\|_{\theta, (0, \infty)}}{\|f\|_{\infty, v_1, \mathbb{R}^n}} \\ &= \sup_{g \neq 0} \frac{\|(S(|g|))\omega\|_{\theta, (0, \infty)}}{\|gw\|_{\infty, \mathbb{R}^n}}, \end{aligned}$$

where  $w = v_1/v_2$ . The statement follows by Theorem 2.3.  $\square$

**THEOREM 3.2.** *Let  $0 < p_1, \theta \leq \infty, 0 < p_2 < \infty, v_1, v_2 \in W(\mathbb{R}^n)$  and  $\omega \in \mathring{\Omega}_\theta$ .*

(i) *If  $p_2 < p_1 \leq \theta < \infty$ , then*

$$\|I\|_{L_{p_1}(\mathbb{R}^n, v_1) \rightarrow {}^cLM_{p_2 \theta, \omega}(\mathbb{R}^n, v_2)} \approx \sup_{t > 0} \|\omega\|_{\theta, (0, t)} \left\| v_1^{-1/p_1} v_2^{1/p_2} \right\|_{\frac{p_1 p_2}{p_1 - p_2}, {}^cB(0, t)}$$

*uniformly in  $\omega \in \mathring{\Omega}_\theta$ .*

(ii) *If  $p_2 < p_1 < \infty, 0 < \theta < p_1$ , then*

$$\|I\|_{L_{p_1}(\mathbb{R}^n, v_1) \rightarrow {}^cLM_{p_2 \theta, \omega}(\mathbb{R}^n, v_2)} \approx \left\| \|\omega\|_{\theta, (0, t)}^{\theta/p_1} \left\| v_1^{-1/p_1} v_2^{1/p_2} \right\|_{\frac{p_1 p_2}{p_1 - p_2}, {}^cB(0, t)} \right\|_{\frac{\theta p_1}{p_1 - \theta}, \omega^\theta, (0, \infty)}$$

*uniformly in  $\omega \in \mathring{\Omega}_\theta$ .*

(iii) *If  $p_2 < p_1 < \infty, \theta = \infty$ , then*

$$\|I\|_{L_{p_1}(\mathbb{R}^n, v_1) \rightarrow {}^cLM_{p_2 \theta, \omega}(\mathbb{R}^n, v_2)} \approx \sup_{t > 0} \|\omega\|_{\theta, (0, t)} \left\| v_1^{-1/p_1} v_2^{1/p_2} \right\|_{\frac{p_1 p_2}{p_1 - p_2}, {}^cB(0, t)}$$

*uniformly in  $\omega \in \mathring{\Omega}_\theta$ .*

(iv) *If  $p_1 = \theta = \infty, 0 < p_2 < \infty$ , then*

$$\|I\|_{L_{p_1}(\mathbb{R}^n, v_1) \rightarrow {}^cLM_{p_2 \theta, \omega}(\mathbb{R}^n, v_2)} \approx \sup_{t > 0} \|\omega\|_{\theta, (0, t)} \left\| v_1^{-1} v_2^{1/p_2} \right\|_{p_2, {}^cB(0, t)}$$

uniformly in  $\omega \in {}^c\Omega_\theta$ .

(v) If  $p_1 = \infty$ ,  $0 < p_2 < \infty$ ,  $0 < \theta < \infty$ , then

$$\|I\|_{L_{p_1}(\mathbb{R}^n, v_1) \rightarrow {}^cLM_{p_2, \theta, \omega}(\mathbb{R}^n, v_2)} \approx \left\| \|\omega(t)\| \left\| v_1^{-1} v_2^{1/p_2} \right\|_{p_2, {}^cB(0,t)} \right\|_{\theta, (0, \infty)}$$

uniformly in  $\omega \in {}^c\Omega_\theta$ .

(vi) If  $p := p_1 = p_2 \leq \theta < \infty$ , then

$$\|I\|_{L_{p_1}(\mathbb{R}^n, v_1) \rightarrow {}^cLM_{p_2, \theta, \omega}(\mathbb{R}^n, v_2)} \approx \sup_{t>0} \|\omega\|_{\theta, (0,t)} \left\| v_1^{-1/p} v_2^{1/p} \right\|_{\infty, {}^cB(0,t)}$$

uniformly in  $\omega \in {}^c\Omega_\theta$ .

(vii) If  $0 < p := p_1 = p_2 < \infty$ ,  $0 < \theta < p$ , then

$$\|I\|_{L_{p_1}(\mathbb{R}^n, v_1) \rightarrow {}^cLM_{p_2, \theta, \omega}(\mathbb{R}^n, v_2)} \approx \left\| \|\omega\|_{\theta, (0,t)}^{\theta/p} \left\| v_1^{-1/p} v_2^{1/p} \right\|_{\infty, {}^cB(0,t)} \right\|_{\frac{\theta p}{p-\theta}, \omega^\theta, (0, \infty)}$$

uniformly in  $\omega \in {}^c\Omega_\theta$ .

(viii) If  $0 < p := p_1 = p_2 < \infty$ ,  $\theta = \infty$ , then

$$\|I\|_{L_{p_1}(\mathbb{R}^n, v_1) \rightarrow {}^cLM_{p_2, \theta, \omega}(\mathbb{R}^n, v_2)} \approx \sup_{t>0} \|\omega\|_{\theta, (0,t)} \left\| v_1^{-1/p} v_2^{1/p} \right\|_{\infty, {}^cB(0,t)}$$

uniformly in  $\omega \in {}^c\Omega_\theta$ .

(ix) If  $p := p_1 = p_2 = \infty$ ,  $0 < \theta < \infty$ , then

$$\|I\|_{L_{p_1}(\mathbb{R}^n, v_1) \rightarrow {}^cLM_{p_2, \theta, \omega}(\mathbb{R}^n, v_2)} \approx \left\| \|\omega(t)\| \left\| v_1^{-1} v_2 \right\|_{\infty, {}^cB(0,t)} \right\|_{\theta, (0, \infty)}$$

uniformly in  $\omega \in {}^c\Omega_\theta$ .

Theorem 1.9 reduces the problem of boundedness of  $M$  from  $L_{p_1}(\mathbb{R}^n, v_1)$  to  ${}^cLM_{p_2, \theta_2, \omega_2}(\mathbb{R}^n, v_2)$  and from  $L_{p_1}(\mathbb{R}^n, v_1)$  to  ${}^cLM_{p_2, \theta_2, \omega_2}(\mathbb{R}^n, v_2)$  to the characterizations of (1.9) and (1.10), respectively, when we know the boundedness of  $M$  on  $L_{p_1}(\mathbb{R}^n, v_1)$ . The latter happens exactly when  $v_1 \in A_{p_1}$ ,  $1 < p_1 < \infty$ .

Let  $w$  be a weight function and  $1 < p < \infty$ . We say that  $w \in A_p$  if there exists a constant  $c_p > 0$  such that, for every ball  $B \subset \mathbb{R}^n$ ,

$$\left( \int_B w(x) dx \right) \left( \int_B w(x)^{1-p'} dx \right)^{p-1} \leq c_p |B|^p.$$

It is well known that the Muckenhoupt classes characterize the boundedness of  $M$  on weighted Lebesgue spaces. Namely,  $M$  is bounded on  $L_p(\mathbb{R}^n, w)$  if and only if  $w \in A_p$ ,  $1 < p < \infty$  (see, for instance, [22]).

The following statements are consequences of combination of Theorems 3.1 and 3.2 with Theorem 1.9.

COROLLARY 3.3. Let  $1 < p_1 < \infty$ ,  $0 < p_2 < \infty$ ,  $0 < \theta \leq \infty$ ,  $v_2 \in W(\mathbb{R}^n)$  and  $\omega \in \Omega_\theta$ . Moreover, assume that  $v_1 \in A_{p_1}$ .

(i) If  $p_2 < p_1 \leq \theta < \infty$ , then

$$\|M\|_{L_{p_1}(\mathbb{R}^n, v_1) \rightarrow LM_{p_2, \theta, \omega}(\mathbb{R}^n, v_2)} \approx \sup_{t>0} \|\omega\|_{\theta, (t, \infty)} \left\| v_1^{-1/p_1} v_2^{1/p_2} \right\|_{\frac{p_1 p_2}{p_1 - p_2}, B(0, t)}$$

uniformly in  $\omega \in \Omega_\theta$ .

(ii) If  $p_2 < p_1 < \infty$ ,  $0 < \theta < p_1$ , then

$$\|M\|_{L_{p_1}(\mathbb{R}^n, v_1) \rightarrow LM_{p_2, \theta, \omega}(\mathbb{R}^n, v_2)} \approx \left\| \|\omega\|_{\theta, (t, \infty)}^{\theta/p_1} \right\| v_1^{-1/p_1} v_2^{1/p_2} \left\| \right\|_{\frac{p_1 p_2}{p_1 - p_2}, B(0, t)} \left\| \right\|_{\frac{\theta p_1}{p_1 - \theta}, \omega^\theta, (0, \infty)}$$

uniformly in  $\omega \in \Omega_\theta$ .

(iii) If  $p_2 < p_1 < \infty$ ,  $\theta = \infty$ , then

$$\|M\|_{L_{p_1}(\mathbb{R}^n, v_1) \rightarrow LM_{p_2, \theta, \omega}(\mathbb{R}^n, v_2)} \approx \sup_{t>0} \|\omega\|_{\theta, (t, \infty)} \left\| v_1^{-1/p_1} v_2^{1/p_2} \right\|_{\frac{p_1 p_2}{p_1 - p_2}, B(0, t)}$$

uniformly in  $\omega \in \Omega_\theta$ .

(iv) If  $p := p_1 = p_2 \leq \theta < \infty$ , then

$$\|M\|_{L_{p_1}(\mathbb{R}^n, v_1) \rightarrow LM_{p_2, \theta, \omega}(\mathbb{R}^n, v_2)} \approx \sup_{t>0} \|\omega\|_{\theta, (t, \infty)} \|v_1^{-1/p} v_2^{1/p}\|_{\infty, B(0, t)}$$

uniformly in  $\omega \in \Omega_\theta$ .

(v) If  $p := p_1 = p_2$ ,  $0 < \theta < p$ , then

$$\|M\|_{L_{p_1}(\mathbb{R}^n, v_1) \rightarrow LM_{p_2, \theta, \omega}(\mathbb{R}^n, v_2)} \approx \left\| \|\omega\|_{\theta, (t, \infty)}^{\theta/p} \right\| v_1^{-1/p} v_2^{1/p} \left\| \right\|_{\infty, B(0, t)} \left\| \right\|_{\frac{\theta p}{p - \theta}, \omega^\theta, (0, \infty)}$$

uniformly in  $\omega \in \Omega_\theta$ .

(vi) If  $p := p_1 = p_2$ ,  $\theta = \infty$ , then

$$\|M\|_{L_{p_1}(\mathbb{R}^n, v_1) \rightarrow LM_{p_2, \theta, \omega}(\mathbb{R}^n, v_2)} \approx \sup_{t>0} \|\omega\|_{\theta, (t, \infty)} \|v_1^{-1/p} v_2^{1/p}\|_{\infty, B(0, t)}$$

uniformly in  $\omega \in \Omega_\theta$ .

COROLLARY 3.4. Let  $1 < p_1 < \infty$ ,  $0 < p_2 < \infty$ ,  $0 < \theta \leq \infty$ ,  $v_2 \in W(\mathbb{R}^n)$  and  $\omega \in {}^c\Omega_\theta$ . Moreover, assume that  $v_1 \in A_{p_1}$ .

(i) If  $p_2 < p_1 \leq \theta < \infty$ , then

$$\|M\|_{L_{p_1}(\mathbb{R}^n, v_1) \rightarrow {}^cLM_{p_2, \theta, \omega}(\mathbb{R}^n, v_2)} \approx \sup_{t>0} \|\omega\|_{\theta, (0, t)} \left\| v_1^{-1/p_1} v_2^{1/p_2} \right\|_{\frac{p_1 p_2}{p_1 - p_2}, {}^cB(0, t)}$$

uniformly in  $\omega \in {}^c\Omega_\theta$ .

(ii) If  $p_2 < p_1 < \infty$ ,  $0 < \theta < p_1$ , then

$$\|M\|_{L_{p_1}(\mathbb{R}^n, v_1) \rightarrow {}^cLM_{p_2, \theta, \omega}(\mathbb{R}^n, v_2)} \approx \left\| \|\omega\|_{\theta, (0, t)}^{\theta/p_1} \left\| v_1^{-1/p_1} v_2^{1/p_2} \right\|_{\frac{p_1 p_2}{p_1 - p_2}, {}^cB(0, t)} \right\|_{\frac{\theta p_1}{p_1 - \theta}, \omega^\theta, (0, \infty)}$$

uniformly in  $\omega \in {}^c\Omega_\theta$ .

(iii) If  $p_2 < p_1 < \infty$ ,  $\theta = \infty$ , then

$$\|M\|_{L_{p_1}(\mathbb{R}^n, v_1) \rightarrow {}^cLM_{p_2, \theta, \omega}(\mathbb{R}^n, v_2)} \approx \sup_{t > 0} \|\omega\|_{\theta, (0, t)} \left\| v_1^{-1/p_1} v_2^{1/p_2} \right\|_{\frac{p_1 p_2}{p_1 - p_2}, {}^cB(0, t)}$$

uniformly in  $\omega \in {}^c\Omega_\theta$ .

(iv) If  $p := p_1 = p_2 \leq \theta < \infty$ , then

$$\|M\|_{L_{p_1}(\mathbb{R}^n, v_1) \rightarrow {}^cLM_{p_2, \theta, \omega}(\mathbb{R}^n, v_2)} \approx \sup_{t > 0} \|\omega\|_{\theta, (0, t)} \left\| v_1^{-1/p} v_2^{1/p} \right\|_{\infty, {}^cB(0, t)}$$

uniformly in  $\omega \in {}^c\Omega_\theta$ .

(v) If  $p := p_1 = p_2$ ,  $0 < \theta < p$ , then

$$\|M\|_{L_{p_1}(\mathbb{R}^n, v_1) \rightarrow {}^cLM_{p_2, \theta, \omega}(\mathbb{R}^n, v_2)} \approx \left\| \|\omega\|_{\theta, (0, t)}^{\theta/p} \left\| v_1^{-1/p} v_2^{1/p} \right\|_{\infty, {}^cB(0, t)} \right\|_{\frac{\theta p}{p - \theta}, \omega^\theta, (0, \infty)}$$

uniformly in  $\omega \in {}^c\Omega_\theta$ .

(vi) If  $p := p_1 = p_2$ ,  $\theta = \infty$ , then

$$\|M\|_{L_{p_1}(\mathbb{R}^n, v_1) \rightarrow {}^cLM_{p_2, \theta, \omega}(\mathbb{R}^n, v_2)} \approx \left\| \|\omega\|_{\theta, (0, t)} \left\| v_1^{-1/p} v_2^{1/p} \right\|_{\infty, {}^cB(0, t)} \right\|_{\infty, (0, \infty)}$$

uniformly in  $\omega \in {}^c\Omega_\theta$ .

#### 4. Characterizations of $LM_{p_2, \theta, \omega}(\mathbb{R}^n, v_2) \hookrightarrow L_{p_1}(\mathbb{R}^n, v_1)$ and ${}^cLM_{p_2, \theta, \omega}(\mathbb{R}^n, v_2) \hookrightarrow L_{p_1}(\mathbb{R}^n, v_1)$

In this section we characterize the embeddings (1.11) and (1.12).

**THEOREM 4.1.** *Let  $0 < p_1 \leq p_2 < \infty$ ,  $0 < \theta \leq \infty$ ,  $v_1, v_2 \in W(\mathbb{R}^n)$  and  $\omega \in \Omega_\theta$ .*

(a) *If  $0 < \theta \leq p_1 < p_2 < \infty$ , then*

$$\|I\|_{LM_{p_2, \theta, \omega}(\mathbb{R}^n, v_2) \rightarrow L_{p_1}(\mathbb{R}^n, v_1)} \approx \sup_{t > 0} \|\omega\|_{\theta, (t, \infty)}^{-1} \left\| v_1^{1/p_1} v_2^{-1/p_2} \right\|_{\frac{p_1 p_2}{p_2 - p_1}, {}^cB(0, t)}$$

uniformly in  $\omega \in \Omega_\theta$ .

(b) *If  $0 < \theta \leq p_1 = p_2 =: p < \infty$ , then*

$$\|I\|_{LM_{p_2, \theta, \omega}(\mathbb{R}^n, v_2) \rightarrow L_{p_1}(\mathbb{R}^n, v_1)} \approx \sup_{t > 0} \|\omega\|_{\theta, (t, \infty)}^{-1} \left\| v_1 v_2^{-1} \right\|_{\infty, {}^cB(0, t)}^{1/p}$$

uniformly in  $\omega \in \Omega_\theta$ .

(c) If  $0 < p_1 < p_2 < \infty$ ,  $p_1 < \theta \leq \infty$ , then

$$\begin{aligned} \|I\|_{LM_{p_2, \theta, \omega}(\mathbb{R}^n, v_2) \rightarrow L_{p_1}(\mathbb{R}^n, v_1)} &\approx \left( \int_{(0, \infty)} \left\| v_1^{1/p_1} v_2^{-1/p_2} \left\| \frac{p_1 \theta}{\theta - p_1} \right\|_{\frac{p_1 p_2}{p_2 - p_1}, c_{B(0, t)}} d \left( \|\omega\|_{\theta, (t-, \infty)}^{-\frac{p_1 \theta}{\theta - p_1}} \right) \right)^{\frac{\theta - p_1}{p_1 \theta}} \\ &\quad + \|\omega\|_{\theta, (0, \infty)}^{-1} \left\| v_1^{1/p_1} v_2^{-1/p_2} \right\|_{\frac{p_1 p_2}{p_2 - p_1}, \mathbb{R}^n} \end{aligned}$$

uniformly in  $\omega \in \Omega_\theta$ .

(d) If  $0 < p_1 = p_2 =: p < \theta \leq \infty$ , then

$$\begin{aligned} \|I\|_{LM_{p_2, \theta, \omega}(\mathbb{R}^n, v_2) \rightarrow L_{p_1}(\mathbb{R}^n, v_1)} &\approx \left( \int_{(0, \infty)} \|v_1 v_2^{-1}\|_{\frac{\theta}{\theta - p}, c_{B(0, t)}}^{\frac{\theta}{\theta - p}} d \left( \|\omega\|_{\theta, (t-, \infty)}^{-\frac{p \theta}{\theta - p}} \right) \right)^{\frac{\theta - p}{p \theta}} \\ &\quad + \|\omega\|_{\theta, (0, \infty)}^{-1} \|v_1 v_2^{-1}\|_{\infty, \mathbb{R}^n}^{1/p} \end{aligned}$$

uniformly in  $\omega \in \Omega_\theta$ .

*Proof.* Since

$$\begin{aligned} \|I\|_{LM_{p_2, \theta, \omega}(\mathbb{R}^n, v_2) \rightarrow L_{p_1}(\mathbb{R}^n, v_1)} &= \sup_{f \neq 0} \frac{\|f\|_{p_1, v_1, \mathbb{R}^n}}{\|w(r)\|_{p_2, v_2, B(0, r)} \|f\|_{\theta, (0, \infty)}} \\ &= \left( \sup_{g \neq 0} \frac{\|g\|_{q_1, w_1, \mathbb{R}^n}}{\|H(|g|)\|_{q_2, w_2, (0, \infty)}} \right)^{1/p_2}, \end{aligned}$$

it remains to apply Theorem 2.7.  $\square$

**THEOREM 4.2.** Let  $0 < p_1 \leq p_2 < \infty$ ,  $0 < \theta \leq \infty$ ,  $v_1, v_2 \in W(\mathbb{R}^n)$  and  $\omega \in {}^c\Omega_\theta$ .

(a) If  $0 < \theta \leq p_1 < p_2 < \infty$ , then

$$\|I\|_{c_{LM_{p_2, \theta, \omega}(\mathbb{R}^n, v_2) \rightarrow L_{p_1}(\mathbb{R}^n, v_1)}} \approx \sup_{t > 0} \|\omega\|_{\theta, (0, t)}^{-1} \left\| v_1^{1/p_1} v_2^{-1/p_2} \right\|_{\frac{p_1 p_2}{p_2 - p_1}, B(0, t)}$$

uniformly in  $\omega \in {}^c\Omega_\theta$ .

(b) If  $0 < \theta \leq p_1 = p_2 =: p < \infty$ , then

$$\|I\|_{LM_{p_2, \theta, \omega}(\mathbb{R}^n, v_2) \rightarrow L_{p_1}(\mathbb{R}^n, v_1)} \approx \sup_{t > 0} \|\omega\|_{\theta, (0, t)}^{-1} \|v_1 v_2^{-1}\|_{\infty, B(0, t)}^{1/p}$$

uniformly in  $\omega \in \Omega_\theta$ .

(c) If  $0 < p_1 < p_2 < \infty$ ,  $p_1 < \theta \leq \infty$ , then

$$\begin{aligned} \|I\|_{c_{LM_{p_2, \theta, \omega}(\mathbb{R}^n, v_2) \rightarrow L_{p_1}(\mathbb{R}^n, v_1)}} &\approx \left( \int_{(0, \infty)} \left\| v_1^{1/p_1} v_2^{-1/p_2} \left\| \frac{p_1 \theta}{\theta - p_1} \right\|_{\frac{p_1 p_2}{p_2 - p_1}, B(0, t)} d \left( -\|\omega\|_{\theta, (0, t+)}^{-\frac{p_1 \theta}{\theta - p_1}} \right) \right)^{\frac{\theta - p_1}{p_1 \theta}} \\ &\quad + \|\omega\|_{\theta, (0, \infty)}^{-1} \left\| v_1^{1/p_1} v_2^{-1/p_2} \right\|_{\frac{p_1 p_2}{p_2 - p_1}, \mathbb{R}^n} \end{aligned}$$

uniformly in  $\omega \in {}^c\Omega_\theta$ .

(d) If  $0 < p_1 = p_2 =: p < \theta \leq \infty$ , then

$$\begin{aligned} \|I\|_{LM_{p_2, \theta, \omega}(\mathbb{R}^n, v_2) \rightarrow L_{p_1}(\mathbb{R}^n, v_1)} &\approx \left( \int_{(0, \infty)} \|v_1 v_2^{-1}\|_{\infty, B(0, t)}^{\frac{\theta}{\theta-p}} d \left( \|\omega\|_{\theta, (0, t+)}^{-\frac{p\theta}{\theta-p}} \right) \right)^{\frac{\theta-p}{p\theta}} \\ &\quad + \|\omega\|_{\theta, (0, \infty)}^{-1} \|v_1 v_2^{-1}\|_{\infty, \mathbb{R}^n}^{1/p} \end{aligned}$$

uniformly in  $\omega \in \Omega_\theta$ .

*Proof.* It suffices to note that

$$\begin{aligned} \|I\|_{{}^cLM_{p_2, \theta, \omega}(\mathbb{R}^n, v_2) \rightarrow L_{p_1}(\mathbb{R}^n, v_1)} &= \sup_{f \neq 0} \frac{\|f\|_{p_1, v_1, \mathbb{R}^n}}{\left\| w(r) \|f\|_{p_2, v_2, {}^cB(0, r)} \right\|_{\theta, (0, \infty)}} \\ &= \left( \sup_{g \neq 0} \frac{\|g\|_{q_1, w_1, \mathbb{R}^n}}{\|H^*(|g|)\|_{q_2, w_2, (0, \infty)}} \right)^{1/p_2}. \end{aligned}$$

and apply Theorem 2.8.  $\square$

DEFINITION 4.3. Let  $X$  be a set of functions from  $\mathfrak{M}(\mathbb{R}^n)$ , endowed with a positively homogeneous functional  $\|\cdot\|_X$ , defined for every  $f \in \mathfrak{M}(\mathbb{R}^n)$  and such that  $f \in X$  if and only if  $\|f\|_X < \infty$ . We define the associate space  $X'$  of  $X$  as the set of all functions  $f \in \mathfrak{M}(\mathbb{R}^n)$  such that  $\|f\|_{X'} < \infty$ , where

$$\|f\|_{X'} = \sup \left\{ \int_{\mathbb{R}^n} |f(x)g(x)| dx : \|g\|_X \leq 1 \right\}.$$

In [16], the associate spaces of local Morrey-type and complementary local Morrey-type spaces were calculated. In particular, Theorems 4.1 and 4.2 allow us to give a characterization of the associate spaces of weighted local Morrey-type and complementary local Morrey-type spaces.

THEOREM 4.4. Assume  $1 \leq p < \infty$ ,  $0 < \theta \leq \infty$ . Let  $\omega \in \Omega_\theta$  and  $v \in W(\mathbb{R}^n)$ . Set

$$X = LM_{p\theta, \omega}(\mathbb{R}^n, v).$$

(i) Let  $0 < \theta \leq 1$ . Then

$$\|f\|_{X'} \approx \sup_{t \in (0, \infty)} \|f\|_{p', v^{1-p'}, {}^cB(0, t)} \|\omega\|_{L_\theta(t, \infty)}^{-1},$$

with the positive constants in equivalence independent of  $f$ .

(ii) Let  $1 < \theta \leq \infty$ . Then

$$\|f\|_{X'} \approx \left( \int_{(0, \infty)} \|f\|_{p', v^{1-p'}, {}^cB(0, t)}^{\theta'} d \left( \|\omega\|_{\theta, (t, \infty)}^{-\theta'} \right) \right)^{1/\theta'} + \frac{\|f\|_{p', v^{1-p'}, \mathbb{R}^n}}{\|\omega\|_{\theta, (0, \infty)}},$$

with the positive constants in equivalence independent of  $f$ .

THEOREM 4.5. Assume  $1 \leq p < \infty$ ,  $0 < \theta \leq \infty$ . Let  $\omega \in {}^c\Omega_\theta$  and  $v \in W(\mathbb{R}^n)$ .

Set

$$X = {}^cLM_{p,\theta,\omega}(\mathbb{R}^n, v).$$

(i) Let  $0 < \theta \leq 1$ . Then

$$\|f\|_{X'} \approx \sup_{t \in (0, \infty)} \|f\|_{p', v^{1-p'}, B(0,t)} \|\omega\|_{L_\theta(0,t)}^{-1},$$

with the positive constants in equivalence independent of  $f$ .

(ii) Let  $1 < \theta \leq \infty$ . Then

$$\|f\|_{X'} \approx \left( \int_{(0, \infty)} \|f\|_{p', v^{1-p'}, B(0,t)}^{\theta'} d \left( \|\omega\|_{\theta, (0,t+)}^{-\theta'} \right) \right)^{1/\theta'} + \frac{\|f\|_{p', v^{1-p'}, \mathbb{R}^n}}{\|\omega\|_{\theta, (0, \infty)}},$$

with the positive constants in equivalence independent of  $f$ .

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