

## CAUCHY'S ERROR REPRESENTATION OF LIDSTONE INTERPOLATING POLYNOMIAL AND RELATED RESULTS

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*Abstract.* In this paper we consider the Cauchy's error representation of Lidstone interpolating polynomial and as a consequence the results concerning to the Hermite-Hadamard inequalities. Using these inequalities, we produce new exponentially convex functions. Also, we give several examples of the families of functions for which the obtained results can be applied.

### 1. Introduction

*Lidstone series* is generalization of Taylor's series. It approximates to a given function in the neighborhood of two points (instead of one). Such series have been studied by G. J. Lidstone (1929), H. Poritsky (1932), J. M. Wittaker (1934) and others.

DEFINITION 1. Let  $f \in C^\infty([0, 1])$ , then Lidstone series has the form

$$\sum_{k=0}^{\infty} \left( f^{(2k)}(0) \Lambda_k(1-x) + f^{(2k)}(1) \Lambda_k(x) \right),$$

where  $\Lambda_n$  is a polynomial of degree  $2n + 1$  defined by the relations

$$\begin{aligned} \Lambda_0(t) &= t, \\ \Lambda_n''(t) &= \Lambda_{n-1}(t), \\ \Lambda_n(0) = \Lambda_n(1) &= 0, \quad n \geq 1. \end{aligned} \tag{1.1}$$

Another explicit representations of Lidstone polynomial are given by [1] and [10],

$$\Lambda_n(t) = (-1)^n \frac{2}{\pi^{2n+1}} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^{2n+1}} \sin k\pi t,$$

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$$\Lambda_n(t) = \frac{1}{6} \left[ \frac{6t^{2n+1}}{(2n+1)!} - \frac{t^{2n-1}}{(2n-1)!} \right] - \sum_{k=0}^{n-2} \frac{2(2^{2k+3}-1)}{(2k+4)!} B_{2k+4} \frac{t^{2n-2k-3}}{(2n-2k-3)!}, \quad n = 1, 2, \dots,$$

$$\Lambda_n(t) = \frac{2^{2n+1}}{(2n+1)!} B_{2n+1} \left( \frac{1+t}{2} \right), \quad n = 1, 2, \dots,$$

where  $B_{2k+4}$  is the  $(2k+4)$ -th Bernoulli number and  $B_{2n+1} \left( \frac{1+t}{2} \right)$  is a Bernoulli polynomial.

In [11], Widder proved the fundamental lemma:

LEMMA 1. *If  $f \in C^{(2n)}([0, 1])$ , then*

$$f(t) = \sum_{k=0}^{n-1} \left[ f^{(2k)}(0)\Lambda_k(1-t) + f^{(2k)}(1)\Lambda_k(t) \right] + \int_0^1 G_n(t,s)f^{(2n)}(s)ds, \quad (1.2)$$

where

$$G_1(t,s) = G(t,s) = \begin{cases} (t-1)s, & \text{if } s \leq t, \\ (s-1)t, & \text{if } t \leq s, \end{cases} \quad (1.3)$$

is the homogeneous Green's function of the differential operator  $\frac{d^2}{ds^2}$  on  $[0, 1]$ , and with the successive iterates of  $G(t, s)$

$$G_n(t,s) = \int_0^1 G_1(t,p)G_{n-1}(p,s)dp, \quad n \geq 2. \quad (1.4)$$

Lidstone polynomial can be expressed, in terms of  $G_n(t, s)$  as

$$\Lambda_n(t) = \int_0^1 G_n(t,s)sds.$$

DEFINITION 2. Let  $f$  be a real-valued function defined on the segment  $[a, b]$ . The *divided difference* of order  $n$  of the function  $f$  at distinct points  $x_0, \dots, x_n \in [a, b]$ , is defined recursively (see [2], [8]) by

$$f[x_i] = f(x_i), \quad (i = 0, \dots, n)$$

and

$$f[x_0, \dots, x_n] = \frac{f[x_1, \dots, x_n] - f[x_0, \dots, x_{n-1}]}{x_n - x_0}.$$

The value  $f[x_0, \dots, x_n]$  is independent of the order of the points  $x_0, \dots, x_n$ .

The definition may be extended to include the case that some (or all) of the points coincide. Assuming that  $f^{(j-1)}(x)$  exists, we define

$$f[\underbrace{x, \dots, x}_{j\text{-times}}] = \frac{f^{(j-1)}(x)}{(j-1)!}. \quad (1.5)$$

The notion of  $n$ -convexity goes back to Popoviciu ([9]). We follow the definition given by Karlin ([5]):

DEFINITION 3. A function  $f : [\alpha, \beta] \rightarrow \mathbb{R}$  is said to be  $n$ -convex on  $[\alpha, \beta]$ ,  $n \geq 0$ , if for all choices of  $(n + 1)$  distinct points in  $[\alpha, \beta]$ ,  $n$ -th order divided difference of  $f$  satisfies

$$f[x_0, \dots, x_n] \geq 0.$$

In fact, Popoviciu proved that each continuous  $n$ -convex function on  $[0, 1]$  is the uniform limit of the sequence of corresponding Bernstein's polynomials (see for example [8, p. 293]). Also, Bernstein's polynomials of continuous  $n$ -convex function are also  $n$ -convex functions. Therefore, when stating our results for a continuous  $n$ -convex function  $f$ , without any loss in generality we assume that  $f^{(n)}$  exists and is continuous.

Many related results, as well as some important inequalities due to Favard, Berwald and Steffensen can be found in [6].

### 2. Cauchy's error representation

In [1] the following theorem is proved:

THEOREM 1. If  $f \in C^{(2n)}([0, 1])$  then

$$\int_0^1 G_n(t, s) f^{(2n)}(s) ds = \frac{1}{(2n)!} E_{2n}(t) f^{(2n)}(\xi), \tag{2.1}$$

where  $\xi \in (0, 1)$ .

For the Euler polynomials it is known that

$$(-1)^n E_{2n}(t) \geq 0. \tag{2.2}$$

Euler polynomials can be expressed in terms of Bernoulli polynomials as

$$E_{2n}(t) = \frac{2^{2n+1}}{2n+1} \left[ B_{2n+1} \left( \frac{1+t}{2} \right) - B_{2n+1} \left( \frac{t}{2} \right) \right]. \tag{2.3}$$

For the Bernoulli polynomials it is known that

$$B_{2n+1} \left( 1 - \frac{t}{2} \right) = -B_{2n+1} \left( \frac{t}{2} \right). \tag{2.4}$$

The Bernoulli polynomials  $B_k(t), k \geq 0$  are uniquely determined by the following identities

$$B'_k(t) = kB_{k-1}(t), \quad k \geq 1; \quad B_0(t) = 1 \tag{2.5}$$

and

$$B_k(t+1) - B_k(t) = kt^{k-1}, \quad k \geq 0. \tag{2.6}$$

By Lemma 1 and Theorem 1 we can represent associated error of the Lidstone interpolating polynomial of the function  $f \in C^{(2n)}([\alpha, \beta])$  in the form:

$$\begin{aligned} f(x) &= \sum_{k=0}^{n-1} (\beta - \alpha)^{2k} \left[ f^{(2k)}(\alpha) \Lambda_k \left( \frac{\beta - x}{\beta - \alpha} \right) + f^{(2k)}(\beta) \Lambda_k \left( \frac{x - \alpha}{\beta - \alpha} \right) \right] \\ &= \frac{(\beta - \alpha)^{2n-1}}{(2n)!} E_{2n} \left( \frac{x - \alpha}{\beta - \alpha} \right) f^{(2n)}(\xi), \quad \xi \in [\alpha, \beta]. \end{aligned} \tag{2.7}$$

Motivated by (2.7) we define functionals  $\Phi_1(f)$  by

$$\Phi_1(f) = f(x) - \sum_{k=0}^{n-1} (\beta - \alpha)^{2k} \left[ f^{(2k)}(\alpha) \Lambda_k \left( \frac{\beta - x}{\beta - \alpha} \right) + f^{(2k)}(\beta) \Lambda_k \left( \frac{x - \alpha}{\beta - \alpha} \right) \right] \tag{2.8}$$

COROLLARY 1. For every  $(4m)$ -convex function  $f : [\alpha, \beta] \rightarrow \mathbb{R}$  holds

$$\Phi_1(f) \geq 0. \tag{2.9}$$

*Proof.* For  $n = 2m$  in (2.7) we have

$$\begin{aligned} \Phi_1(f) &= \frac{(\beta - \alpha)^{4m-1}}{(4m)!} E_{4m} \left( \frac{x - \alpha}{\beta - \alpha} \right) f^{(4m)}(\xi) \\ &= \frac{(\beta - \alpha)^{4m-1}}{(4m)!} (-1)^{2m} E_{4m} \left( \frac{x - \alpha}{\beta - \alpha} \right) f^{(4m)}(\xi). \end{aligned}$$

By (2.2) for the  $(4m)$ -convex function  $f : [\alpha, \beta] \rightarrow \mathbb{R}$ , (2.9) obviously holds.  $\square$

The following error estimates is proved in [1]:

THEOREM 2. If  $f \in C^{(2n)}([0, 1])$ , then the following hold

$$\begin{aligned} & \left| f^{(2i)}(x) - \sum_{k=0}^{n-1-i} \left[ f^{(2k+2i)}(0) \Lambda_k(1-x) + f^{(2k+2i)}(1) \Lambda_k(x) \right] \right| \\ & \leq \frac{(-1)^{n-i}}{(2n-2i)!} E_{2n-2i}(x) M_{2n} \leq \frac{(-1)^{n-i} E_{2n-2i}}{2^{2n-2i} [(2n-2i)!]^2} M_{2n}, \quad 0 \leq i \leq n-1 \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} & \left| f^{(2i+1)}(x) - \sum_{k=0}^{n-1-i} \left[ f^{(2k+2i)}(0) \Lambda'_k(1-x) + f^{(2k+2i)}(1) \Lambda'_k(x) \right] \right| \\ & \leq (-1)^{n-i} \left[ 2 \frac{E_{2n-2i}(x)}{(2n-2i)!} + (1-2x) \frac{E_{2n-2i-1}(x)}{(2n-2i-1)!} \right] M_{2n} \\ & \leq (-1)^{n-i+1} \frac{2(2^{2n-2i}-1)}{(2n-2i)!} B_{2n-2i} M_{2n}, \quad 0 \leq i \leq n-1 \end{aligned} \tag{2.11}$$

where  $M_{2n} = \max_{0 \leq x \leq 1} |f^{(2n)}(x)|$ .

REMARK 1. Inequalities (2.10) and (2.11) are the best possible, as throughout the equality holds for the function  $f(x) = \frac{E_{2n}(x)}{(2n)!}$ .

### 3. Generalization of the Hermite-Hadamard inequality

The classical Hermite-Hadamard inequality states that for a convex function  $f : [\alpha, \beta] \rightarrow \mathbb{R}$  the following estimation holds:

$$f\left(\frac{\alpha + \beta}{2}\right) \leq \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} f(x) dx \leq \frac{f(\alpha) + f(\beta)}{2}. \tag{3.1}$$

As a consequences of our results given in Section 2, here we give the generalized Hermite-Hadamard inequality.

**THEOREM 3.** *Let  $n = 2m + 1$ ,  $m \in \mathbb{N}_0$ . Then for every  $(2n)$ -convex function  $f : [\alpha, \beta] \rightarrow \mathbb{R}$  holds*

$$\begin{aligned} & f\left(\frac{\alpha + \beta}{2}\right) + \sum_{k=1}^{n-1} \frac{(\beta - \alpha)^{2k}}{(2k+1)!} \left[ f^{(2k)}(\alpha) + f^{(2k)}(\beta) \right] \\ & \times \left( \frac{2^{2k+3} - 2}{2k+2} B_{2k+2} + 2^{2k+1} B_{2k+1} \left( \frac{1}{4} \right) \right) \\ & \leq \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} f(x) dx \\ & \leq \frac{1}{2} [f(\alpha) + f(\beta)] + \sum_{k=1}^{n-1} \frac{(\beta - \alpha)^{2k}}{(2k+2)!} \left[ f^{(2k)}(\alpha) + f^{(2k)}(\beta) \right] (2^{2k+3} - 2) B_{2k+2}. \end{aligned} \tag{3.2}$$

If  $n = 2m$  or  $f$  is  $(2n)$ -concave, the inequalities are reversed.

*Proof.* The integration of the identity (2.7) on  $[\alpha, \beta]$  gives

$$\begin{aligned} & \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} f(x) dx \\ & - \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} \sum_{k=0}^{n-1} (\beta - \alpha)^{2k} \left[ f^{(2k)}(\alpha) \Lambda_k \left( \frac{\beta - x}{\beta - \alpha} \right) + f^{(2k)}(\beta) \Lambda_k \left( \frac{x - \alpha}{\beta - \alpha} \right) \right] dx \\ & = \frac{1}{(2n)!(\beta - \alpha)} \int_{\alpha}^{\beta} (\beta - \alpha)^{2n-1} E_{2n} \left( \frac{x - \alpha}{\beta - \alpha} \right) f^{(2n)}(\xi) dx, \quad \xi \in [\alpha, \beta]. \end{aligned} \tag{3.3}$$

We have

$$\begin{aligned} \int_{\alpha}^{\beta} \Lambda_k \left( \frac{x - \alpha}{\beta - \alpha} \right) dx &= \int_{\alpha}^{\beta} \Lambda''_{k+1} \left( \frac{x - \alpha}{\beta - \alpha} \right) dx = (\beta - \alpha) [\Lambda'_{k+1}(1) - \Lambda'_{k+1}(0)] \\ &= (\beta - \alpha) \frac{2^{2k+3} - 2}{(2k+2)!} B_{2k+2}, \\ \int_{\alpha}^{\beta} \Lambda_k \left( \frac{\beta - x}{\beta - \alpha} \right) dx &= \int_{\alpha}^{\beta} \Lambda''_{k+1} \left( \frac{\beta - x}{\beta - \alpha} \right) dx = (\beta - \alpha) [\Lambda'_{k+1}(1) - \Lambda'_{k+1}(0)] \\ &= (\beta - \alpha) \frac{2^{2k+3} - 2}{(2k+2)!} B_{2k+2} \end{aligned}$$

and by (2.3) and (2.5) we have

$$\int_{\alpha}^{\beta} E_{2n} \left( \frac{x - \alpha}{\beta - \alpha} \right) dx = (\beta - \alpha) \frac{2^{2n+4} - 4}{(2n + 1)(2n + 2)} B_{2n+2}.$$

So, using the fact that  $(-1)^n B_{2n+2} > 0$  we have that for  $(2n)$ -convex function the second inequality in (3.2) is valid when  $n = 2m + 1, m \in \mathbb{N}_0$ .

Using (2.7) for  $x = \frac{\alpha + \beta}{2}$  we get the following

$$\begin{aligned} f \left( \frac{\alpha + \beta}{2} \right) & - \sum_{k=0}^{n-1} (\beta - \alpha)^{2k} \left[ f^{(2k)}(\alpha) \Lambda_k \left( \frac{1}{2} \right) + f^{(2k)}(\beta) \Lambda_k \left( \frac{1}{2} \right) \right] \\ & = \frac{(\beta - \alpha)^{2n-1}}{(2n)!} E_{2n} \left( \frac{1}{2} \right) f^{(2n)}(\xi). \end{aligned}$$

From (1.2) and (2.3) we get

$$\begin{aligned} & f \left( \frac{\alpha + \beta}{2} \right) - \frac{f(\alpha) + f(\beta)}{2} \\ & + \sum_{k=1}^{n-1} (\beta - \alpha)^{2k} \left[ f^{(2k)}(\alpha) + f^{(2k)}(\beta) \right] \frac{2^{2k+1}}{(2k + 1)!} B_{2k+1} \left( \frac{1}{4} \right) \\ & = -(\beta - \alpha)^{2n-1} \frac{2^{2n+2}}{(2n + 1)!} B_{2n+1} \left( \frac{1}{4} \right) f^{(2n)}(\xi). \end{aligned} \tag{3.4}$$

By subtracting the identities (3.3) and (3.4), we get that for  $(2n)$ -convex function the first inequality in (3.2) is valid when

$$\frac{2^{2n+2} - 1}{2n + 2} B_{2n+2} + 2^{2n} B_{2n+1} \left( \frac{1}{4} \right) \geq 0. \tag{3.5}$$

Using the expansion of Bernoulli polynomials in Fourier series we have

$$B_{2k+1} \left( \frac{1}{4} \right) = (-1)^{k-1} (2k + 1)! 2^{-2k} \pi^{-2k-1} (1 - 3^{-2k-1} + 5^{-2k-1} - \dots)$$

and

$$B_{2k} = (-1)^{k-1} (2k)! 2^{-2k+1} \pi^{-2k} (1 + 2^{-2k} + 3^{-2k} + \dots).$$

Now, for  $n = 2m + 1, m \in \mathbb{N}_0$  in (3.5), we get

$$\begin{aligned} & 2^{4m+2} B_{4m+3} \left( \frac{1}{4} \right) + \frac{2^{4m+4} - 1}{4m + 4} B_{4m+4} \\ & = 2^{4m+2} (4m + 3)! 2^{-4m-2} \pi^{-4m-3} (1 - 3^{-4m-3} + 5^{-4m-3} - \dots) \\ & \quad - \frac{2^{4m+4} - 1}{4m + 4} (4m + 4)! 2^{-4m-3} \pi^{-4m-4} (1 + 2^{-4m-4} + 3^{-4m-4} + \dots) \\ & \geq (4m + 3)! \pi^{-4m-4} (\pi - \pi 3^{-4m-3} - 2) \geq 0. \end{aligned}$$

So, the first inequality also valid.  $\square$

**4.  $n$ -exponential convexity**

Motivated by the results from Theorem 3, we define functionals  $\Phi_2(f)$  and  $\Phi_3(f)$  by

$$\begin{aligned} \Phi_2(f) &= \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} f(x) dx - f\left(\frac{\alpha + \beta}{2}\right) \\ &\quad - \sum_{k=1}^{n-1} \frac{(\beta - \alpha)^{2k}}{(2k + 1)!} \left[ f^{(2k)}(\alpha) + f^{(2k)}(\beta) \right] \left( \frac{2^{2k+3} - 2}{2k + 2} B_{2k+2} + 2^{2k+1} B_{2k+1} \left(\frac{1}{4}\right) \right) \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} \Phi_3(f) &= \frac{1}{2} [f(\alpha) + f(\beta)] - \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} f(x) dx \\ &\quad + \sum_{k=1}^{n-1} \frac{(\beta - \alpha)^{2k}}{(2k + 2)!} \left[ f^{(2k)}(\alpha) + f^{(2k)}(\beta) \right] (2^{2k+3} - 2) B_{2k+2}. \end{aligned} \tag{4.2}$$

**THEOREM 4.** *Let  $f : [\alpha, \beta] \rightarrow \mathbb{R}$ ,  $f \in C^{(2n)}([\alpha, \beta])$ ,  $n \in \mathbb{N}$ . Then there exists  $\xi \in [\alpha, \beta]$  such that*

$$\Phi_i(f) = f^{(2n)}(\xi) \Phi_i(\varphi), \quad i = 1, 2, 3, \tag{4.3}$$

where  $\varphi(x) = \frac{x^{2n}}{(2n)!}$ .

*Proof.* Let us denote  $m = \min f^{(2n)}$  and  $M = \max f^{(2n)}$ . We first consider the following function  $\phi_1(x) = \frac{Mx^{2n}}{(2n)!} - f(x)$ . Then  $\phi_1^{(2n)}(x) = M - f^{(2n)}(x) \geq 0$ ,  $x \in [\alpha, \beta]$ , so  $\phi_1$  is a  $(2n)$ -convex function. Similarly, a function  $\phi_2(x) = f(x) - \frac{mx^{2n}}{(2n)!}$  is a  $(2n)$ -convex function. Now, we use inequalities from (2.9) and (3.2) for  $(2n)$ -convex functions  $\phi_1$  and  $\phi_2$ . So, we can conclude that there exists  $\xi \in [\alpha, \beta]$  that we are looking for in (4.3).  $\square$

**COROLLARY 2.** *Let  $f, h : [\alpha, \beta] \rightarrow \mathbb{R}$  such that  $f, h \in C^{(2n)}([\alpha, \beta])$ . Then there exists  $\xi \in [\alpha, \beta]$  such that*

$$\frac{\Phi_i(f)}{\Phi_i(h)} = \frac{f^{(2n)}(\xi)}{h^{(2n)}(\xi)}, \quad i = 1, 2, 3, \tag{4.4}$$

provided that the denominator of the left-hand side is non-zero.

*Proof.* We use the following standard technique: Let us define the linear functional  $L(\chi) = \Phi_i(\chi)$ ,  $i = 1, 2, 3$ . Next, we define  $\chi(t) = f(t)L(h) - h(t)L(f)$ . According to Theorems 4, applied on  $\chi$ , there exists  $\xi \in [\alpha, \beta]$  so that

$$L(\chi) = \chi^{(2n)}(\xi) \Phi_i(\varphi), \quad \varphi(x) = \frac{x^{2n}}{(2n)!}, \quad i = 1, 2, 3.$$

From  $L(\chi) = 0$ , it follows  $f^{(2n)}(\xi)L(h) - h^{(2n)}(\xi)L(f) = 0$  and (4.4) is proved.  $\square$

Now, let us recall some definitions and facts about exponentially convex functions (see [4]):

DEFINITION 4. A function  $\psi : I \rightarrow \mathbb{R}$  is  $n$ -exponentially convex in the Jensen sense on  $I$  if

$$\sum_{i,j=1}^n \xi_i \xi_j \psi \left( \frac{x_i + x_j}{2} \right) \geq 0,$$

hold for all choices  $\xi_1, \dots, \xi_n \in \mathbb{R}$  and all choices  $x_1, \dots, x_n \in I$ .

A function  $\psi : I \rightarrow \mathbb{R}$  is  $n$ -exponentially convex if it is  $n$ -exponentially convex in the Jensen sense and continuous on  $I$ .

REMARK 2. It is clear from the definition that 1-exponentially convex functions in the Jensen sense are in fact nonnegative functions. Also,  $n$ -exponentially convex function in the Jensen sense are  $k$ -exponentially convex in the Jensen sense for every  $k \in \mathbb{N}$ ,  $k \leq n$ .

By definition of positive semi-definite matrices and some basic linear algebra we have the following proposition:

PROPOSITION 1. If  $\psi$  is an  $n$ -exponentially convex in the Jensen sense, then the matrix  $\left[ \psi \left( \frac{x_i + x_j}{2} \right) \right]_{i,j=1}^k$  is positive semi-definite matrix for all  $k \in \mathbb{N}$ ,  $k \leq n$ . Particularly,  $\det \left[ \psi \left( \frac{x_i + x_j}{2} \right) \right]_{i,j=1}^k \geq 0$  for all  $k \in \mathbb{N}$ ,  $k \leq n$ .

DEFINITION 5. A function  $\psi : I \rightarrow \mathbb{R}$  is exponentially convex in the Jensen sense on  $I$  if it is  $n$ -exponentially convex in the Jensen sense for all  $n \in \mathbb{N}$ .

A function  $\psi : I \rightarrow \mathbb{R}$  is exponentially convex if it is exponentially convex in the Jensen sense and continuous.

REMARK 3. It is known (and easy to show) that  $\psi : I \rightarrow \mathbb{R}$  is a log-convex in the Jensen sense if and only if

$$\alpha^2 \psi(x) + 2\alpha\beta \psi \left( \frac{x+y}{2} \right) + \beta^2 \psi(y) \geq 0$$

holds for every  $\alpha, \beta \in \mathbb{R}$  and  $x, y \in I$ . It follows that a positive function is log-convex in the Jensen sense if and only if it is 2-exponentially convex in the Jensen sense.

A positive function is log-convex if and only if it is 2-exponentially convex.

PROPOSITION 2. If  $f$  is a convex function on  $I$  and if  $x_1 \leq y_1$ ,  $x_2 \leq y_2$ ,  $x_1 \neq x_2$ ,  $y_1 \neq y_2$ , then the following inequality is valid

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(y_2) - f(y_1)}{y_2 - y_1}.$$

If the function  $f$  is concave, the inequality is reversed.

We use an idea from [4] to give an elegant method of producing an  $n$ -exponentially convex functions and exponentially convex functions applying the above functionals on a given family with the same property (see [7])

**THEOREM 5.** *Let  $\Upsilon = \{f_s : s \in J\}$ , where  $J$  an interval in  $\mathbb{R}$ , be a family of functions defined on an interval  $[\alpha, \beta]$  in  $\mathbb{R}$ , such that the function  $s \mapsto f_s[z_0, \dots, z_{2l}]$  is  $n$ -exponentially convex in the Jensen sense on  $J$  for every  $(2l + 1)$  mutually different points  $z_0, \dots, z_{2l} \in [\alpha, \beta]$ . Let  $\Phi_i(f)$ ,  $i = 1, 2, 3$  be linear functional defined as in (2.8), (4.1) and (4.2). Then  $s \mapsto \Phi_i(f_s)$  is an  $n$ -exponentially convex function in the Jensen sense on  $J$ . If the function  $s \mapsto \Phi_i(f_s)$  is continuous on  $J$ , then it is  $n$ -exponentially convex on  $J$ .*

*Proof.* For  $\xi_i \in \mathbb{R}$ ,  $i = 1, \dots, n$  and  $s_i \in J$ ,  $i = 1, \dots, n$ , we define the function

$$h(z) = \sum_{i,j=1}^n \xi_i \xi_j f_{\frac{s_i+s_j}{2}}(z).$$

Using the assumption that the function  $s \mapsto f_s[z_0, \dots, z_{2l}]$  is  $n$ -exponentially convex in the Jensen sense, we have

$$h[z_0, \dots, z_{2l}] = \sum_{i,j=1}^n \xi_i \xi_j f_{\frac{s_i+s_j}{2}}[z_0, \dots, z_{2l}] \geq 0,$$

which in turn implies that  $h$  is a  $(2l)$ -convex function on  $J$ , so it is  $\Phi_k(h) \geq 0$ ,  $k = 1, 2, 3$ , hence

$$\sum_{i,j=1}^n \xi_i \xi_j \Phi_k \left( f_{\frac{s_i+s_j}{2}} \right) \geq 0.$$

We conclude that the function  $s \mapsto \Phi_k(f_s)$  is  $n$ -exponentially convex on  $J$  in the Jensen sense.

If the function  $s \mapsto \Phi_k(f_s)$  is also continuous on  $J$ , then  $s \mapsto \Phi_k(f_s)$  is  $n$ -exponentially convex by definition.  $\square$

The following corollaries are an immediate consequences of the above theorem:

**COROLLARY 3.** *Let  $\Upsilon = \{f_s : s \in J\}$ , where  $J$  an interval in  $\mathbb{R}$ , be a family of functions defined on an interval  $[\alpha, \beta]$  in  $\mathbb{R}$ , such that the function  $s \mapsto f_s[z_0, \dots, z_{2l}]$  is exponentially convex in the Jensen sense on  $J$  for every  $(2l + 1)$  mutually different points  $z_0, \dots, z_{2l} \in [\alpha, \beta]$ . Let  $\Phi_i(f)$ ,  $i = 1, 2, 3$  be linear functional defined as in (2.8), (4.1) and (4.2). Then  $s \mapsto \Phi_i(f_s)$  is an exponentially convex function in the Jensen sense on  $J$ . If the function  $s \mapsto \Phi_i(f_s)$  is continuous on  $J$ , then it is exponentially convex on  $J$ .*

**COROLLARY 4.** *Let  $\Upsilon = \{f_s : s \in J\}$ , where  $J$  an interval in  $\mathbb{R}$ , be a family of functions defined on an interval  $[\alpha, \beta]$  in  $\mathbb{R}$ , such that the function  $s \mapsto f_s[z_0, \dots, z_{2l}]$  is 2-exponentially convex in the Jensen sense on  $J$  for every  $(2l + 1)$  mutually different points  $z_0, \dots, z_{2l} \in [\alpha, \beta]$ . Let  $\Phi_i(f)$ ,  $i = 1, 2, 3$  be linear functional defined as in (2.8), (4.1) and (4.2). Then the following statements hold:*

(i) If the function  $s \mapsto \Phi_i(f_s)$  is continuous on  $J$ , then it is 2-exponentially convex function on  $J$ . If  $s \mapsto \Phi_i(f_s)$  is additionally strictly positive, then it is also log-convex on  $J$ . Furthermore, the following inequality holds true:

$$[\Phi_i(f_s)]^{t-r} \leq [\Phi_i(f_r)]^{t-s} [\Phi_i(f_t)]^{s-r} \tag{4.5}$$

for every choice  $r, s, t \in J$ , such that  $r < s < t$ .

(ii) If the function  $s \mapsto \Phi_i(f_s)$  is strictly positive and differentiable on  $J$ , then for every  $s, q, u, v \in J$ , such that  $s \leq u$  and  $q \leq v$ , we have

$$\mu_{s,q}(\Phi_i, \Upsilon) \leq \mu_{u,v}(\Phi_i, \Upsilon), \tag{4.6}$$

where

$$\mu_{s,q}(\Phi_i, \Upsilon) = \begin{cases} \left( \frac{\Phi_i(f_s)}{\Phi_i(f_q)} \right)^{\frac{1}{s-q}}, & s \neq q, \\ \exp \left( \frac{d}{ds} \frac{\Phi_i(f_s)}{\Phi_i(f_q)} \right), & s = q, \end{cases} \tag{4.7}$$

for  $f_s, f_q \in \Upsilon$ .

*Proof.*

(i) This is an immediate consequence of Theorem 5 and Remark 3.

(ii) Since by (i) the function  $s \mapsto \Phi_i(f_s)$ ,  $i = 1, 2, 3$  is log-convex on  $J$ , that is, the function  $s \mapsto \log \Phi_i(f_s)$  is convex on  $J$ . So, we get

$$\frac{\log \Phi_i(f_s) - \log \Phi_i(f_q)}{s - q} \leq \frac{\log \Phi_i(f_u) - \log \Phi_i(f_v)}{u - v} \tag{4.8}$$

for  $s \leq u, q \leq v, s \neq q, u \neq v$ , and there from conclude that

$$\mu_{s,q}(\Phi_i, \Upsilon) \leq \mu_{u,v}(\Phi_i, \Upsilon).$$

Cases  $s = q$  and  $u = v$  follows from (4.8) as limit cases.  $\square$

REMARK 4. Note that the results from above theorem and corollaries still hold when two of the points  $z_0, \dots, z_{2l} \in [\alpha, \beta]$  coincide, say  $z_1 = z_0$ , for a family of differentiable functions  $f_s$  such that the function  $s \mapsto f_s[z_0, \dots, z_{2l}]$  is  $n$ -exponentially convex in the Jensen sense (exponentially convex in the Jensen sense, log-convex in the Jensen sense), and furthermore, they still hold when all  $(2l + 1)$  points coincide for a family of  $2l$  differentiable functions with the same property. The proofs are obtained by (1.5) and suitable characterization of convexity.

### 5. Applications to Stolarsky type means

In this section, we present several families of functions which fulfil the conditions of Theorem 5, Corollary 3, Corollary 4, and Remark 4. This enable us to construct a large families of functions which are exponentially convex. For a discussion related to this problem see [3].

EXAMPLE 1. Consider a family of functions

$$\Omega_1 = \{l_s : \mathbb{R} \rightarrow [0, \infty) : s \in \mathbb{R}\}$$

defined by

$$l_s(x) = \begin{cases} \frac{e^{sx}}{s^{2n}}, & s \neq 0, \\ \frac{x^{2n}}{(2n)!}, & s = 0. \end{cases}$$

We have  $\frac{d^{2n}l_s}{dx^{2n}}(x) = e^{sx} > 0$  which shows that  $l_s$  is  $(2n)$ -convex on  $\mathbb{R}$  for every  $s \in \mathbb{R}$  and  $s \mapsto \frac{d^{2n}l_s}{dx^{2n}}(x)$  is exponentially convex by definition. Using analogous arguing as in the proof of Theorem 5 we also have that  $s \mapsto l_s[z_0, \dots, z_{2n}]$  is exponentially convex (and so exponentially convex in the Jensen sense). Using Corollary 3 we conclude that  $s \mapsto \Phi_i(l_s)$ ,  $i = 1, 2, 3$  are exponentially convex in the Jensen sense. It is easy to verify that this mapping is continuous (although mapping  $s \mapsto l_s$  is not continuous for  $s = 0$ ), so it is exponentially convex.

For this family of functions,  $\mu_{s,q}(\Phi_i, \Omega_1)$ ,  $i = 1, 2, 3$  from (4.7), becomes

$$\mu_{s,q}(\Phi_i, \Omega_1) = \begin{cases} \left(\frac{\Phi_i(l_s)}{\Phi_i(l_q)}\right)^{\frac{1}{s-q}}, & s \neq q, \\ \exp\left(\frac{\Phi_i(id \cdot l_s)}{\Phi_i(l_s)} - \frac{2n}{s}\right), & s = q \neq 0, \\ \exp\left(\frac{1}{2n+1} \frac{\Phi_i(id \cdot l_0)}{\Phi_i(l_0)}\right), & s = q = 0. \end{cases}$$

Now, using (4.6) it is monotonous function in parameters  $s$  and  $q$ . For  $i = 1, 2, 3$  we have

$$\Phi_1(l_s) = \frac{e^{sx}}{s^{2n}} - \sum_{k=0}^{n-1} \frac{(b-a)^{2k}}{s^{2n-2k}} \left[ e^{sa} \Lambda_k \left( \frac{b-x}{b-a} \right) + e^{sb} \Lambda_k \left( \frac{x-a}{b-a} \right) \right],$$

$$\Phi_1(id \cdot l_s) = \frac{x e^{sx}}{s^{2n}} - \sum_{k=0}^{n-1} \frac{(b-a)^{2k}}{s^{2n-2k+1}} \left[ e^{sa} (2k+as) \Lambda_k \left( \frac{b-x}{b-a} \right) + e^{sb} (2k+bs) \Lambda_k \left( \frac{x-a}{b-a} \right) \right],$$

$$\Phi_1(l_0) = \frac{x^{2n}}{(2n)!} - \sum_{k=0}^{n-1} \frac{(b-a)^{2k}}{(2n-2k)!} \left[ a^{2n-2k} \Lambda_k \left( \frac{b-x}{b-a} \right) + b^{2n-2k} \Lambda_k \left( \frac{x-a}{b-a} \right) \right],$$

$$\Phi_1(id \cdot l_0) = \frac{x^{2n+1}}{(2n)!} - \sum_{k=0}^{n-1} \frac{(b-a)^{2k} (2n+1)}{(2n-2k+1)!} \left[ a^{2n-2k+1} \Lambda_k \left( \frac{b-x}{b-a} \right) + b^{2n-2k+1} \Lambda_k \left( \frac{x-a}{b-a} \right) \right],$$

$$\begin{aligned} \Phi_2(l_s) &= \frac{e^{sb} - e^{sa}}{(b-a)s^{2n+1}} - \frac{1}{s^{2n}} e^{s\left(\frac{a+b}{2}\right)} \\ &\quad - \sum_{k=1}^{n-1} \frac{(b-a)^{2k} (e^{sb} + e^{sa})}{(2k+1)!s^{2n-2k}} \left( \frac{2^{2k+3} - 2}{2k+2} B_{2k+2} + 2^{2k+1} B_{2k+1} \left( \frac{1}{4} \right) \right), \end{aligned}$$

$$\begin{aligned} \Phi_2(id \cdot l_s) &= \frac{1}{(b-a)s^{2n+1}} \left( be^{sb} - ae^{sa} - \frac{1}{s} (e^{sb} - e^{sa}) \right) - \frac{a+b}{2s^{2n}} e^{s\left(\frac{a+b}{2}\right)} \\ &\quad - \sum_{k=1}^{n-1} \frac{(b-a)^{2k}}{(2k+1)!s^{2n-2k}} \left[ ae^{sa} + be^{sb} + \frac{2k}{s} (e^{sb} + e^{sa}) \right] \\ &\quad \left( \frac{2^{2k+3} - 2}{2k+2} B_{2k+2} + 2^{2k+1} B_{2k+1} \left( \frac{1}{4} \right) \right), \end{aligned}$$

$$\begin{aligned} \Phi_2(l_0) &= \frac{b^{2n+1} - a^{2n+1}}{(b-a)(2n+1)!} - \frac{1}{(2n)!} \left( \frac{a+b}{2} \right)^{2n} - \sum_{k=1}^{n-1} \frac{(b-a)^{2k} (a^{2n-2k} + b^{2n-2k})}{(2k+1)!(2n-2k)!} \\ &\quad \left( \frac{2^{2k+3} - 2}{2k+2} B_{2k+2} + 2^{2k+1} B_{2k+1} \left( \frac{1}{4} \right) \right), \end{aligned}$$

$$\begin{aligned} \Phi_2(id \cdot l_0) &= \frac{b^{2n+2} - a^{2n+2}}{(b-a)(2n)!(2n+2)} - \frac{1}{(2n)!} \left( \frac{a+b}{2} \right)^{2n+1} \\ &\quad - \sum_{k=1}^{n-1} \frac{(b-a)^{2k} (2n+1) (a^{2n-2k+1} + b^{2n-2k+1})}{(2k+1)!(2n-2k+1)!} \\ &\quad \times \left( \frac{2^{2k+3} - 2}{2k+2} B_{2k+2} + 2^{2k+1} B_{2k+1} \left( \frac{1}{4} \right) \right), \end{aligned}$$

$$\Phi_3(l_s) = \frac{e^{sa} - e^{sb}}{(b-a)s^{2n+1}} + \sum_{k=0}^{n-1} \frac{(b-a)^{2k} (e^{sa} + e^{sb})}{(2k+2)!s^{2n-2k}} (2^{2k+3} - 2) B_{2k+2},$$

$$\begin{aligned} \Phi_3(id \cdot l_s) &= -\frac{1}{(b-a)s^{2n+1}} \left[ be^{sb} - ae^{sa} - \frac{1}{s} (e^{sb} - e^{sa}) \right] \\ &\quad + \sum_{k=0}^{n-1} \frac{(b-a)^{2k}}{(2k+2)!s^{2n-2k}} \left[ ae^{sa} + be^{sb} + \frac{2k}{s} (e^{sa} + e^{sb}) \right] (2^{2k+3} - 2) B_{2k+2}, \end{aligned}$$

$$\Phi_3(l_0) = \frac{a^{2n+1} - b^{2n+1}}{(b-a)(2n+1)!} + \sum_{k=0}^{n-1} \frac{(b-a)^{2k} (a^{2n-2k} + b^{2n-2k})}{(2k+2)!(2n-2k)!} (2^{2k+3} - 2) B_{2k+2},$$

$$\begin{aligned} \Phi_3(id \cdot l_0) &= \frac{a^{2n+2} - b^{2n+2}}{(b-a)(2n)!(2n+2)} \\ &\quad + \sum_{k=0}^{n-1} \frac{(b-a)^{2k} (2n+1) (a^{2n-2k+1} + b^{2n-2k+1})}{(2k+2)!(2n-2k+1)!} (2^{2k+3} - 2) B_{2k+2}. \end{aligned}$$

We observe here that  $\left( \frac{\frac{d^{2n} l_s}{dx^{2n}}}{\frac{d^{2n} l_q}{dx^{2n}}} \right)^{\frac{1}{s-q}} (\ln x) = x$  so using Corollary 2 it follows that:

$$M_{s,q}(\Phi_i, \Omega_1) = \ln \mu_{s,q}(\Phi_i, \Omega_1), \quad i = 1, 2, 3$$

satisfy

$$\alpha \leq M_{s,q}(\Phi_i, \Omega_1) \leq \beta, \quad i = 1, 2, 3.$$

EXAMPLE 2. Consider a family of functions

$$\Omega_2 = \{f_s : (0, \infty) \rightarrow \mathbb{R} : s \in \mathbb{R}\}$$

defined by

$$f_s(x) = \begin{cases} \frac{x^s}{s(s-1)\dots(s-2n+1)}, & s \notin \{0, 1, \dots, 2n-1\}, \\ \frac{x^j \ln x}{(-1)^{2n-1-j} j! (2n-1-j)!}, & s = j \in \{0, 1, \dots, 2n-1\}. \end{cases}$$

Here,  $\frac{d^{2n} f_s}{dx^{2n}}(x) = x^{s-2n} = e^{(s-2n) \ln x} > 0$  which shows that  $f_s$  is  $(2n)$ -convex for  $x > 0$  and  $s \mapsto \frac{d^{2n} f_s}{dx^{2n}}(x)$  is exponentially convex by definition. Arguing as in Example 1 we get that the mappings  $s \mapsto \Phi_i(f_s)$ ,  $i = 1, 2, 3$  are exponentially convex. In this case we assume that  $[\alpha, \beta] \in \mathbb{R}^+$ . Functions (4.7) now are equal to:

$$\mu_{s,q}(\Phi_i, \Omega_2) = \begin{cases} \left(\frac{\Phi_i(f_s)}{\Phi_i(f_q)}\right)^{\frac{1}{s-q}}, & s \neq q, \\ \exp\left(- (2n-1)! \frac{\Phi_i(f_0 f_s)}{\Phi_i(f_s)} + \sum_{k=0}^{2n-1} \frac{1}{k-s}\right), & s = q \notin \{0, 1, \dots, 2n-1\}, \\ \exp\left(- (2n-1)! \frac{\Phi_i(f_0 f_s)}{2\Phi_i(f_s)} + \sum_{\substack{k=0 \\ k \neq s}}^{2n-1} \frac{1}{k-s}\right), & s = q \in \{0, 1, \dots, 2n-1\}. \end{cases}$$

For  $i = 1, 2, 3$  and  $s \notin \{0, 1, \dots, 2n-1\}$  we have

$$\Phi_1(f_s) = \frac{x^s}{s(s-1)\dots(s-2n+1)} - \sum_{k=0}^{n-1} \frac{(b-a)^{2k}}{(s-2k)\dots(s-2n+1)} \left[ a^{s-2k} \Lambda_k\left(\frac{b-x}{b-a}\right) + b^{s-2k} \Lambda_k\left(\frac{x-a}{b-a}\right) \right],$$

$$\begin{aligned} \Phi_1(f_0 \cdot f_s) &= \frac{x^s \ln x}{(-1)^{2n-1} (2n-1)! s(s-1)\dots(s-2n+1)} \\ &- \sum_{k=0}^{n-1} (b-a)^{2k} \left[ \frac{a^{s-2k} \left( \ln a + \sum_{j=0}^{2k+1} \frac{1}{s+2-j} - \frac{1}{s+1} - \frac{1}{s+2} \right) \Lambda_k\left(\frac{b-x}{b-a}\right)}{(-1)^{2n-1} (2n-1)! (s-2k)\dots(s-2n+1)} \right. \\ &\left. + \frac{b^{s-2k} \left( \ln b + \sum_{j=0}^{2k+1} \frac{1}{s+2-j} - \frac{1}{s+1} - \frac{1}{s+2} \right) \Lambda_k\left(\frac{x-a}{b-a}\right)}{(-1)^{2n-1} (2n-1)! (s-2k)\dots(s-2n+1)} \right], \end{aligned}$$

$$\begin{aligned} \Phi_2(f_s) &= \frac{b^{s+1} - a^{s+1}}{(b-a)(s+1)s(s-1)\dots(s-2n+1)} - \frac{\left(\frac{a+b}{2}\right)^s}{s(s-1)\dots(s-2n+1)} \\ &- \sum_{k=1}^{n-1} \frac{(b-a)^{2k} (a^{s-2k} + b^{s-2k})}{(2k+1)!(s-2k)\dots(s-2n+1)} \left( \frac{2^{2k+3}-2}{2k+2} B_{2k+2} + 2^{2k+1} B_{2k+1} \left(\frac{1}{4}\right) \right), \end{aligned}$$

$$\begin{aligned} \Phi_2(f_0 \cdot f_s) &= \frac{b^{s+1}((s+1)\ln b - 1) - a^{s+1}((s+1)\ln a - 1)}{(b-a)(-1)^{2n-1}(2n-1)!s(s-1)\cdots(s-2n+1)(s+1)^2} \\ &\quad - \frac{\left(\frac{a+b}{2}\right)^s \ln\left(\frac{a+b}{2}\right)}{(-1)^{2n-1}(2n-1)!s(s-1)\cdots(s-2n+1)} - \sum_{k=1}^{n-1} \frac{(b-a)^{2k}}{(2k+1)!} \\ &\quad \times \frac{a^{s-2k}\ln a + b^{s-2k}\ln b + \left(\sum_{j=0}^{2k+1} \frac{1}{s+2-j} - \frac{1}{s+1} - \frac{1}{s+2}\right)(a^{s-2k} + b^{s-2k})}{(-1)^{2n-1}(2n-1)!(s-2k)\cdots(s-2n+1)} \\ &\quad \times \left(\frac{2^{2k+3} - 2}{2k+2} B_{2k+2} - 2^{2k+1} B_{2k+1} \left(\frac{1}{4}\right)\right), \\ \Phi_3(f_s) &= \sum_{k=0}^{n-1} \frac{(b-a)^{2k}(a^{s-2k} + b^{s-2k})}{(2k+2)!(s-2k)\cdots(s-2n+1)} (2^{2k+3} - 2) B_{2k+2} \\ &\quad - \frac{b^{s+1} - a^{s+1}}{(b-a)(s+1)s(s-1)\cdots(s-2n+1)}, \\ \Phi_3(f_0 \cdot f_s) &= \sum_{k=0}^{n-1} \frac{(b-a)^{2k}}{(2k+2)!} \\ &\quad \times \frac{a^{s-2k}\ln a + b^{s-2k}\ln b + \left(\sum_{j=0}^{2k+1} \frac{1}{s+2-j} - \frac{1}{s+1} - \frac{1}{s+2}\right)(a^{s-2k} + b^{s-2k})}{(-1)^{2n-1}(2n-1)!(s-2k)\cdots(s-2n+1)} \\ &\quad \times \left(2^{2k+3} - 2\right) B_{2k+2} - \frac{b^{s+1}((s+1)\ln b - 1) - a^{s+1}((s+1)\ln a - 1)}{(b-a)(-1)^{2n-1}(2n-1)!s(s-1)\cdots(s-2n+1)(s+1)^2} \end{aligned}$$

For  $s = j \in \{0, 1, \dots, 2n - 1\}$ :

$$\begin{aligned} \Phi_1(f_s) &= \frac{x^s \ln x}{(-1)^{2n-1-s} s! (2n-1-s)!} - \sum_{k=0}^{n-1} (b-a)^{2k} \\ &\quad \times \left[ \frac{a^{s-2k} \left( \ln a + \sum_{j=0}^{2k+1} \frac{1}{s+2-j} - \frac{1}{s+1} - \frac{1}{s+2} \right) \Lambda_k \left( \frac{b-x}{b-a} \right)}{(-1)^{2n-1-s} (s-2k)! (2n-1-s)!} \right. \\ &\quad \left. + \frac{b^{s-2k} \left( \ln b + \sum_{j=0}^{2k+1} \frac{1}{s+2-j} - \frac{1}{s+1} - \frac{1}{s+2} \right) \Lambda_k \left( \frac{x-a}{b-a} \right)}{(-1)^{2n-1-s} (s-2k)! (2n-1-s)!} \right], \\ \Phi_1(f_0 \cdot f_s) &= \frac{x^s \ln^2(x)}{(-1)^s (2n-1)! s! (2n-1-s)!} - \sum_{k=0}^{n-1} \frac{(b-a)^{2k}}{(-1)^s (2n-1)! (s-2k)! (2n-1-s)!} \\ &\quad \times \left\{ \left[ a^{s-2k} \ln^2 a + 2 \left( \sum_{j=0}^{2k+1} \frac{1}{s+2-j} - \frac{1}{s+1} - \frac{1}{s+2} \right) a^{s-2k} \ln a \right. \right. \\ &\quad \left. \left. + \left( \sum_{j=0}^{2k+1} \sum_{\substack{l=0 \\ l \neq j}}^{2k+1} \frac{1}{(s+2-j)(s+2-l)} - 2 \left( \frac{1}{s+1} + \frac{1}{s+2} \right) \right) \sum_{j=0}^{2k+1} \frac{1}{s+2-j} \right] \right\} \end{aligned}$$

$$\begin{aligned}
 &+ 2 \left( \frac{1}{(s+1)^2} + \frac{1}{(s+1)(s+2)} + \frac{1}{(s+2)^2} \right) a^{s-2k} \Big] \Lambda_k \left( \frac{b-x}{b-a} \right) \\
 &+ \left[ b^{s-2k} \ln^2 b + 2 \left( \sum_{j=0}^{2k+1} \frac{1}{s+2-j} - \frac{1}{s+1} - \frac{1}{s+2} \right) b^{s-2k} \ln b \right. \\
 &+ \left( \sum_{j=0}^{2k+1} \sum_{\substack{l=0 \\ l \neq j}}^{2k+1} \frac{1}{(s+2-j)(s+2-l)} - 2 \left( \frac{1}{s+1} + \frac{1}{s+2} \right) \sum_{j=0}^{2k+1} \frac{1}{s+2-j} \right. \\
 &\left. \left. + 2 \left( \frac{1}{(s+1)^2} + \frac{1}{(s+1)(s+2)} + \frac{1}{(s+2)^2} \right) \right) b^{s-2k} \Big] \Lambda_k \left( \frac{x-a}{b-a} \right) \Big\}, \\
 \Phi_2(f_s) &= \frac{b^{s+1}((s+1)\ln b - 1) - a^{s+1}((s+1)\ln a - 1)}{(b-a)(-1)^{2n-1-s}s!(2n-1-s)!(s+1)^2} \\
 &- \frac{\left(\frac{a+b}{2}\right)^s \ln\left(\frac{a+b}{2}\right)}{(-1)^{2n-1-s}s!(2n-1-s)!} - \sum_{k=1}^{n-1} \frac{(b-a)^{2k}}{(2k+1)!} \\
 &\times \frac{a^{s-2k} \ln a + b^{s-2k} \ln b + \left(\sum_{j=0}^{2k+1} \frac{1}{s+2-j} - \frac{1}{s+1} - \frac{1}{s+2}\right) (a^{s-2k} + b^{s-2k})}{(-1)^{2n-1-s}(s-2k)!(2n-1-s)!} \\
 &\times \left( \frac{2^{2k+3} - 2}{2k+2} B_{2k+2} - 2^{2k+1} B_{2k+1} \left(\frac{1}{4}\right) \right), \\
 \Phi_2(f_0 \cdot f_s) &= \frac{b^{s+1}((s+1)^2 \ln^2 b - 2(s+1)\ln b + 2) - a^{s+1}((s+1)^2 \ln^2 a - 2(s+1)\ln a + 2)}{(b-a)(-1)^s(2n-1)!s!(2n-1-s)!(s+1)^3} \\
 &- \frac{\left(\frac{a+b}{2}\right)^s \ln^2\left(\frac{a+b}{2}\right)}{(-1)^s(2n-1)!s!(2n-1-s)!} - \sum_{k=1}^{n-1} \frac{(b-a)^{2k}}{(2k+1)!} \left\{ b^{s-2k} \ln^2 b + a^{s-2k} \ln^2 a \right. \\
 &+ 2 \left( \sum_{j=0}^{2k+1} \frac{1}{s+2-j} - \frac{1}{s+1} - \frac{1}{s+2} \right) (a^{s-2k} \ln a + b^{s-2k} \ln b) \\
 &+ \left( \sum_{j=0}^{2k+1} \sum_{\substack{l=0 \\ l \neq j}}^{2k+1} \frac{1}{(s+2-j)(s+2-l)} - 2 \left( \frac{1}{s+1} + \frac{1}{s+2} \right) \sum_{j=0}^{2k+1} \frac{1}{s+2-j} \right. \\
 &\left. \left. + 2 \left( \frac{1}{(s+1)^2} + \frac{1}{(s+1)(s+2)} + \frac{1}{(s+2)^2} \right) \right) (a^{s-2k} + b^{s-2k}) \right\} \\
 &\times \frac{1}{(-1)^s(2n-1)!(s-2k)!(2n-1-s)!} \left( \frac{2^{2k+3} - 2}{2k+2} B_{2k+2} + 2^{2k+1} B_{2k+1} \left(\frac{1}{4}\right) \right), \\
 \Phi_3(f_s) &= \sum_{k=0}^{n-1} \frac{(b-a)^{2k}}{(2k+2)!} \cdot \frac{a^{s-2k} \ln a + b^{s-2k} \ln b + \left(\sum_{j=0}^{2k+1} \frac{1}{s+2-j} - \frac{1}{s+1} - \frac{1}{s+2}\right) (a^{s-2k} + b^{s-2k})}{(-1)^{2n-1-s}(s-2k)!(2n-1-s)!} \\
 &\times \left( 2^{2k+3} - 2 \right) B_{2k+2} - \frac{b^{s+1}((s+1)\ln b - 1) - a^{s+1}((s+1)\ln a - 1)}{(b-a)(-1)^{2n-1-s}s!(2n-1-s)!(s+1)^2},
 \end{aligned}$$

$$\begin{aligned} \Phi_3(f_0 \cdot f_s) &= \sum_{k=0}^{n-1} \frac{(b-a)^{2k}}{(2k+2)!} \left\{ a^{s-2k} \ln^2 a + b^{s-2k} \ln^2 b \right. \\ &\quad + 2 \left( \sum_{j=0}^{2k+1} \frac{1}{s+2-j} - \frac{1}{s+1} - \frac{1}{s+2} \right) (a^{s-2k} \ln a + b^{s-2k} \ln b) \\ &\quad + \left( \sum_{j=0}^{2k+1} \sum_{\substack{l=0 \\ l \neq j}}^{2k+1} \frac{1}{(s+2-j)(s+2-l)} - 2 \left( \frac{1}{s+1} + \frac{1}{s+2} \right) \sum_{j=0}^{2k+1} \frac{1}{s+2-j} \right. \\ &\quad \left. \left. + 2 \left( \frac{1}{(s+1)^2} + \frac{1}{(s+1)(s+2)} + \frac{1}{(s+2)^2} \right) \right) (a^{s-2k} + b^{s-2k}) \right\} \\ &\quad \times \frac{(2^{2k+3} - 2) B_{2k+2}}{(-1)^s (2n-1)! (s-2k)! (2n-1-s)!} \\ &\quad - \frac{b^{s+1} ((s+1)^2 \ln^2 b - 2(s+1) \ln b + 2) - a^{s+1} ((s+1)^2 \ln^2 a - 2(s+1) \ln a + 2)}{(b-a)(-1)^s (2n-1)! s! (2n-1-s)! (s+1)^3}. \end{aligned}$$

We observe that  $\left( \frac{d^{2n} f_s}{dx^{2n}} \right)^{\frac{1}{s-q}}(x) = x$ , so if  $\Phi_i$  ( $i = 1, 2, 3$ ) are positive, then Corollary 2 yield that there exist some  $\xi_i \in [\alpha, \beta]$ ,  $i = 1, 2, 3$  such that

$$\xi_i^{s-q} = \frac{\Phi_i(f_s)}{\Phi_i(f_q)}, \quad i = 1, 2, 3.$$

Since the function  $\xi \mapsto \xi^{s-q}$  is invertible for  $s \neq q$ , we then have

$$\alpha \leq \left( \frac{\Phi_i(f_s)}{\Phi_i(f_q)} \right)^{\frac{1}{s-q}} \leq \beta, \quad i = 1, 2, 3. \tag{5.1}$$

EXAMPLE 3. Consider a family of functions

$$\Omega_3 = \{h_s : (0, \infty) \rightarrow (0, \infty) : s \in (0, \infty)\}$$

defined by

$$h_s(x) = \begin{cases} \frac{s^{-x}}{(\ln s)^{2n}}, & s \neq 1 \\ \frac{x^{2n}}{(2n)!}, & s = 1. \end{cases}$$

Since  $\frac{d^{2n} h_s}{dx^{2n}}(x) = s^{-x}$  is the Laplace transform of a non-negative function (see [12]) it is exponentially convex. Obviously  $h_s$  are  $(2n)$ -convex functions for every  $s > 0$ . For this family of functions,  $\mu_{s,q}(\Phi_i, \Omega_3)$ ,  $i = 1, 2, 3$ , in this case for  $[\alpha, \beta] \in \mathbb{R}^+$ , from (4.7) becomes

$$\mu_{s,q}(\Phi_i, \Omega_3) = \begin{cases} \left( \frac{\Phi_i(h_s)}{\Phi_i(h_q)} \right)^{\frac{1}{s-q}}, & s \neq q, \\ \exp \left( -\frac{\Phi_i(id \cdot h_s)}{s \Phi_i(h_s)} - \frac{2n}{s \ln s} \right), & s = q \neq 1, \\ \exp \left( -\frac{1}{2n+1} \frac{\Phi_i(id \cdot h_1)}{\Phi_i(h_1)} \right), & s = q = 1. \end{cases}$$

These are monotonous function in parameters  $s$  and  $q$  by (4.6). For  $i = 1, 2, 3$  we have

$$\begin{aligned} \Phi_1(h_s) &= \frac{s^{-x}}{(\ln s)^{2n}} - \sum_{k=0}^{n-1} \frac{(b-a)^{2k}}{(\ln s)^{2n-2k}} \left[ s^{-a} \Lambda_k \left( \frac{b-x}{b-a} \right) + s^{-b} \Lambda_k \left( \frac{x-a}{b-a} \right) \right], \\ \Phi_1(id \cdot h_s) &= \frac{xs^{-x}}{(\ln s)^{2n}} - \sum_{k=0}^{n-1} \frac{(b-a)^{2k} (2k-a \ln s) s^{-a} \Lambda_k \left( \frac{b-x}{b-a} \right) + (2k-b \ln s) s^{-b} \Lambda_k \left( \frac{x-a}{b-a} \right)}{(-\ln s)^{2n-2k+1}}, \\ \Phi_1(h_1) &= \Phi_1(l_0), \quad \Phi_1(id \cdot h_1) = \Phi_1(id \cdot l_0), \\ \Phi_2(h_s) &= \frac{s^{-b} - s^{-a}}{(b-a)(-\ln s)^{2n+1}} - \frac{s^{-\frac{a+b}{2}}}{(\ln s)^{2n}} \\ &\quad - \sum_{k=1}^{n-1} \frac{(b-a)^{2k} (s^{-a} + s^{-b})}{(2k+1)! (\ln s)^{2n-2k}} \left( \frac{2^{2k+3} - 2}{2k+2} B_{2k+2} + 2^{2k+1} B_{2k+1} \left( \frac{1}{4} \right) \right) \\ \Phi_2(id \cdot h_s) &= \frac{\ln s (as^{-a} - bs^{-b}) + s^{-a} - s^{-b}}{(b-a)(\ln s)^{2n+2}} - \frac{(a+b)s^{-\frac{a+b}{2}}}{2(\ln s)^{2n}} \\ &\quad - \sum_{k=1}^{n-1} \frac{(b-a)^{2k}}{(2k+1)!} \left[ \frac{2k(s^{-a} + s^{-b}) - \ln s (as^{-a} + bs^{-b})}{(-\ln s)^{2n-2k+1}} \right] \\ &\quad \times \left( \frac{2^{2k+3} - 2}{2k+2} B_{2k+2} + 2^{2k+1} B_{2k+1} \left( \frac{1}{4} \right) \right), \\ \Phi_2(h_1) &= \Phi_2(l_0), \quad \Phi_2(id \cdot h_1) = \Phi_2(id \cdot l_0), \\ \Phi_3(h_s) &= \sum_{k=0}^{n-1} \frac{(b-a)^{2k} (s^{-a} + s^{-b})}{(2k+2)! (\ln s)^{2n-2k}} (2^{2k+3} - 2) B_{2k+2} - \frac{s^{-b} - s^{-a}}{(b-a)(-\ln s)^{2n+1}} \\ \Phi_3(id \cdot h_s) &= \sum_{k=0}^{n-1} \frac{(b-a)^{2k}}{(2k+2)!} \left[ \frac{2k(s^{-a} + s^{-b}) - \ln s (as^{-a} + bs^{-b})}{(-\ln s)^{2n-2k+1}} \right] (2^{2k+3} - 2) B_{2k+2} \\ &\quad - \frac{\ln s (as^{-a} - bs^{-b}) + s^{-a} - s^{-b}}{(b-a)(\ln s)^{2n+2}} \\ \Phi_3(h_1) &= \Phi_3(l_0), \quad \Phi_3(id \cdot h_1) = \Phi_3(id \cdot l_0). \end{aligned}$$

Using Corollary 2 it follows that

$$M_{s,q}(\Phi_i, \Omega_3) = -L(s, q) \ln \mu_{s,q}(\Phi_i, \Omega_3), \quad i = 1, 2, 3$$

satisfy

$$\alpha \leq M_{s,q}(\Phi_i, \Omega_3) \leq \beta, \quad i = 1, 2, 3.$$

$L(s, q)$  is logarithmic mean defined by

$$L(s, q) = \begin{cases} \frac{s-q}{\log s - \log q}, & s \neq q \\ s, & s = q. \end{cases}$$

EXAMPLE 4. Consider a family of functions

$$\Omega_4 = \{k_s : (0, \infty) \rightarrow (0, \infty) : s \in (0, \infty)\}$$

defined by

$$k_s(x) = \frac{e^{-x\sqrt{s}}}{s^n}.$$

Since  $\frac{d^{2n}}{dx^{2n}}(x) = e^{-x\sqrt{s}}$  is the Laplace transform of a non-negative function (see [12]) it is exponentially convex. Obviously  $k_s$  are  $(2n)$ -convex functions for every  $s > 0$ . For this family of functions,  $\mu_{s,q}(\Phi_i, \Omega_4)$ ,  $i = 1, 2, 3$ , in this case for  $[\alpha, \beta] \in \mathbb{R}^+$ , from (4.7) becomes

$$\mu_{s,q}(\Phi_i, \Omega_4) = \begin{cases} \left( \frac{\Phi_i(k_s)}{\Phi_i(k_q)} \right)^{\frac{1}{s-q}}, & s \neq q, \\ \exp\left(-\frac{\Phi_i(id \cdot k_s)}{2\sqrt{s}\Phi_i(k_s)} - \frac{n}{s}\right), & s = q. \end{cases}$$

These are monotonous function in parameters  $s$  and  $q$  by (4.6). For  $i = 1, 2, 3$  we have

$$\begin{aligned} \Phi_1(k_s) &= \frac{e^{-x\sqrt{s}}}{s^n} - \sum_{k=0}^{n-1} \frac{(b-a)^{2k}}{s^{n-k}} \left[ e^{-a\sqrt{s}} \Lambda_k\left(\frac{b-x}{b-a}\right) + e^{-b\sqrt{s}} \Lambda_k\left(\frac{x-a}{b-a}\right) \right], \\ \Phi_1(id \cdot k_s) &= \frac{xe^{-x\sqrt{s}}}{s^n} - \sum_{k=0}^{n-1} (b-a)^{2k} \frac{(2k-\sqrt{s}a)e^{-a\sqrt{s}} \Lambda_k\left(\frac{b-x}{b-a}\right) + (2k-\sqrt{s}b)e^{-b\sqrt{s}} \Lambda_k\left(\frac{x-a}{b-a}\right)}{(-\sqrt{s})^{2n-2k+1}}, \\ \Phi_2(k_s) &= \frac{e^{-b\sqrt{s}} - e^{-a\sqrt{s}}}{(b-a)(-\sqrt{s})^{2n+1}} - \frac{e^{-\frac{a+b}{2}\sqrt{s}}}{s^n} - \sum_{k=1}^{n-1} \frac{(b-a)^{2k}}{(2k+1)!} \left[ \frac{e^{-a\sqrt{s}}}{s^{n-k}} + \frac{e^{-b\sqrt{s}}}{s^{n-k}} \right] \\ &\quad \times \left( \frac{2^{2k+3} - 2}{2k+2} B_{2k+2} + 2^{2k+1} B_{2k+1} \left( \frac{1}{4} \right) \right), \\ \Phi_2(id \cdot k_s) &= \frac{be^{-b\sqrt{s}} - ae^{-a\sqrt{s}} + \frac{1}{\sqrt{s}}(e^{-b\sqrt{s}} - e^{-a\sqrt{s}})}{(b-a)(-\sqrt{s})^{2n+1}} - \frac{(a+b)e^{-\frac{a+b}{2}\sqrt{s}}}{2s^n} \\ &\quad - \sum_{k=1}^{n-1} \frac{(b-a)^{2k}}{(2k+1)!} \left[ \frac{2k(e^{-a\sqrt{s}} + e^{-b\sqrt{s}}) - \sqrt{s}(ae^{-a\sqrt{s}} + be^{-b\sqrt{s}})}{(-\sqrt{s})^{2n-2k+1}} \right] \\ &\quad \times \left( \frac{2^{2k+3} - 2}{2k+2} B_{2k+2} + 2^{2k+1} B_{2k+1} \left( \frac{1}{4} \right) \right), \\ \Phi_3(k_s) &= \sum_{k=0}^{n-1} \frac{(b-a)^{2k}}{(2k+2)!} \left[ \frac{e^{-a\sqrt{s}}}{s^{n-k}} + \frac{e^{-b\sqrt{s}}}{s^{n-k}} \right] (2^{2k+3} - 2) B_{2k+2} - \frac{e^{-b\sqrt{s}} - e^{-a\sqrt{s}}}{(b-a)(-\sqrt{s})^{2n+1}}, \\ \Phi_3(id \cdot k_s) &= \sum_{k=0}^{n-1} \frac{(b-a)^{2k}}{(2k+2)!} \left[ \frac{2k(e^{-a\sqrt{s}} + e^{-b\sqrt{s}}) - \sqrt{s}(ae^{-a\sqrt{s}} + be^{-b\sqrt{s}})}{(-\sqrt{s})^{2n-2k+1}} \right] \\ &\quad \times (2^{2k+3} - 2) B_{2k+2} - \frac{be^{-b\sqrt{s}} - ae^{-a\sqrt{s}} + \frac{1}{\sqrt{s}}(e^{-b\sqrt{s}} - e^{-a\sqrt{s}})}{(b-a)(-\sqrt{s})^{2n+1}}. \end{aligned}$$

Using Corollary 2 it follows that

$$M_{s,q}(\Phi_i, \Omega_4) = -(\sqrt{s} + \sqrt{q}) \ln \mu_{s,q}(\Phi_i, \Omega_4), \quad i = 1, 2, 3$$

satisfy

$$\alpha \leq M_{s,q}(\Phi_i, \Omega_4) \leq \beta, \quad i = 1, 2, 3.$$

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