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GENERALIZED ČEBYŠEV AND KY FAN IDENTITIES AND INEQUALITIES

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Abstract. We give generalization of Čebyšev and Ky Fan integral identities and inequalities for functions of two variables by using higher order derivatives. Generalized discrete Čebyšev identity and inequality is also discussed.

1. Introduction

A. M. Ostrowski [1] gave the following generalization of Čebyšev inequality for monotonic functions f and g such that f' and g' are continuous on $[a,b]$ and p is a positive integrable function on $[a,b]$

$$T(f,g,p) = f'(\xi)g'(\eta)T(x-a,x-a,p), \tag{1}$$

where

$$T(f,g,p) = \int_a^b p(x)dx \int_a^b p(x)f(x)g(x)dx - \int_a^b p(x)f(x)dx \int_a^b p(x)g(x)dx. \tag{2}$$

For other generalizations of this result [2] can be seen. In [3] J. Pečarić gave further generalizations of this by using the following functional

$$C(f,p) = \int_a^b \int_a^b p(x,y)f(x,x)dy\,dx - \int_a^b \int_a^b p(x,y)f(x,y)dy\,dx, \tag{3}$$

where p and f are integrable functions on $I^2 = [a,b]\times[a,b]$.

PROPOSITION 1. *Let* $p : I^2 \to \mathbb{R}$ *be an integrable function such that*

$$X(x,x) = \overline{X}(x,x) \quad \forall x \in [a,b]$$

and either

$$X(x,y) \geqslant 0, \quad a \leqslant y \leqslant x \leqslant b; \quad \overline{X}(x,y) \geqslant 0, \quad a \leqslant x \leqslant y \leqslant b$$

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or its reverse inequalities are valid. Where

$$X(x,y) = \int_x^b \int_a^y p(s,t)\,dt\,ds$$

and

$$\overline{X}(x,y) = \int_a^x \int_y^b p(s,t)\,dt\,ds.$$

If $f : I^2 \to \mathbb{R}$ has continuous partial derivatives $f_{(1,0)} = \frac{\partial}{\partial x} f(x,y)$, $f_{(0,1)} = \frac{\partial}{\partial y} f(x,y)$, and $f_{(1,1)} = \frac{\partial^2}{\partial x \partial y} f(x,y)$. Then there exist $\xi, \eta \in [a,b]$ such that

$$C(f,p) = f_{(1,1)}(\xi,\eta)C((x-a)(y-a),p). \tag{4}$$

In [5] Ky Fan considered the inequality

$$\int_a^b \int_a^b w(x,y) f(x) g(y)\,dx\,dy \leqslant B \int_a^b f(x) g(x)\,dx$$

for non-negative and non-increasing functions f and g and some real constant B such that $\int_a^b w(x,y)dx \leqslant B$ for $a \leqslant y \leqslant b$ and also $\int_a^b w(x,y)dy \leqslant B$ for $a \leqslant x \leqslant b$ where $w : [a,b] \times [a,b] \to R$ is an integrable function. For some generalization of this result J. Pečarić [3] considered the following expression

$$K(f,p,q) = \int_a^b q(x) f(x,x)\,dx - \int_a^b \int_a^b p(x,y) f(x,y)\,dx\,dy \tag{5}$$

where f, p, and q are integrable functions and by using this he gave the following result.

PROPOSITION 2. *Let $p : I^2 \to \mathbb{R}$ and $q : I \to \mathbb{R}$ be integrable functions such that*

$$P(a,y) = Q(y), \quad P(x,a) = Q(x), \quad P(x,y) \leqslant Q(max\{x,y\}), \quad \forall x, y \in [a,b]$$

where $Q(x) = \int_x^b q(t)dt$, $\quad P(x,y) = \int_x^b \int_y^b p(s,t)\,dt\,ds$.
If $f : I^2 \to \mathbb{R}$ has the continuous partial derivatives $f_{(1,0)}$, $f_{(0,1)}$, and $f_{(1,1)}$ on I^2. Then

$$K(f,p,q) = f_{(1,1)}(\xi,\eta)K((x-a)(y-a),p,q) \quad \textit{for} \quad \xi,\eta \in [a,b]. \tag{6}$$

Let $p : I^2 \to \mathbb{R}$ and $q : I \to \mathbb{R}$ be two integrable functions. Then we define the following notations for simplification of statements of the theorems:

$$\overline{P}^{(i,j)}(x,y) = \int_x^b \int_y^b p(s,t) \frac{(s-x)^i}{i!} \frac{(s-y)^j}{j!}\,dt\,ds, \tag{7}$$

$$P^{(i,j)}(x,y) = \int_x^b \int_y^b p(s,t) \frac{(s-x)^i}{i!} \frac{(t-y)^j}{j!}\,dt\,ds, \tag{8}$$

$$Q^{(i,j)}(x) = \int_x^b q(s)\frac{(s-x)^i}{i!}\frac{(s-a)^j}{j!}ds, \tag{9}$$

$$\begin{aligned} R(x,y) &= \int_{max\{x,y\}}^b \int_a^b p(s,t)\frac{(s-x)^N}{N!}\frac{(s-y)^M}{M!}dt\,ds - \\ &- \int_x^b \int_y^b p(s,t)\frac{(s-x)^N}{N!}\frac{(t-y)^M}{M!}dt\,ds, \end{aligned} \tag{10}$$

$$\begin{aligned} \overline{R}(x,y) &= \int_{max\{x,y\}}^b q(s)\frac{(s-x)^N}{N!}\frac{(s-y)^M}{M!}ds - \\ &- \int_x^b \int_y^b p(s,t)\frac{(s-x)^N}{N!}\frac{(t-y)^M}{M!}dt\,ds, \end{aligned} \tag{11}$$

$$G_0(x,y) = \frac{(x-a)^{N+1}(y-a)^{M+1}}{(N+1)!(M+1)!}. \tag{12}$$

At this point we are needed a useful definition from [6] and for that we are needed some notations as follows: let $D = [a,b]\times[c,d]$ denote a rectangle in $\mathbb{R}^2$ and $S(D)$ denote the system of all rectangles $[x_1,x_2]\times[y_1,y_2]$ contained in D. Having a function $u : D \to \mathbb{R}$, we put $F_u([x_1,x_2]\times[y_1,y_2]) = u(x_1,y_1) - u(x_2,y_1) - u(x_1,y_2) + u(x_2,y_2)$ for $[x_1,x_2]\times[y_1,y_2] \in S(D)$. The function of rectangles $F_u : S(D) \to \mathbb{R}$ just defined is said to be a function of rectangles associated with u.

DEFINITION 1. We say that a function $u : D \to \mathbb{R}$ is absolutely continuous on D in the sense of Carathéodory if the following two conditions hold:

(a) the function of rectangles F_u associated with u is absolutely continuous, i. e. for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $P_1,\dots,P_k \in S(D)$ are mutually non-overlapping rectangles with the property $\sum_{i=1}^k |P_i| \leqslant \delta$, where $|\cdot|$ denotes the area of a rectangle, then $\sum_{i=1}^k F_u(P_i) \leqslant \varepsilon$

(b) the functions $u(a,\cdot) : [a,b] \to \mathbb{R}$ and $u(\cdot,c) : [c;d] \to \mathbb{R}$ are absolutely continuous.

If $u : D \to \mathbb{R}$ is absolutely continuous in the sense of Carathéodory, then for every $(x,y) \in D$ it admits the integral representation

$$u(x,y) = u(a,c) + \int_a^x u_{(1,0)}(s,c)\,ds + \int_c^y u_{(0,1)}(a,t)\,dt + \int_a^x \int_c^y u_{(1,1)}(s,t)\,dtds, \tag{13}$$

where the partial derivatives in (13) exist almost everywhere (for details see [6]).

Let $f,p : I^2 \to \mathbb{R}$ and $q : I \to \mathbb{R}$ be three functions such that p,q are integrable and $f_{(N,M)}$ exists and is absolutely continuous in the sense of Carathéodory. Then $\overline{C}(f,p)$ and $\overline{K}(f,p,q)$, given below are well defined:

$$\begin{aligned} \overline{C}(f,p) = C(f,p) &- \sum_{i=0}^N \sum_{j=0}^M f_{(i,j)}(a,a)\left[\overline{P}^{(i,j)}(a,a) - P^{(i,j)}(a,a)\right] \\ &- \sum_{j=0}^M \int_a^b f_{(N+1,j)}(x,a)\left[\overline{P}^{(N,j)}(x,a) - P^{(N,j)}(x,a)\right]dx \end{aligned}$$

$$- \sum_{i=0}^{N} \int_a^b f_{(i,M+1)}(a,y) \left[\overline{P}^{(i,M)}(a,y) - P^{(i,M)}(a,y) \right] dy, \tag{14}$$

where $C(f,p)$ is defined in (3).

$$\begin{aligned} \overline{K}(f,p,q) &= K(f,p,q) - \sum_{j=0}^{M} \sum_{i=0}^{N} f_{(i,j)}(a,a) \left[Q^{(i,j)}(a) - P^{(i,j)}(a,a) \right] \\ &\quad - \sum_{j=0}^{M} \int_a^b f_{(N+1,j)}(x,a) \left[Q^{(N,j)}(x) - P^{(N,j)}(x,a) \right] dx \\ &\quad - \sum_{i=0}^{N} \int_a^b f_{(i,M+1)}(a,y) \left[Q^{(M,i)}(y) - P^{(i,M)}(a,y) \right] dy, \end{aligned} \tag{15}$$

where $K(f,p,q)$ is defined in (5).

For discrete identity and inequality we are needed some notions and notations which may be stated as (see [7]): Let I,J be two real intervals of $\mathbb{R}$ and f be a real-valued function defined on I. The n-th order divided difference of f at distinct points $x_i, x_{i+1}, \ldots, x_{i+n}$ in I is defined recursively by:

$$[x_j; f] = f(x_j), \quad i \leqslant j \leqslant i+n$$

$$[x_i, \ldots, x_{i+n}; f] = \frac{[x_{i+1}, \ldots, x_{i+n}; f] - [x_i, \ldots, x_{i+n-1}; f]}{x_{i+n} - x_i}.$$

We denote $[x_i, \ldots, x_{i+n}; f]$ by $\Delta_{(n)} f(x_i)$.
We say that $f : I \to \mathbb{R}$ is a convex function of order n (or n-convex function) if for all choices of $(n+1)$ distinct points $x_i, \ldots, x_{i+n}$ inequality $\Delta_{(n)} f(x_i) \geqslant 0$ holds. It is well-known that if $f^{(n)}$ exists, then f is n-convex if and only if $f^{(n)} \geqslant 0$.

Let f be a real-valued function defined on $I \times J$. Then the (n,m) order divided difference of the function f at distinct points $x_i, \ldots, x_{i+n} \in I$, $y_j, \ldots, y_{j+m} \in J$ is defined by

$$\Delta_{(n,m)} f(x_i, y_j) = [x_i, \ldots, x_{i+n}; [y_j, \ldots, y_{j+m}; f]].$$

A function $f : I \times J \to \mathbb{R}$ is said to be convex of order (n,m) or (n,m)-convex if inequality

$$\Delta_{(n,m)} f(x_i, y_j) \geqslant 0$$

holds for all distinct points $x_i, \ldots, x_{i+n} \in I$, $y_j, \ldots, y_{j+m} \in J$. It is known that if the partial derivative $\frac{\partial^{n+m} f}{\partial x^n \partial y^m}$ exists, then f is (n,m)-convex iff $\frac{\partial^{n+m} f}{\partial x^n \partial y^m} \geqslant 0$.

We also define (n,m) order finite difference of the function f for $x \in I$, $y \in J$, $h,k \in \mathbb{R}$, as follows

$$\begin{aligned} \Delta_{h,k}^{n,m} f(x,y) &= \Delta_h^n (\Delta_k^m f(x,y)) = \Delta_k^m (\Delta_h^n f(x,y)) \\ &= \sum_{i=0}^{n} \sum_{j=0}^{m} (-1)^{n+m-i-j} \binom{n}{i} \binom{m}{j} f(x+ih, y+jk), \end{aligned}$$

provided $x+ih \in I$ for $i=0,1,...,n$ and $y+jk \in J$ for $j=0,1,...,m$.

Divided and finite order differences of a sequence a_{ij} are defined as $\Delta_{(n,m)}a_{ij} = \Delta_{(n,m)}f(x_i,y_j)$ and $\Delta^{(n,m)}a_{ij} = \Delta_{1,1}^{(n,m)}f(x_i,y_j)$, respectively, where $x_i = i$, $y_j = j$ and $f:\{1,\dots,n_1\}\times\{1,\dots,n_2\}\to\mathbb{R}$ is the function $f(i,j)=a_{ij}$.

Now we are ready to state discrete Čebyšev identity and inequality as follows (see [3]). Let

$$C_\Delta(a,p) = \sum_{i=1}^{N}\sum_{j=1}^{N} p_{ij}a_{ii} - \sum_{i=1}^{N}\sum_{j=1}^{N} p_{ij}a_{ij}, \tag{16}$$

where a_{ij} and p_{ij} $(1 \leqslant i,j \leqslant N)$ are real numbers.

PROPOSITION 3. *The inequality*

$$C_\Delta(a,p) \geqslant 0 \tag{17}$$

holds for all real numbers a_{ij} $i,j=1,\dots,n$ *such that* $\Delta_1\Delta_2\, a_{ij} \geqslant 0$ $i,j=1,\dots,N-1$ *if and only if*

$$X_{j+1,j} = \overline{X}_{j,j+1} \quad (j=1,\dots,N-1)$$

and

$$X_{ij} \geqslant 0, \quad j+1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant N-1$$

$$\overline{X}_{ij} \geqslant 0, \quad 1 \leqslant i \leqslant j-1,\ 2 \leqslant j \leqslant N$$

holds. If $\Delta_1\Delta_2 a_{ij} \leqslant 0\,(i,j=1,\dots,n-1)$, *then the reverse inequality in* (17) *is valid, where* $X_{ij} = \sum_{r=i}^{N}\sum_{s=1}^{j} p_{rs}$ *and* $\overline{X}_{ij} = \sum_{r=1}^{i}\sum_{s=j}^{N} p_{rs}$.

This paper consist of four main sections. In the second and the third sections generalization of Čebyšev and Ky Fan identities and inequalities are given respectively. Last section is devoted to generalized discrete Čebyšev identity and inequality.

2. Generalized Čebyšev Identity and Inequality

To prove the main theorem we use the following lemma (see [4]).

LEMMA 1. *Let* $p,f : I^2 \to \mathbb{R}$ *be two functions such that* p *is integrable and* $f_{(N+1,M)}$ *and* $f_{(N,M+1)}$ *exist and are absolutely continuous. Then we have*

$$\begin{aligned}
&\int_a^b\int_a^b p(x,y)f(x,y)\,dy\,dx \\
&= \sum_{i=0}^{N}\sum_{j=0}^{M} f_{(i,j)}(a,a)P^{(i,j)}(a,a) \\
&+ \sum_{j=0}^{M}\int_a^b f_{(N+1,j)}(x,a)P^{(N,j)}(x,a)\,dx
\end{aligned}$$

$$+\sum_{i=0}^{N}\int_a^b f_{(i,M+1)}(a,y)P^{(i,M)}(a,y)\,dy$$
$$+\int_a^b\int_a^b f_{(N+1,M+1)}(x,y)P^{(N,M)}(x,y)\,dy\,dx,$$

where $P^{(i,j)}$ *is defined in* (8).

THEOREM 1. *Let* $p,f:I^2\to\mathbb{R}$ *be two functions such that* p *is integrable and* $f_{(N+1,M)}$ *and* $f_{(N,M+1)}$ *exist and are absolutely continuous. Then we have*

$$\begin{aligned}
C(f,p) &= \int_a^b\int_a^b p(x,y)f(x,x)dy\,dx - \int_a^b\int_a^b p(x,y)f(x,y)dy\,dx \\
&= \sum_{i=0}^{N}\sum_{j=0}^{M} f_{(i,j)}(a,a)\left[\overline{P}^{(i,j)}(a,a)-P^{(i,j)}(a,a)\right] \\
&+ \sum_{j=0}^{M}\int_a^b f_{(N+1,j)}(x,a)\left[\overline{P}^{(N,j)}(x,a)-P^{(N,j)}(x,a)\right]dx \\
&+ \sum_{i=0}^{N}\int_a^b f_{(i,M+1)}(a,y)\left[\overline{P}^{(i,M)}(a,y)-P^{(i,M)}(a,y)\right]dy \\
&+ \int_a^b\int_a^b f_{(N+1,M+1)}(x,y)R(x,y)dy\,dx, \qquad (18)
\end{aligned}$$

where $\overline{P}^{(i,j)}$, $P^{(i,j)}$, *and* $R(x,y)$ *are defined in* (7), (8), *and* (10) *respectively.*

Proof. To prove this identity first we find expression for $\int_a^b\int_a^b p(x,y)f(x,x)dy\,dx$ as follows. First we expand $f(x,x)$ in Taylor expansion of two variables and multiply it with $p(x,y)$ and integrate it over $[a,b]\times[a,b]$ by variables x and y to get

$$\begin{aligned}
&\int_a^b\int_a^b p(x,y)f(x,x)dy\,dx \\
&= \int_a^b\left[\sum_{j=0}^{M}\left(\sum_{i=0}^{N} f_{(i,j)}(a,a)\frac{(x-a)^i}{i!}\right)\int_a^b p(x,y)\frac{(x-a)^j}{j!}dy\right]dx \\
&+ \int_a^b\left[\sum_{j=0}^{M}\left(\int_a^x f_{(N+1,j)}(s,a)\frac{(x-s)^N}{N!}ds\right)\int_a^b p(x,y)\frac{(x-a)^j}{j!}dy\right]dx \\
&+ \int_a^b\left[\int_a^b\int_a^x p(x,y)\left(\sum_{i=0}^{N} f_{(i,M+1)}(a,t)\frac{(x-a)^i}{i!}\right)\frac{(x-t)^M}{M!}dt\,dy\right]dx \\
&+ \int_a^b\left[\int_a^b\int_a^x\left(\int_a^x p(x,y)f_{(N+1,M+1)}(s,t)\frac{(x-s)^N}{N!}ds\right)\frac{(x-t)^M}{M!}dt\,dy\right]dx.
\end{aligned}$$

In the first summand we change the order of summation, use linearity of integral and get

$$\sum_{i=0}^{N}\sum_{j=0}^{M}\int_a^b\int_a^b p(x,y)f_{(i,j)}(a,a)\frac{(x-a)^i}{i!}\frac{(x-a)^j}{j!}dydx.$$

By using Fubini's theorem, the second summand is rewritten as:

$$\begin{aligned}
&\int_a^b\left[\sum_{j=0}^{M}\left(\int_a^x f_{(N+1,j)}(s,a)\frac{(x-s)^N}{N!}ds\right)\int_a^b p(x,y)\frac{(x-a)^j}{j!}dy\right]dx\\
&=\int_a^b\left[\sum_{j=0}^{M}\left(\int_a^x\int_a^b p(x,y)\frac{(x-a)^j}{j!}f_{(N+1,j)}(s,a)\frac{(x-s)^N}{N!}dyds\right)\right]dx\\
&=\sum_{j=0}^{M}\int_a^b\int_a^x\int_a^b p(x,y)f_{(N+1,j)}(s,a)\frac{(x-s)^N}{N!}\frac{(x-a)^j}{j!}dydsdx\\
&=\sum_{j=0}^{M}\int_a^b\int_s^b\int_a^b p(x,y)f_{(N+1,j)}(s,a)\frac{(x-s)^N}{N!}\frac{(x-a)^j}{j!}dydxds.
\end{aligned}$$

Similarly, the third summand is rewritten as:

$$\begin{aligned}
&\int_a^b\left[\int_a^b\int_a^x p(x,y)\left(\sum_{i=0}^{N}f_{(i,M+1)}(a,t)\frac{(x-a)^i}{i!}\right)\frac{(x-t)^M}{M!}dt\,dy\right]dx\\
&=\sum_{i=0}^{N}\int_a^b\int_a^b\int_a^x p(x,y)f_{(i,M+1)}(a,t)\frac{(x-a)^i}{i!}\frac{(x-t)^M}{M!}dt\,dydx\\
&=\sum_{i=0}^{N}\int_a^b\int_a^b\int_t^b p(x,y)f_{(i,M+1)}(a,t)\frac{(x-a)^i}{i!}\frac{(x-t)^M}{M!}dydxdt.
\end{aligned}$$

Finally, the fourth summand is rewritten as:

$$\begin{aligned}
&\int_a^b\left[\int_a^b\int_a^x\left(\int_a^x p(x,y)f_{(N+1,M+1)}(s,t)\frac{(x-s)^N}{N!}ds\right)\frac{(x-t)^M}{M!}dt\,dy\right]dx\\
&=\int_a^b\int_a^b\int_a^x\int_a^x p(x,y)f_{(N+1,M+1)}(s,t)\frac{(x-s)^N}{N!}\frac{(x-t)^M}{M!}dsdt\,dydx\\
&=\int_a^b\int_a^b\int_{max\{s,t\}}^b\int_a^b p(x,y)f_{(N+1,M+1)}(s,t)\frac{(x-s)^N}{N!}\frac{(x-t)^M}{M!}dydxdt\,ds,
\end{aligned}$$

we add up all these results to get

$$\begin{aligned}
&\int_a^b\int_a^b p(x,y)f(x,x)dydx\\
&=\sum_{i=0}^{N}\sum_{j=0}^{M}\int_a^b\int_a^b p(x,y)f_{(i,j)}(a,a)\frac{(x-a)^i}{i!}\frac{(x-a)^j}{j!}dydx
\end{aligned}$$

$$= \sum_{j=0}^{M} \int_a^b \int_s^b \int_a^b p(x,y) f_{(N+1,j)}(s,a) \frac{(x-s)^N}{N!} \frac{(x-a)^j}{j!} dy\,dx\,ds$$
$$= \sum_{i=0}^{N} \int_a^b \int_a^b \int_t^b p(x,y) f_{(i,M+1)}(a,t) \frac{(x-a)^i}{i!} \frac{(x-t)^M}{M!} dy\,dx\,dt$$
$$= \int_a^b \int_a^b \int_{max\{s,t\}}^b \int_a^b p(x,y) f_{(N+1,M+1)}(s,t) \frac{(x-s)^N}{N!} \frac{(x-t)^M}{M!} dy\,dx\,dt\,ds.$$

When we change the names of variables on the right hand side $x \leftrightarrow s$, $y \leftrightarrow t$ then we get:

$$\begin{aligned}
&\int_a^b \int_a^b p(x,y) f(x,x)\,dy\,dx \\
&= \sum_{i=0}^{N} \sum_{j=0}^{M} \int_a^b \int_a^b p(s,t) f_{(i,j)}(a,a) \frac{(s-a)^{i+j}}{i!j!} dt\,ds \\
&+ \sum_{j=0}^{M} \int_a^b \int_x^b \int_a^b p(s,t) f_{(N+1,j)}(x,a) \frac{(s-x)^N}{N!} \frac{(s-a)^j}{j!} dt\,ds\,dx \\
&+ \sum_{i=0}^{N} \int_a^b \int_a^b \int_x^b p(s,t) f_{(i,M+1)}(a,y) \frac{(s-a)^i}{i!} \frac{(s-y)^M}{M!} dt\,ds\,dy \\
&+ \int_a^b \int_a^b \int_{max\{x,y\}}^b \int_a^b p(s,t) f_{(N+1,M+1)}(x,y) \frac{(s-x)^N}{N!} \frac{(s-y)^M}{M!} dt\,ds\,dy\,dx.
\end{aligned}$$

By using defined notations we finally get

$$\begin{aligned}
&\int_a^b \int_a^b p(x,y) f(x,x)\,dy\,dx \\
&= \sum_{i=0}^{N} \sum_{j=0}^{M} f_{(i,j)}(a,a) \overline{P}^{(i,j)}(a,a) \\
&+ \sum_{j=0}^{M} \int_a^b f_{(N+1,j)}(x,a) \overline{P}^{(N,j)}(x,a)\,dx \\
&+ \sum_{i=0}^{N} \int_a^b f_{(i,M+1)}(a,y) \overline{P}^{(i,M)}(a,y)\,dy \\
&+ \int_a^b \int_a^b f_{(N+1,M+1)}(x,y) \int_{max\{x,y\}}^b \int_a^b p(s,t) \frac{(s-x)^N}{N!} \frac{(s-y)^M}{M!}\,dt\,ds\,dy\,dx,
\end{aligned}$$

where $\overline{P}^{(i,j)}$ is defined in (8). Using the above expression for $\int_a^b \int_a^b p(x,y) f(x,x) dy\,dx$ and Lemma 1 in

$$C(f,p) = \int_a^b \int_a^b p(x,y) f(x,x)\,dy\,dx - \int_a^b \int_a^b p(x,y) f(x,y)\,dy\,dx,$$

we get our required identity.

REMARK 1. If in Theorem 1, we put $f(x,y) = f(x)g(y)$ and $p(x,y) = p(x)p(y)$, then we may state the following corollary.

COROLLARY 1. *Let $p, f, g : I \to \mathbb{R}$ be three functions such that p is integrable and $f^{(N)}$ and $g^{(M)}$ exist and are absolutely continuous. Then we have*

$$\begin{aligned}
&T(f,g,p) = T(P_N(f),P_M(g),p) + T(R_N(f),P_M(g),p) + T(P_N(f),R_M(g),p)\\
&+\int_a^b p(x)\,dx \int_a^b \int_a^b \int_{max\{x,y\}}^b \frac{f^{(N+1)}(x)(s-x)^N}{N!}\frac{g^{(M+1)}(y)(s-y)^M}{M!}\,p(s)\,ds\,dy\,dx\\
&-\int_a^b R_N(f)(x)p(x)\,dx \int_a^b R_M(g)(x)p(x)\,dx, \qquad (19)
\end{aligned}$$

where $P_k(h)(x) = \sum_{i=0}^k \frac{h^{(i)}(a)(x-a)^i}{i!}$, $R_k(h)(x) = \int_a^x \frac{h^{(N+1)}(s)(x-s)^N}{N!}\,ds$, $k \in \mathbb{N}$, and h is a function and T(f,g,p) is defined in (2).

REMARK 2. We can get (19) directly by using Taylor formula for functions f and g.

COROLLARY 2. *Let $p, f : I^2 \to \mathbb{R}$ be two functions such that p is integrable and $f_{(N+1,M)}$ and $f_{(N,M+1)}$ exist and are absolutely continuous. Then for $\frac{1}{s} + \frac{1}{t} = 1$; $s,t > 1$; we have*

$$|\,\overline{C}(f,p)\,| \leqslant \left(\int_a^b \int_a^b |f_{(N+1,M+1)}(x,y)|^s\,dy\,dx\right)^{\frac{1}{s}} \left(\int_a^b \int_a^b |R(x,y)|^t\,dy\,dx\right)^{\frac{1}{t}} \qquad (20)$$

where $\overline{C}(f,p)$ and $R(x,y)$ are defined in (14) and (10) respectively.

Proof. We can get (20) easily by using Hölder's inequality for integrals in Theorem 1.

THEOREM 2. *Let $p, f : I^2 \to \mathbb{R}$ be two functions such that p is integrable and f is $(N+1,M+1)$-convex. Then*

$$\overline{C}(f,p) \geqslant 0 \quad \textit{if} \quad R(x,y) \geqslant 0 \quad \forall x,y \in [a,b]$$

where $\overline{C}(f,p)$ and $R(x,y)$ are defined in (14) and (10) respectively.

Proof. If f is $(N+1,M+1)$-convex function it may be approximated uniformly on I^2 by polynomials having non-negative partial derivatives of order $(N+1,M+1)$. Indeed, Bernstein polynomials

$$B^{n,m}(x,y) = \sum_{i=0}^{n}\sum_{j=0}^{m}\binom{n}{i}\binom{m}{j} f(a+ih,a+jk)(x-a)^i(b-x)^{n-i}(y-a)^j(b-y)^{m-j},$$

where $h=(b-a)/n$ and $k=(b-a)/m$, converge uniformly to f on I^2 as $n\to\infty$, $m\to\infty$ provided that f is continuous. Furthermore, the following formula can be proven by induction

$$
\begin{aligned}
&B^{n,m}_{(N+1,M+1)}(x,y)= \\
&(N+1)!(M+1)!\binom{n}{N+1}\binom{m}{M+1}\sum_{i=0}^{n-N-1}\sum_{j=0}^{m-M-1}\binom{n-N-1}{i}\binom{m-M-1}{j}\times \\
&\times\left(\Delta_{h,k}^{(N+1,M+1)}f(a+ih,a+jk)\right)(x-a)^i(b-x)^{n-N-1-i}(y-a)^j(b-y)^{m-M-1-j}.
\end{aligned}
$$

Since f is $(N+1,M+1)$-convex, $\Delta_{h,k}^{(N+1,M+1)}f\geqslant 0$ for $h,k>0$, so $B^{n,m}_{(N+1,M+1)}\geqslant 0$. Since $R(x,y)$ is continuous and $B^{n,m}_{(N+1,M+1)}\geqslant 0$ on I^2 so by (14) we obtain

$$
\begin{aligned}
\overline{C}(B^{n,m},p)=\int_a^b\int_a^b B^{n,m}_{(N+1,M+1)}(x,y)\left[\int_{max\{s,t\}}^b\int_a^b p(s,t)\frac{(x-s)^N}{N!}\frac{(x-t)^M}{M!}dt\,ds-\right.\\
\left.-\int_x^b\int_y^b p(s,t)\frac{(s-x)^N}{N!}\frac{(t-y)^M}{M!}dt\,ds\right]dy\,dx\geqslant 0,
\end{aligned}
$$

or we can write

$$
\overline{C}(B^{n,m},p)=\int_a^b\int_a^b B^{n,m}_{(N+1,M+1)}(x,y)R(x,y)dy\,dx. \tag{21}
$$

Now by letting $n,m\to\infty$ through an appropriate sequence, the uniform convergence of $B^{n,m}_{(N+1,M+1)}$ to $f_{(N+1,M+1)}$ gives our desired result.

THEOREM 3. *Let $p,f:I^2\to\mathbb{R}$ be two functions such that p is integrable and $f_{(N+1,M)}$ and $f_{(N,M+1)}$ exist and are absolutely continuous. If*

$$
R(x,y)\geqslant 0
$$

holds $\forall x,y\in[a,b]$, then there exist $\xi,\eta\in[a,b]$ such that

$$
\overline{C}(f,p)=f_{(N+1,M+1)}(\xi,\eta)C(G_0,p), \tag{22}
$$

where $\overline{C}(f,p)$, $R(x,y)$, and G_0 are defined in (14), (10), and (12) respectively.

Proof. Method 1:
We have

$$
\overline{C}(f,p)=\int_a^b\int_a^b f_{(N+1,M+1)}(x,y)R(x,y)dy\,dx, \tag{23}
$$

using the Mean Value Theorem for double integrals we get

$$
\overline{C}(f,p)=f_{(N+1,M+1)}(\xi,\eta)\int_a^b\int_a^b R(x,y)dy\,dx.
$$

If we put $f(x,y) = G_0(x,y)$ in above expression then we get

$$\overline{C}(G_0, p) = C(G_0, p) = \int_a^b \int_a^b R(x,y)\, dy\, dx,$$

and hence we get what we wanted.

Proof. Method 2:
Let $L = \min_{(x,y)\in I^2} f_{(N+1,M+1)}(x,y)$, $U = \max_{(x,y)\in I^2} f_{(N+1,M+1)}(x,y)$. Then the function

$$G(x,y) = UG_0(x,y) - f(x,y)$$

gives us

$$G_{(N+1,M+1)}(x,y) = U - f_{(N+1,M+1)}(x,y) \geqslant 0,$$

that is G is $(N+1, M+1)$-convex function. Hence $\overline{C}(G,p) \geqslant 0$ by Theorem 2 and we conclude that

$$\overline{C}(f,p) \leqslant U\overline{C}(G_0,p),$$

similarly

$$L\overline{C}(G_0,p) \leqslant \overline{C}(f,p).$$

Combining the two inequalities we get

$$L\overline{C}(G_0,p) \leqslant \overline{C}(f,p) \leqslant U\overline{C}(G_0,p)$$

by using Mean Value Theorem we easily get (22).

REMARK 3. For $N = M = 0$ Theorem 3 is equivalent to Proposition 1. Also if we choose $f(x,y) = f(x)g(y)$ and $p(x,y) = p(x)p(y)$ in Theorem 3 with $N = M = 0$, then we get (1).

THEOREM 4. *Let $p : I^2 \to \mathbb{R}$ be an integrable function and $f, g : I^2 \to \mathbb{R}$ be two functions such that $f_{(N+1,M)}$, $f_{(N,M+1)}$ exist and are absolutely continuous and $g \in C^{(N+1,M+1)}(I^2)$ with $g_{(N+1,M+1)} \neq 0$ on I^2. If*

$$R(x,y) \geqslant 0, \quad \forall x, y \in [a,b]$$

holds then there exist $\xi, \eta \in [a,b]$ such that

$$\overline{C}(f,p) = \frac{f_{(N+1,M+1)}(\xi,\eta)}{g_{(N+1,M+1)}(\xi,\eta)} \overline{C}(g,p),$$

where $\overline{C}(f,p)$ and $R(x,y)$ are defined in (14) and (10) respectively.

Proof. Method 1:
Using (23) and Integral Mean Value Theorem we have

$$\overline{C}(f,p) = \int_a^b \int_a^b \frac{f_{(N+1,M+1)}(x,y)}{g_{(N+1,M+1)}(x,y)} g_{(N+1,M+1)}(x,y) R(x,y)\, dy\, dx$$

$$= \frac{f_{(N+1,M+1)}(\xi,\eta)}{g_{(N+1,M+1)}(\xi,\eta)} \int_a^b \int_a^b g_{(N+1,M+1)}(x,y) R(x,y)\, dy\, dx$$
$$= \frac{f_{(N+1,M+1)}(\xi,\eta)}{g_{(N+1,M+1)}(\xi,\eta)} \overline{C}(g,p).$$

Proof. Method 2:
Let $h \in C^{(N+1,M+1)}(I^2)$ be defined as

$$h = \overline{C}(g,p)f - \overline{C}(f,p)g$$

using Theorem 3 there exist $\xi,\eta \in I$ such that

$$0 = \overline{C}(h,p) = h_{(N+1,M+1)}(\xi,\eta)\overline{C}(G_0,p),$$

or

$$[\overline{C}(g,p)f_{(N+1,M+1)}(\xi,\eta) - \overline{C}(f,p)g_{(N+1,M+1)}(\xi,\eta)]\overline{C}(G_0,p) = 0$$

which gives us the required result.

REMARK 4. For $N = M = 0$ Theorem 4 becomes Theorem 2 of [3].

THEOREM 5. *Let $p, f : I^2 \to \mathbb{R}$ be two functions such that p is integrable and f is $(N+1,M+1)$-convex. Then there exist $\xi,\eta \in [a,b]$ such that*

$$C(f,p) = R(\xi,\eta)\left(f_{(N,M)}(b,b) - f_{(N,M)}(a,b) - f_{(N,M)}(b,a) + f_{(N,M)}(a,a)\right),$$

where $C(f,p)$ and $R(x,y)$ are defined in (3) and (10).

Proof. Since $R(x,y)$ is continuous and $B^{n,m}_{(N+1,M+1)} \geqslant 0$ on I^2, where $B^{n,m}$ is Bernstien polynomial, by same arguments used in the proof of Theorem 3 we obtain

$$C(B^{n,m},p) = \int_a^b \int_a^b R(x,y) B^{n,m}_{(N+1,M+1)}(x,y)\, dy\, dx$$
$$= R(\xi_{n,m},\eta_{n,m}) \int_a^b \int_a^b B^{n,m}_{(N+1,M+1)}(x,y)\, dy\, dx$$
$$= R(\xi_{n,m},\eta_{n,m}) \left(B^{n,m}_{(N,M)}(b,b) - B^{n,m}_{(N,M)}(a,b) - B^{n,m}_{(N,M)}(b,a) + B^{n,m}_{(N,M)}(a,a)\right).$$

The points $x_{n,m} = (\xi_{n,m},\eta_{n,m})$ have a limit point (ξ,η) in I^2 as $n \to \infty$, $m \to \infty$, so letting $n,m \to \infty$ through an appropriate sequence, the uniform convergence of $B^{n,m}_{(N,M)}$ to $f_{(N,M)}$ gives our desired result.

REMARK 5. In Theorem 5 for the case $N = M = 0$ we obtain Theorem 6 of [3].

3. Generalized Ky Fan Identity and Inequality

THEOREM 6. *Let $f,p: I^2 \to \mathbb{R}$ and $q: I \to \mathbb{R}$ be three functions such that p and q are integrable and $f_{(N+1,M)}$ and $f_{(N,M+1)}$ exist and are absolutely continuous. Then we have*

$$\begin{aligned} K(f,p,q) = & \sum_{j=0}^{M}\sum_{i=0}^{N} f_{(i,j)}(a,a)\left[Q^{(i,j)}(a) - P^{(i,j)}(a,a)\right] \\ & + \sum_{j=0}^{M}\int_a^b f_{(N+1,j)}(x,a)\left[Q^{(N,j)}(x) - P^{(N,j)}(x,a)\right] dx \\ & + \sum_{i=0}^{N}\int_a^b f_{(i,M+1)}(a,y)\left[Q^{(M,i)}(y) - P^{(i,M)}(a,y)\right] dy \\ & + \int_a^b\int_a^b f_{(N+1,M+1)}(x,y)\overline{R}(x,y)\,dy\,dx, \end{aligned}$$

where $P^{(i,j)}$, $Q^{(i,j)}$, and $\overline{R}(x,y)$ are defined in (8), (9), and (11) respectively.

REMARK 6. The proof of this theorem is analogous to proof of Theorem 1 if we use the following substitution

$$\int_a^b p(x,y)dy = q(x).$$

REMARK 7. If in Theorem 6, we put $f(x,y) = f(x)g(y)$ and $p(x,y) = \frac{q(x)q(y)}{\int_a^b q(t)\,dt}$ where q is an integrable function such that $\int_a^b q(t)\,dt \neq 0$, then we may state the following Corollary.

COROLLARY 3. *Let $q,f,g: I \to \mathbb{R}$ be three functions such that q is an integrable function such that $\int_a^b q(t)\,dt \neq 0$ and $f^{(N)}$ and $g^{(M)}$ exist and are absolutely continuous. Then we have*

$$\begin{aligned} T(f,g,q) = & \, T(P_N(f),P_M(g),q) + T(R_N(f),P_M(g),q) + T(P_N(f),R_M(g),q) \\ & + \int_a^b\int_a^b\int_{max\{x,y\}}^b \frac{f^{(N+1)}(x)(s-x)^N}{N!}\frac{g^{(M+1)}(y)(s-y)^M}{M!} q(s)\,ds\,dy\,dx \\ & - \int_a^b R_N(f)(x)q(x)\,dx \int_a^b R_M(g)(x)q(x)\,dx, \end{aligned}$$

where $P_k(h)(x) = \sum_{i=0}^{k}\frac{h^{(i)}(a)(x-a)^i}{i!}$, $R_k(h)(x) = \int_a^x \frac{h^{(N+1)}(s)(x-s)^N}{N!}ds$, $k \in \mathbb{N}$, and h is a function and T(f,g,p) is defined in (2).

COROLLARY 4. *Let $f,p: I^2 \to \mathbb{R}$ and $q: I \to \mathbb{R}$ be three functions such that p and q are integrable and $f_{(N+1,M)}$ and $f_{(N,M+1)}$ exist and are absolutely continuous.*

Then for $\frac{1}{s}+\frac{1}{t}=1$*;* $s,t>1$*; we have*

$$|\,\overline{C}(f,p)\,| \leqslant \left(\int_a^b\int_a^b |f_{(N+1,M+1)}(x,y)|^s\,dy\,dx\right)^{\frac{1}{s}}\left(\int_a^b\int_a^b |\overline{R}(x,y)|^t\,dy\,dx\right)^{\frac{1}{t}},$$

where $\overline{C}(f,p)$ *and* $\overline{R}(x,y)$ *are defined in* (14) *and* (11) *respectively.*

THEOREM 7. *Let* $f,p: I^2 \to \mathbb{R}$ *and* $q: I \to \mathbb{R}$ *be three functions such that* p *and* q *are integrable and* $f_{(N+1,M)}$ *and* $f_{(N,M+1)}$ *exist and are absolutely continuous. Then*

$$\overline{K}(f,p,q) \geqslant 0 \quad \textit{if} \quad \overline{R}(x,y) \geqslant 0, \quad \forall x,\, y \in [a,b],$$

where $\overline{K}(f,p,q)$ *and* $\overline{R}(x,y)$ *are defined in* (15) *and* (11) *respectively.*

REMARK 8. The proof of this theorem is analogous to proof of Theorem 2.

THEOREM 8. *Let* $f,p: I^2 \to \mathbb{R}$ *and* $q: I \to \mathbb{R}$ *be three functions such that* p *and* q *are integrable and* f *is* $(N+1,M)-$*convex function . We also assume that* $\forall x,\, y \in [a,b]$

$$\overline{R}(x,y) \geqslant 0.$$

Then there exist $\xi,\,\eta \in [a,b]$ *such that*

$$\overline{K}(f,p,q) = f_{(N+1,M+1)}(\xi,\eta)K\left(G_0,p,q\right),$$

where $\overline{K}(f,p,q)$ *and* G_0 *are defined in* (15) *and* (12) *respectively.*

REMARK 9. The proof of this theorem may also be given in two different ways analogous to the proof of Theorem 3 and this theorem gives us Proposition 2 for $N=M=0$.

THEOREM 9. *Let* $f,g,p: I^2 \to \mathbb{R}$ *and* $q: I \to \mathbb{R}$ *be four functions such that* p *and* q *are integrable and* $f_{(N+1,M)}$, $f_{(N,M+1)}$ *exist and are absolutely continuous and* $g \in C^{(N+1,M+1)}(I^2)$ *with* $g_{(N+1,M+1)} \neq 0$ *on* I^2*, we also assume that* $\forall x,\, y \in [a,b]$

$$\overline{R}(x,y) \geqslant 0.$$

Then there exist $\xi,\,\eta \in [a,b]$ *such that*

$$\overline{K}(f,p,q) = \frac{f_{(N+1,M+1)}(\xi,\eta)}{g_{(N+1,M+1)}(\xi,\eta)}\overline{K}(g,p,q).$$

REMARK 10. The proof of Theorem 9 may also be given in two different ways analogous to the proof of Theorem 4. Also for $N=M=0$ we get Theorem 16 of [3].

4. Generalized Discrete Čebyšev's Identity and Inequality

Now we state discrete Čebyšev identity's and inequality as follows: We will make use of the following notation

$$(x_i - x_k)^{(l+1)} = (x_i - x_k)(x_i - x_{k+1}) \cdots (x_i - x_{k+l}) \text{ for } l \geqslant 0 \text{ and } (x_i - x_k)^{(0)} = 1.$$

To state our main identity of this section we are needed the following lemma from [4].

LEMMA 2. *Let $(x_i, y_j) \in I^2 = [a,b] \times [a,b]$ $(i, j = 1, \dots, N)$, be mutually distinct elements and let f be a real-valued function on I^2. Let p_{ij} $(i, j = 1, \dots, N)$, be real numbers. Then the following identity holds:*

$$\begin{aligned}
&\sum_{i=1}^{N}\sum_{j=1}^{N} p_{ij} f(x_i, y_j) \\
&= \sum_{k=0}^{m-1}\sum_{t=0}^{n-1}\sum_{s=t+1}^{N}\sum_{r=k+1}^{N} p_{sr}(x_s - x_1)^{(t)}(y_r - y_1)^{(k)}\Delta_{(t,k)} f(x_1, y_1) \\
&+ \sum_{k=0}^{m-1}\sum_{t=n+1}^{N}\sum_{s=t}^{N}\sum_{r=k+1}^{N} p_{sr}(x_s - x_{t-n+1})^{(n-1)}(y_r - y_1)^{(k)}\Delta_{(n,k)} f(x_{t-n}, y_1)(x_t - x_{t-n}) \\
&+ \sum_{k=m+1}^{N}\sum_{t=0}^{n-1}\sum_{s=t+1}^{N}\sum_{r=k}^{N} p_{sr}(x_s - x_1)^{(t)}(y_r - y_{k-m+1})^{(m-1)}\Delta_{(t,m)} f(x_1, y_{k-m})(y_k - y_{k-m}) \\
&+ \sum_{k=m+1}^{N}\sum_{t=n+1}^{N}\Big(\sum_{s=t}^{N}\sum_{r=k}^{N} p_{sr}(x_s - x_{t-n+1})^{(n-1)}(y_r - y_{k-m+1})^{(m-1)}\Big)\times \\
&\times \Delta_{(n,m)} f(x_{t-n}, y_{k-m})(x_t - x_{t-n})(y_k - y_{k-m}),
\end{aligned}$$

where $\Delta_{(t,k)} f(x,y)$ represents divided difference of function f of two variables.

THEOREM 10. *Let $(x_i, y_j) \in I^2 = [a,b] \times [a,b]$ $(i, j = 1, \dots, N)$, be mutually distinct elements and let f be a real-valued function on I^2. Let p_{ij} $(i, j = 1, \dots, N)$, be real numbers. Then the following identity*

$$\begin{aligned}
C_\Delta(f,p) &= \sum_{i=1}^{N}\sum_{j=1}^{N} p_{ij} f(x_i, y_i) - \sum_{i=1}^{N}\sum_{j=1}^{N} p_{ij} f(x_i, y_j) \qquad (24)\\
&= \sum_{k=0}^{m-1}\sum_{t=0}^{n-1} \Delta_{(t,k)} f(x_1, y_1) \times \\
&\times \Bigg[\sum_{s=max\{t,k\}+1}^{N}\sum_{r=1}^{N} p_{sr}(x_s - x_1)^{(t)}(y_s - y_1)^{(k)} \\
&- \sum_{s=t+1}^{N}\sum_{r=k+1}^{N} p_{sr}(x_s - x_1)^{(t)}(y_r - y_1)^{(k)}\Bigg]
\end{aligned}$$

$$+\sum_{k=0}^{m-1}\sum_{t=n+1}^{N}\Delta_{(n,k)}f(x_{t-n},y_1)(x_t-x_{t-n})\times$$
$$\times\left[\sum_{s=max\{t,k+1\}}^{N}\sum_{r=1}^{N}p_{sr}(x_s-x_{t-n+1})^{(n-1)}(y_s-y_1)^{(k)}\right.$$
$$\left.-\sum_{s=t}^{N}\sum_{r=k+1}^{N}p_{sr}(x_s-x_{t-n+1})^{(n-1)}(y_r-y_1)^{(k)}\right]$$
$$+\sum_{k=m+1}^{N}\sum_{t=0}^{n-1}\Delta_{(t,m)}f(x_1,y_{k-m})(y_k-y_{k-m})\times$$
$$\times\left[\sum_{s=max\{t+1,k\}}^{N}\sum_{r=1}^{N}p_{sr}(x_s-x_1)^{(t)}(y_s-y_{k-m+1})^{(m-1)}\right.$$
$$\left.-\sum_{s=t+1}^{N}\sum_{r=k}^{N}p_{sr}(x_s-x_1)^{(t)}(y_r-y_{k-m+1})^{(m-1)}\right]$$
$$+\sum_{k=m+1}^{N}\sum_{t=n+1}^{N}\Delta_{(n,m)}f(x_{t-n},y_{k-m})(x_t-x_{t-n})(y_k-y_{k-m})\times$$
$$\times\left[\sum_{s=max\{t,k\}}^{N}\sum_{r=1}^{N}p_{sr}(x_s-x_{t-n+1})^{(n-1)}(y_s-y_{k-m+1})^{(m-1)}\right.$$
$$\left.-\sum_{s=t}^{N}\sum_{r=k}^{N}p_{sr}(x_s-x_{t-n+1})^{(n-1)}(y_r-y_{k-m+1})^{(m-1)}\right],$$

holds, where $\Delta_{(t,k)}f(x,y)$ represents divided difference of function f of two variables.

Proof. We start the proof by considering the expression

$$\sum_{i=1}^{N}\sum_{j=1}^{N}\tilde{p}_{ij}f(x_i,y_i)$$

where $\tilde{p}_{ij}$ is defined as

$$\tilde{p}_{ij}=\begin{cases}\sum_{r=1}^{N}p_{ir}, & i=j,\\ 0, & i\neq j.\end{cases}$$

So using this definition and Lemma 2, we get

$$\sum_{i=1}^{N}\sum_{j=1}^{N}\tilde{p}_{ij}f(x_i,y_i)=\sum_{i=1}^{N}\sum_{j=1}^{N}p_{ij}f(x_i,y_i)$$
$$=\sum_{k=0}^{m-1}\sum_{t=0}^{n-1}\Delta_{(t,k)}f(x_1,y_1)\sum_{s=max\{t+1,k+1\}}^{N}\sum_{r=1}^{N}p_{sr}(x_s-x_1)^{(t)}(y_s-y_1)^{(k)}$$

$$+\sum_{k=0}^{m-1}\sum_{t=n+1}^{N}\Delta_{(n,k)}f(x_{t-n},y_1)(x_t-x_{t-n})\sum_{s=max\{t,k+1\}}^{N}\sum_{r=1}^{N}p_{sr}(x_s-x_{t-n+1})^{(n-1)}(y_s-y_1)^{(k)}$$

$$+\sum_{k=m+1}^{N}\sum_{t=0}^{n-1}\Delta_{(t,m)}f(x_1,y_{k-m})(y_k-y_{k-m})\times$$

$$\times\sum_{s=max\{t+1,k\}}^{N}\sum_{r=1}^{N}p_{sr}(x_s-x_1)^{(t)}(y_s-y_{k-m+1})^{(m-1)}$$

$$+\sum_{k=m+1}^{N}\sum_{t=n+1}^{N}\Delta_{(n,m)}f(x_{t-n},y_{k-m})(x_t-x_{t-n})(y_k-y_{k-m})\times$$

$$\times\sum_{s=max\{t,k\}}^{N}\sum_{r=1}^{N}p_{sr}(x_s-x_{t-n+1})^{(n-1)}(y_s-y_{k-m+1})^{(m-1)}.$$

So, we easily get our required result by putting the expressions of $\sum_{i=1}^{N}\sum_{j=1}^{N}p_{ij}\times$ $\times f(x_i,y_i)$ and $\sum_{i=1}^{N}\sum_{j=1}^{N}p_{ij}f(x_i,y_j)$ in $C_\Delta(f,p)=\sum_{i=1}^{N}\sum_{j=1}^{N}p_{ij}f(x_i,y_i)$ $-\sum_{i=1}^{N}\sum_{j=1}^{N}p_{ij}f(x_i,y_j)$.

REMARK 11. If we put $x_i=i,\ y_j=j$ and $f(x_i,y_j)=f(i,j)=a_{ij}$ in Theorem 10 then we get following corollary.

COROLLARY 5. *Let p_{ij},a_{ij} $(i,j=1,\dots,N)$, be real numbers. Then the following identity holds:*

$$C_\Delta(a,p)=\sum_{i=1}^{N}\sum_{j=1}^{N}p_{ij}a_{ii}-\sum_{i=1}^{N}\sum_{j=1}^{N}p_{ij}a_{ij}$$

$$=\sum_{k=0}^{m-1}\sum_{t=0}^{n-1}\Delta^{(t,k)}a_{11}\left[\sum_{s=max\{t,k\}+1}^{N}\sum_{r=1}^{N}p_{sr}\binom{s-1}{t}\binom{s-1}{k}\right.$$

$$\left.-\sum_{s=t+1}^{N}\sum_{r=k+1}^{N}p_{sr}\binom{s-1}{t}\binom{r-1}{k}\right]$$

$$+\sum_{k=0}^{m-1}\sum_{t=n+1}^{N}\Delta^{(n,k)}a_{(t-n)1}$$

$$\left[\sum_{s=max\{t,k+1\}}^{N}\sum_{r=1}^{N}p_{sr}\binom{s-t+n-1}{n-1}\binom{s-1}{k}-\sum_{s=t}^{N}\sum_{r=k+1}^{N}p_{sr}\binom{s-t+n-1}{n-1}\binom{r-1}{k}\right]$$

$$+\sum_{k=m+1}^{N}\sum_{t=0}^{n-1}\Delta^{(t,m)}a_{1(k-m)}$$

$$\left[\sum_{s=max\{t+1,k\}}^{N}\sum_{r=1}^{N}p_{sr}\binom{s-1}{t}\binom{s-k+m-1}{m-1}-\sum_{s=t+1}^{N}\sum_{r=k}^{N}p_{sr}\binom{s-1}{t}\binom{r-k+m-1}{m-1}\right]$$

$$+\sum_{k=m+1}^{N}\sum_{t=n+1}^{N}\Delta^{(n,m)}a_{(t-n)(k-m)}\left[\sum_{s=max\{t,k\}}^{N}\sum_{r=1}^{N}p_{sr}\binom{s-t+n-1}{n-1}\binom{s-k+m-1}{m-1}\right.$$
$$\left.-\sum_{s=t}^{N}\sum_{r=k}^{N}p_{sr}\binom{s-t+n-1}{n-1}\binom{r-k+m-1}{m-1}\right],$$

where $\Delta^{(t,k)}a_{ij}$ *represents finite difference of order* (t,k) *of the sequence* a_{ij}.

REMARK 12. If we put $n=m=1$, this Corollary gives us Theorem 3 of [1].

Before we state our next theorem, under the assumptions of Theorem 10 we define some notations as follows:

$$\overline{C}_{\Delta}(f,p)=C_{\Delta}(f,p)-\sum_{k=0}^{m-1}\sum_{t=0}^{n-1}\Delta_{(t,k)}f(x_1,y_1)\times \tag{25}$$
$$\times\left[\sum_{s=max\{t+1,k+1\}}^{N}\sum_{r=1}^{N}p_{sr}(x_s-x_1)^{(t)}(y_s-y_1)^{(k)}\right.$$
$$\left.-\sum_{s=t+1}^{N}\sum_{r=k+1}^{N}p_{sr}(x_s-x_1)^{(t)}(y_r-y_1)^{(k)}\right]$$
$$-\sum_{k=0}^{m-1}\sum_{t=n+1}^{N}\Delta_{(n,k)}f(x_{t-n},y_1)(x_t-x_{t-n})\times$$
$$\times\left[\sum_{s=max\{t,k+1\}}^{N}\sum_{r=1}^{N}p_{sr}(x_s-x_{t-n+1})^{(n-1)}(y_s-y_1)^{(k)}\right.$$
$$\left.-\sum_{s=t}^{N}\sum_{r=k+1}^{N}p_{sr}(x_s-x_{t-n+1})^{(n-1)}(y_r-y_1)^{(k)}\right]$$
$$-\sum_{k=m+1}^{N}\sum_{t=0}^{n-1}\Delta_{(t,m)}f(x_1,y_{k-m})(y_k-y_{k-m})\times$$
$$\times\left[\sum_{s=max\{t+1,k\}}^{N}\sum_{r=1}^{N}p_{sr}(x_s-x_1)^{(t)}(y_s-y_{k-m+1})^{(m-1)}\right.$$
$$\left.-\sum_{s=t+1}^{N}\sum_{r=k}^{N}p_{sr}(x_s-x_1)^{(t)}(y_r-y_{k-m+1})^{(m-1)}\right],$$

$$R_{\Delta}(t,k)=\left[\sum_{s=max\{t,k\}}^{N}\sum_{r=1}^{N}p_{sr}(x_s-x_{t-n+1})^{(n-1)}(y_s-y_{k-m+1})^{(m-1)}\right.$$
$$\left.-\sum_{s=t}^{N}\sum_{r=k}^{N}p_{sr}(x_s-x_{t-n+1})^{(n-1)}(y_r-y_{k-m+1})^{(m-1)}\right]. \tag{26}$$

THEOREM 11. *Let* x_i, y_j $(i,j=1,\ldots,N)$ *be two real sequences that are either both strictly increasing or both strictly decreasing and let* f *be* (n,m)*-convex function and* p_{ij} $(i,j=1,\ldots,N)$ *be real numbers. Then*

$$\overline{C}_\Delta(f,p) \geqslant 0 \quad \text{if} \quad R_\Delta(t,k) \geqslant 0 \quad t=n+1,\ldots,N; k=m+1,\ldots,N.$$

where $\overline{C}_\Delta(f,p)$ *and* $R_\Delta(t,k)$ *are defined respectively in* (25) *and* (26).

REMARK 13. This result is easily followed by using identity (24).

REMARK 14. If we put $x_i = i$, $y_j = j$ and $f(x_i,y_j) = f(i,j) = a_{ij}$ in previous theorem for $n=m=1$ then we get Theorem 3 of paper [3] and hence in this result for $a_{ij} = f(a_i,b_j)$ we get Corollary 2 of paper [3].

THEOREM 12. *Let* p_{ij} $(i,j=1,\ldots,N)$*, be real numbers. Let* $(x_i,y_j) \in I^2 = [a,b]\times[a,b]$ *(* $i,j=1,\ldots,N$ *), be mutually distinct elements and let* f,g *be a two real-valued functions on* I^2 *such that inequalities*

$$R_\Delta(t,k) \geqslant 0, \quad t=n+1,\ldots,N; k=m+1,\ldots,N. \tag{27}$$

and

$$L\Delta_{(n,m)}g(x_i,y_j) \leqslant \Delta_{(n,m)}f(x_i,y_j) \leqslant U\Delta_{(n,m)}g(x_i,y_j) \tag{28}$$

hold or reverse inequalities to (27) *and* (28) *hold. Then*

$$L\overline{C}_\Delta(g,p) \leqslant \overline{C}_\Delta(f,p) \leqslant U\overline{C}_\Delta(g,p). \tag{29}$$

If the reverse inequality to (27) *holds, then the reverse inequalities in* (29) *are valid, where* $R_\Delta(t,k)$ *is defined in* (26)

Proof. Let $F_1(x_i,y_j) = f(x_i,y_j) - Lg(x_i,y_j)$ and $F_2(x_i,y_j) = Ug(x_i,y_j) - f(x_i,y_j)$ then $\Delta_{(n,m)}F_1(x_i,y_j) \geqslant 0$ and $\Delta_{(n,m)}F_2(x_i,y_j) \geqslant 0$, so, from Theorem 11 we easily obtain Theorem 12.

REMARK 15. If we put $x_i = i$, $y_j = j$ and $f(x_i,y_j) = f(i,j) = a_{ij}$ and $g(i,j) = b_{ij}$ in previous theorem then we get Theorem 4 of paper [3].

REFERENCES

[1] A. M. OSTROWSKI, *On the Integral Inequality*, Aequationes Math. **4** (1970), 358–373.
[2] J. PEČARIĆ, *On the Ostrowski's Generalization of Čebyšev's Inequality*, J. Math. Anal. Appl. **102** (1984) 479–487.
[3] J. PEČARIĆ, *Some Further Remarks on the Ostrowski's Generalization of Čebyšev's Inequality*, J. Math. Anal. Appl. **123** (1987) 18–33.
[4] A. R. KHAN, J. PEČARIĆ AND S. VAROŠANEC, *Popoviciu Type Characterization of Positivity of Sums and Integrals for Convex Functions of Higher Order*, J. Math. Inequal. **7** (2) (2013) 195–212.
[5] K. FAN, *Problem 4471*, Amer. Math. Monthly **60** (1953), 195–197.

[6] J. ŠREMR, *Absolutely Continuous Functions of Two Variables in the Sense of Carathéodory,* Electron. J. Differential Equations, **2010** (154) (2010), 1–11.

[7] J. PEČARIĆ, F. PROSCHAN AND Y. L. TONG, *Convex functions, partial orderings, and statistical applications*, Mathematics in science and engineering, vol. 187, Academic Press, 1992.

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