

EXTENSIONS OF THE HERMITE–HADAMARD INEQUALITY FOR CONVEX FUNCTIONS VIA FRACTIONAL INTEGRALS

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Abstract. The main aim of this paper is to give extension and refinement of the Hermite-Hadamard inequality for convex functions via Riemann-Liouville fractional integrals. We show how to relax the convexity property of the function f . Obtained results in this work involve a larger class of functions.

1. Introduction

Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a convex function and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(t) dt \leq \frac{f(a)+f(b)}{2} \quad (1)$$

is known as the Hermite-Hadamard inequality.

It was published in 1893 and since then many refinements of the Hermite-Hadamard inequality for convex functions have been considered by a number of authors (e.g., [1], [2], [3], [4], [5], [6], [7], and [9]).

M. Z. Sarikaya et al. [8] proved the following interesting inequalities of Hermite-Hadamard type involving Riemann-Liouville fractional integrals.

THEOREM 1.1. ([8]) *Let $f : [a, b] \rightarrow \mathbb{R}$ be a positive function with $a < b$ and $f \in L_1[a, b]$. If f is a convex function on $[a, b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)] \leq \frac{f(a)+f(b)}{2}, \quad (2)$$

with $\alpha > 0$.

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We remark that the symbols $J_{a^+}^\alpha$ and $J_{b^-}^\alpha f$ denote the left-sided and right-sided Riemann-Liouville fractional integrals of the order $\alpha > 0$ which are defined by

$$J_{a^+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \quad x > a,$$

and

$$J_{b^-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt, \quad x < b,$$

respectively. Here, $\Gamma(\alpha)$ is the Gamma function defined by $\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt$.

Apparently, inequality (2) reduces to inequality (1) if we put $\alpha = 1$. In this paper, we present some new refinements of the inequalities (1) and (2).

Obviously, inequality (2) can be rewritten as

$$\begin{aligned} f\left(\frac{a+b}{2}\right) - \frac{f(a)+f(b)}{2} &\leq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)] - \frac{f(a)+f(b)}{2} \\ &\leq 0 \end{aligned}$$

and

$$\begin{aligned} 0 &\leq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)] - f\left(\frac{a+b}{2}\right) \\ &\leq \frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right), \end{aligned}$$

which says

$$\begin{aligned} &\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)] - \frac{f(a)+f(b)}{2} \leq 0 \\ &\leq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)] - f\left(\frac{a+b}{2}\right). \end{aligned}$$

We observe that both inequalities (1) and (2) require function f to be convex. As a consequence, it is natural to assume that f is a twice differentiable function. Consequently, $f'' \geq 0$. Our first result concerns the case when f'' is bounded in $[a, b]$. Namely, we do not require $f'' \geq 0$. That is, we can prove the following result:

THEOREM 1.2. *Let $f : [a, b] \rightarrow R$ be a positive, twice differentiable function with $a < b$ and $f \in L_1[a, b]$. If f'' is bounded in $[a, b]$. Then we have*

$$\begin{aligned} &\frac{m\alpha}{2(b-a)^\alpha} \int_a^{\frac{a+b}{2}} \left(\frac{a+b}{2} - x\right)^2 [(x-a)^{\alpha-1} + (b-x)^{\alpha-1}] dx \\ &\leq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)] - f\left(\frac{a+b}{2}\right) \\ &\leq \frac{M\alpha}{2(b-a)^\alpha} \int_a^{\frac{a+b}{2}} \left(\frac{a+b}{2} - x\right)^2 [(x-a)^{\alpha-1} + (b-x)^{\alpha-1}] dx, \end{aligned} \quad (3)$$

and

$$\begin{aligned} & \frac{-M\alpha}{2(b-a)^\alpha} \int_a^{\frac{a+b}{2}} (x-a)(b-x)[(x-a)^{\alpha-1} + (b-x)^{\alpha-1}] dx \\ & \leq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)] - \frac{f(a)+f(b)}{2} \\ & \leq \frac{-m\alpha}{2(b-a)^\alpha} \int_a^{\frac{a+b}{2}} (x-a)(b-x)[(x-a)^{\alpha-1} + (b-x)^{\alpha-1}] dx, \end{aligned} \quad (4)$$

with $\alpha > 0$, where $m = \inf_{t \in [a,b]} f''(t)$, $M = \sup_{t \in [a,b]} f''(t)$.

REMARK 1. Applying Theorem 1.2 with function f such that $f'' \geq 0$ and we obtain refinements of inequality (2).

REMARK 2. Applying Theorem 1.2 with function f such that $f'' \geq 0$ and $\alpha = 1$, we obtain refinements of inequality (1).

Additionally, it is obvious that inequality $f'' \geq 0$ implies that f' is non-decreasing. Therefore, in the following result, we assume that

$$f'(a+b-x) \geq f'(x), \quad (5)$$

for all $x \in [a, \frac{a+b}{2}]$.

Clearly, if f' is non-decreasing, then inequality (5) holds. However, it is easy to see that the reverse statement is not true. We relax the assumption of Theorem 1.1 by establishing the following result.

THEOREM 1.3. Let $f: [a, b] \rightarrow \mathbb{R}$ be a positive, differentiable function with $a < b$ and $f \in L_1[a, b]$. If $f'(a+b-x) \geq f'(x)$ for all $x \in [a, \frac{a+b}{2}]$. Then the following inequalities for fractional integrals hold

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)] \leq \frac{f(a)+f(b)}{2}, \quad (6)$$

with $\alpha > 0$.

2. Proof of the theorems

Proof of Theorem 1.2. Firstly, we give the proof of Theorem 1.2. We first prove (3). We have

$$\begin{aligned} \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)] &= \frac{\alpha}{2(b-a)^\alpha} \left[\int_a^b (b-x)^{\alpha-1} f(x) dx + \int_a^b (x-a)^{\alpha-1} f(x) dx \right] \\ &= \frac{\alpha}{2(b-a)^\alpha} \int_a^b f(x) [(x-a)^{\alpha-1} + (b-x)^{\alpha-1}] dx \\ &= \frac{\alpha}{2(b-a)^\alpha} \int_a^b f(a+b-x) [(x-a)^{\alpha-1} + (b-x)^{\alpha-1}] dx. \end{aligned}$$

So

$$\begin{aligned} & \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [J_{a^+}^\alpha f(b) + J_b^\alpha f(a)] \\ &= \frac{\alpha}{4(b-a)^\alpha} \int_a^b [f(x) + f(a+b-x)] [(x-a)^{\alpha-1} + (b-x)^{\alpha-1}] dx, \end{aligned} \quad (7)$$

which gives

$$\begin{aligned} & \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [J_{a^+}^\alpha f(b) + J_b^\alpha f(a)] - f\left(\frac{a+b}{2}\right) \\ &= \frac{\alpha}{4(b-a)^\alpha} \int_a^b \left[f(x) + f(a+b-x) - 2f\left(\frac{a+b}{2}\right) \right] [(x-a)^{\alpha-1} + (b-x)^{\alpha-1}] dx. \end{aligned}$$

Since

$$\left[f(x) + f(a+b-x) - 2f\left(\frac{a+b}{2}\right) \right] [(x-a)^{\alpha-1} + (b-x)^{\alpha-1}]$$

is symmetric about $x = \frac{a+b}{2}$, one has

$$\begin{aligned} & \frac{\alpha}{4(b-a)^\alpha} \int_a^b \left[f(x) + f(a+b-x) - 2f\left(\frac{a+b}{2}\right) \right] [(x-a)^{\alpha-1} + (b-x)^{\alpha-1}] dx \\ &= \frac{\alpha}{2(b-a)^\alpha} \int_a^{\frac{a+b}{2}} \left[f(x) + f(a+b-x) - 2f\left(\frac{a+b}{2}\right) \right] [(x-a)^{\alpha-1} + (b-x)^{\alpha-1}] dx, \end{aligned}$$

which implies

$$\begin{aligned} & \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [J_{a^+}^\alpha f(b) + J_b^\alpha f(a)] - f\left(\frac{a+b}{2}\right) \\ &= \frac{\alpha}{2(b-a)^\alpha} \int_a^{\frac{a+b}{2}} \left[f(x) + f(a+b-x) - 2f\left(\frac{a+b}{2}\right) \right] [(x-a)^{\alpha-1} + (b-x)^{\alpha-1}] dx. \end{aligned} \quad (8)$$

Since

$$f(a+b-x) - f\left(\frac{a+b}{2}\right) = \int_{\frac{a+b}{2}}^{a+b-x} f'(t) dt$$

and

$$f\left(\frac{a+b}{2}\right) - f(x) = \int_x^{\frac{a+b}{2}} f'(t) dt,$$

then

$$\begin{aligned} f(x) + f(a+b-x) - 2f\left(\frac{a+b}{2}\right) &= \int_{\frac{a+b}{2}}^{a+b-x} f'(t) dt - \int_x^{\frac{a+b}{2}} f'(t) dt \\ &= \int_x^{\frac{a+b}{2}} f'(a+b-t) dt - \int_x^{\frac{a+b}{2}} f'(t) dt \\ &= \int_x^{\frac{a+b}{2}} [f'(a+b-t) - f'(t)] dt. \end{aligned} \quad (9)$$

Since

$$f'(a+b-t) - f'(t) = \int_t^{a+b-t} f''(y)dy,$$

then for $t \in [a, \frac{a+b}{2}]$, we get

$$m(a+b-2t) \leq f'(a+b-t) - f'(t) \leq M(a+b-2t).$$

Thus,

$$\int_x^{\frac{a+b}{2}} m(a+b-2t)dt \leq f(x) + f(a+b-x) - 2f\left(\frac{a+b}{2}\right) \leq \int_x^{\frac{a+b}{2}} M(a+b-2t)dt.$$

That is

$$m\left(\frac{a+b}{2} - x\right)^2 \leq f(x) + f(a+b-x) - 2f\left(\frac{a+b}{2}\right) \leq M\left(\frac{a+b}{2} - x\right)^2.$$

Then

$$\begin{aligned} & \frac{m\alpha}{2(b-a)^\alpha} \int_a^{\frac{a+b}{2}} \left(\frac{a+b}{2} - x\right)^2 [(x-a)^{\alpha-1} + (b-x)^{\alpha-1}] dx \\ & \leq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)] - f\left(\frac{a+b}{2}\right) \\ & \leq \frac{M\alpha}{2(b-a)^\alpha} \int_a^{\frac{a+b}{2}} \left(\frac{a+b}{2} - x\right)^2 [(x-a)^{\alpha-1} + (b-x)^{\alpha-1}] dx, \end{aligned}$$

which completes the proof of (3). We now prove (4). By using (7), one has

$$\begin{aligned} & \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)] - \frac{f(a) + f(b)}{2} \\ & = \frac{\alpha}{4(b-a)^\alpha} \int_a^b [f(x) + f(a+b-x) - (f(a) + f(b))] [(x-a)^{\alpha-1} + (b-x)^{\alpha-1}] dx. \end{aligned}$$

From

$$\left[f(x) + f(a+b-x) - (f(a) + f(b)) \right] [(x-a)^{\alpha-1} + (b-x)^{\alpha-1}]$$

is symmetric about $x = \frac{a+b}{2}$, one gets

$$\begin{aligned} & \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)] - \frac{f(a) + f(b)}{2} \\ & = \frac{\alpha}{2(b-a)^\alpha} \int_a^{\frac{a+b}{2}} [f(x) + f(a+b-x) - (f(a) + f(b))] [(x-a)^{\alpha-1} + (b-x)^{\alpha-1}] dx. \end{aligned} \tag{10}$$

Since

$$f(b) - f(a+b-x) = \int_{a+b-x}^b f'(t)dt$$

and

$$f(x) - f(a) = \int_a^x f'(t)dt,$$

then we have

$$\begin{aligned} f(x) + f(a+b-x) - (f(a) + f(b)) &= \int_a^x f'(t)dt - \int_{a+b-x}^b f'(t)dt \\ &= \int_a^x f'(t)dt - \int_a^x f'(a+b-t)dt \quad (11) \\ &= - \int_a^x [f'(a+b-t) - f'(t)]dt. \end{aligned}$$

We also have

$$f'(a+b-t) - f'(t) = \int_t^{a+b-t} f''(y)dy.$$

Then for $t \in [a, \frac{a+b}{2}]$, we get

$$m(a+b-2t) \leq f'(a+b-t) - f'(t) \leq M(a+b-2t).$$

Hence

$$- \int_a^x M(a+b-2t)dt \leq f(x) + f(a+b-x) - (f(a) + f(b)) \leq - \int_a^x m(a+b-2t)dt.$$

That is,

$$-M(x-a)(b-x) \leq f(x) + f(a+b-x) - (f(a) + f(b)) \leq -m(x-a)(b-x)$$

and

$$\begin{aligned} &\frac{-M\alpha}{2(b-a)^\alpha} \int_a^{\frac{a+b}{2}} (x-a)(b-x)[(x-a)^{\alpha-1} + (b-x)^{\alpha-1}]dx \\ &\leq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)] - \frac{f(a) + f(b)}{2} \\ &\leq \frac{-m\alpha}{2(b-a)^\alpha} \int_a^{\frac{a+b}{2}} (x-a)(b-x)[(x-a)^{\alpha-1} + (b-x)^{\alpha-1}]dx, \end{aligned}$$

we have completed the proof. \square

Proof of Theorem 1.3. From the assumption of Theorem 1.3, (8) and (9), one has

$$\begin{aligned} &\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)] - f\left(\frac{a+b}{2}\right) \\ &= \frac{\alpha}{2(b-a)^\alpha} \int_a^{\frac{a+b}{2}} \left[f(x) + f(a+b-x) - 2f\left(\frac{a+b}{2}\right) \right] [(x-a)^{\alpha-1} + (b-x)^{\alpha-1}]dx \\ &= \frac{\alpha}{2(b-a)^\alpha} \int_a^{\frac{a+b}{2}} \left[\int_x^{\frac{a+b}{2}} [f'(a+b-t) - f'(t)]dt \right] [(x-a)^{\alpha-1} + (b-x)^{\alpha-1}]dx \\ &\geq 0. \end{aligned}$$

Similarly, from (10) and (11), one gets

$$\begin{aligned} & \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)] - \frac{f(a)+f(b)}{2} \\ &= \frac{\alpha}{2(b-a)^\alpha} \int_a^{\frac{a+b}{2}} \left[-\int_a^x [f'(a+b-t) - f'(t)] dt \right] [(x-a)^{\alpha-1} + (b-x)^{\alpha-1}] dx \\ &\leq 0. \end{aligned}$$

We have completed the proof. \square

3. Conclusion

In this note, we obtain some new Hermite-Hadamard type inequalities for convex functions via Riemann-Liouville fractional integrals. We conclude that the results obtained in this work are the extensions and refinements of the earlier results. An interesting topic is whether we can use the methods in this paper to extend the Hermite-Hadamard inequalities for convex functions on the co-ordinates via Riemann-Liouville fractional integrals.

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