

NEW INEQUALITIES OF THE HERMITE–HADAMARD TYPE FOR n -TIME DIFFERENTIABLE FUNCTIONS WHICH ARE QUASICONVEX

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Abstract. In this paper, by using an integral identity and the Hölder integral inequality we establish several new inequalities for n -time differentiable mappings that are connected with the Hermite-Hadamard inequality.

1. Introduction

On November 22, 1881, Hermite (1822–1901) sent a letter to the Journal Mathesis. This letter was published in Mathesis 3 (1883, p. 82) and a well-known inequality, nowadays called the Hermite-Hadamard inequality, was presented there.

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a)+f(b)}{2} \quad (1.1)$$

where $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is a convex function on the interval I of a real numbers and $a, b \in I$ with $a < b$. If the function f is concave, the inequality in (1.1) is reversed.

The inequalities in (1.1) have become an important cornerstone in mathematical analysis and optimization. Many uses of these inequalities have been discovered in a variety of settings. Moreover, many inequalities of special means can be obtained for a particular choice of the function f . Due to the rich geometrical significance of Hermite-Hadamard inequality, there is growing literature providing its new proofs, extensions, refinements and generalizations, see for example ([1], [5], [9]–[14], [16]–[19]) and the references therein.

DEFINITION 1. A function $f : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$ is said to be convex if whenever $x, y \in [a, b]$ and $t \in [0, 1]$, the following inequality holds:

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y).$$

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We say that f is concave if $(-f)$ is convex. This definition has its origins in Jensen's results from [8] and has opened up the most extended, useful and multi-disciplinary domain of mathematics, namely, convex analysis. Convex curves and convex bodies have appeared in mathematical literature since antiquity and there are many important results related to them.

We recall that the notion of quasiconvex functions generalizes the notion of convex functions.

DEFINITION 2. A function $f : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$ is said to be quasiconvex on $[a, b]$ if

$$f(tx + (1-t)y) \leq \max \{f(x), f(y)\},$$

for all $x, y \in [a, b]$ and $t \in [0, 1]$.

Clearly, any convex function is quasiconvex. Furthermore, there exist quasiconvex functions which are not convex (see [7], [15]).

For example, consider the following:

Let $f : \mathbb{R}^+ \rightarrow \mathbb{R}$,

$$f(x) = \ln x, \quad x \in \mathbb{R}^+.$$

This function is quasiconvex. However f is not a convex function.

In [1], Alomari *et. al.* proved the following theorem for quasiconvex functions:

THEOREM 1. Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be twice differentiable mapping on I° , $a, b \in I$ with $a < b$ and f'' integrable on $[a, b]$. If $|f''|^q$ is a quasiconvex function on $[a, b]$, $q \geq 1$, then the following inequality holds:

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{(b-a)^2}{12} [\max \{|f''(a)|^q, |f''(b)|^q\}]^{\frac{1}{q}}. \quad (1.2)$$

In [18], Wang *et. al.* proved the following lemma:

LEMMA 1. For $n \in \mathbb{N}$, let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be n -time differentiable. If $a, b \in I$ with $a < b$ and $f^{(n)} \in L[a, b]$, then

$$\begin{aligned} & \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx - \sum_{k=1}^{n-1} \frac{k-1}{2(k+1)!} (a-b)^k f^{(k)}(b) \\ &= \frac{(-1)^n (b-a)^n}{2n!} \int_0^1 (1-t)^{n-1} (2t+n-2) f^{(n)}(ta + (1-t)b) dt, \end{aligned}$$

where an empty sum is understood to be nil.

For other recent results concerning the n -time differentiable functions see [2]–[4], [6], [9], [12], [18] where further references are given.

The main purpose of the present paper is to establish several new inequalities for n -time differentiable mappings that are connected with the Hermite-Hadamard inequality. Also, some applications for special means of real numbers are provided.

2. Main results

THEOREM 2. For $n \geq 2$, let $f : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$ be n -time differentiable and $0 \leq a < b$. If $f^{(n)} \in L[a, b]$ and $|f^{(n)}|^q$ is quasiconvex on $[a, b]$, for $q > 1$, then the following inequality holds:

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx - \sum_{k=1}^{n-1} \frac{k-1}{2(k+1)!} (a-b)^k f^{(k)}(b) \right| \tag{2.1} \\ & \leq \frac{(b-a)^n}{2n!} \left(\frac{q-1}{nq-1} \right)^{1-\frac{1}{q}} \left(\frac{n^{q+1} - (n-2)^{q+1}}{2q+2} \right)^{\frac{1}{q}} \left[\max \left\{ |f^{(n)}(a)|^q, |f^{(n)}(b)|^q \right\} \right]^{\frac{1}{q}}. \end{aligned}$$

Proof. Since $|f^{(n)}|^q$ is quasi-convex on $[a, b]$, for $q > 1$, from Lemma 1 and the Hölder integral inequality, it follows that

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx - \sum_{k=1}^{n-1} \frac{k-1}{2(k+1)!} (a-b)^k f^{(k)}(b) \right| \\ & \leq \frac{(b-a)^n}{2n!} \int_0^1 (1-t)^{n-1} (2t+n-2) |f^{(n)}(ta + (1-t)b)| dt \\ & \leq \frac{(b-a)^n}{2n!} \left(\int_0^1 (1-t)^{\frac{(n-1)q}{q-1}} dt \right)^{1-\frac{1}{q}} \left(\int_0^1 (2t+n-2)^q |f^{(n)}(ta + (1-t)b)|^q dt \right)^{\frac{1}{q}} \\ & \leq \frac{(b-a)^n}{2n!} \left(\int_0^1 (1-t)^{\frac{(n-1)q}{q-1}} dt \right)^{1-\frac{1}{q}} \left(\int_0^1 (2t+n-2)^q dt \right)^{\frac{1}{q}} \\ & \quad \times \left[\max \left\{ |f^{(n)}(a)|^q, |f^{(n)}(b)|^q \right\} \right]^{\frac{1}{q}} \\ & = \frac{(b-a)^n}{2n!} \left(\frac{q-1}{nq-1} \right)^{1-\frac{1}{q}} \left(\frac{n^{q+1} - (n-2)^{q+1}}{2q+2} \right)^{\frac{1}{q}} \left[\max \left\{ |f^{(n)}(a)|^q, |f^{(n)}(b)|^q \right\} \right]^{\frac{1}{q}}. \end{aligned}$$

This completes the proof. \square

COROLLARY 1. Under conditions of Theorem 2, if we choose $n = 2$, then

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{(b-a)^2}{2(q+1)^{\frac{1}{q}}} \left(\frac{q-1}{2q-1} \right)^{1-\frac{1}{q}} \left[\max \left\{ |f''(a)|^q, |f''(b)|^q \right\} \right]^{\frac{1}{q}} \\ & \leq \frac{(b-a)^2}{2} \left(\frac{q-1}{2q-1} \right)^{1-\frac{1}{q}} \left[\max \left\{ |f''(a)|^q, |f''(b)|^q \right\} \right]^{\frac{1}{q}}. \end{aligned}$$

THEOREM 3. For $n \geq 2$, let $f : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$ be n -time differentiable and $0 \leq a < b$. If $f^{(n)} \in L[a, b]$ and $|f^{(n)}|^q$ is quasicconvex on $[a, b]$, then the following inequality holds:

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx - \sum_{k=1}^{n-1} \frac{k-1}{2(k+1)!} (a-b)^k f^{(k)}(b) \right| \quad (2.2) \\ & \leq \frac{(b-a)^n}{2n!} \left(\frac{1}{q(n-1)+1} \right)^{\frac{1}{q}} \\ & \quad \times \left(\frac{(q-1) \left[n \frac{2q-1}{q-1} - (n-2) \frac{2q-1}{q-1} \right]}{2(2q-1)} \right)^{1-\frac{1}{q}} \left[\max \left\{ |f^{(n)}(a)|^q, |f^{(n)}(b)|^q \right\} \right]^{\frac{1}{q}} \end{aligned}$$

where $p = \frac{q}{q-1}$ and $p > 1$.

Proof. From Lemma 1 and the Hölder integral inequality, we obtain

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx - \sum_{k=1}^{n-1} \frac{k-1}{2(k+1)!} (a-b)^k f^{(k)}(b) \right| \\ & \leq \frac{(b-a)^n}{2n!} \int_0^1 (1-t)^{n-1} (2t+n-2) \left| f^{(n)}(ta + (1-t)b) \right| dt \\ & \leq \frac{(b-a)^n}{2n!} \left(\int_0^1 (2t+n-2)^{\frac{q}{q-1}} dt \right)^{1-\frac{1}{q}} \left(\int_0^1 (1-t)^{q(n-1)} \left| f^{(n)}(ta + (1-t)b) \right|^q dt \right)^{\frac{1}{q}}. \end{aligned}$$

Since $|f^{(n)}|^q$ is quasicconvex on $[a, b]$, for $q > 1$, then

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx - \sum_{k=1}^{n-1} \frac{k-1}{2(k+1)!} (a-b)^k f^{(k)}(b) \right| \\ & \leq \frac{(b-a)^n}{2n!} \left(\int_0^1 (2t+n-2)^{\frac{q}{q-1}} dt \right)^{1-\frac{1}{q}} \\ & \quad \times \left(\int_0^1 (1-t)^{q(n-1)} dt \right)^{\frac{1}{q}} \left[\max \left\{ |f^{(n)}(a)|^q, |f^{(n)}(b)|^q \right\} \right]^{\frac{1}{q}} \\ & = \frac{(b-a)^n}{2n!} \left(\frac{1}{q(n-1)+1} \right)^{\frac{1}{q}} \\ & \quad \times \left(\frac{(q-1) \left[n \frac{2q-1}{q-1} - (n-2) \frac{2q-1}{q-1} \right]}{2(2q-1)} \right)^{1-\frac{1}{q}} \left[\max \left\{ |f^{(n)}(a)|^q, |f^{(n)}(b)|^q \right\} \right]^{\frac{1}{q}} \end{aligned}$$

which completes the proof. \square

COROLLARY 2. If we choose $n = 2$ in the inequality (2.2), then we have

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{(b-a)^2}{2} \left(\frac{(q-1)^{q-1}}{(q+1)(2q-1)^{q-1}} \right)^{\frac{1}{q}} [\max \{|f''(a)|^q, |f''(b)|^q\}]^{\frac{1}{q}}. \end{aligned}$$

THEOREM 4. Let $f : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$ be n -time differentiable for $n > 1$ and $0 \leq a < b < \infty$. If $f^{(n)} \in L[a, b]$ and $|f^{(n)}|^q$ is quasi-convex on $[a, b]$, for $q > 1$, then

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx - \sum_{k=1}^{n-1} \frac{k-1}{2(k+1)!} (a-b)^k f^{(k)}(b) \right| \tag{2.3} \\ & \leq \frac{(b-a)^n}{2n(n-2)!} \left(\frac{p(n-2)+2}{[p(n-1)+1][p(n-1)+2]} \right)^{\frac{1}{p}} [\max \{|f^{(n)}(a)|^q, |f^{(n)}(b)|^q\}]^{\frac{1}{q}} \end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $p > 1$.

Proof. Since $|f^{(n)}|^q$ is quasi-convex on $[a, b]$, for $q > 1$, from Lemma 1 and the Hölder integral inequality, it follows that

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx - \sum_{k=1}^{n-1} \frac{k-1}{2(k+1)!} (a-b)^k f^{(k)}(b) \right| \\ & \leq \frac{(b-a)^n}{2n!} \int_0^1 (1-t)^{n-1} (2t+n-2) |f^{(n)}(ta+(1-t)b)| dt \\ & \leq \frac{(b-a)^n}{2n!} \left(\int_0^1 (1-t)^{p(n-1)} (2t+n-2) dt \right)^{\frac{1}{p}} \\ & \quad \times \left(\int_0^1 (2t+n-2) |f^{(n)}(ta+(1-t)b)|^q dt \right)^{\frac{1}{q}} \\ & \leq \frac{(b-a)^n}{2n!} \left(\int_0^1 (1-t)^{p(n-1)} (2t+n-2) dt \right)^{\frac{1}{p}} \\ & \quad \times \left(\int_0^1 (2t+n-2) dt \right)^{\frac{1}{q}} [\max \{|f^{(n)}(a)|^q, |f^{(n)}(b)|^q\}]^{\frac{1}{q}} \\ & = \frac{(b-a)^n}{2n(n-2)!} \left(\frac{p(n-2)+2}{[p(n-1)+1][p(n-1)+2]} \right)^{\frac{1}{p}} [\max \{|f^{(n)}(a)|^q, |f^{(n)}(b)|^q\}]^{\frac{1}{q}}. \end{aligned}$$

On the other hand, we have

$$\int_0^1 (1-t)^{p(n-1)} (2t+n-2) dt = \frac{(n-1)[p(n-2)+2]}{[p(n-1)+1][p(n-1)+2]}.$$

This completes the proof. \square

COROLLARY 3. In Theorem 4, if we choose $n = 2$, we obtain the following inequality:

$$\begin{aligned} & \left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \\ & \leq \frac{(b-a)^2}{4} \left(\frac{2}{(p+1)(p+2)} \right)^{\frac{1}{p}} [\max \{ |f''(a)|^q, |f''(b)|^q \}]^{\frac{1}{q}}. \end{aligned}$$

THEOREM 5. Let $f : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$ be n -time differentiable for $n > 1$ and $0 \leq a < b < \infty$. If $f^{(n)} \in L[a, b]$ and $|f^{(n)}|^q$ is quasicconvex on $[a, b]$, for $q > 1$, then

$$\begin{aligned} & \left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx - \sum_{k=1}^{n-1} \frac{k-1}{2(k+1)!} (a-b)^k f^{(k)}(b) \right| \tag{2.4} \\ & \leq \frac{(b-a)^n}{2n!} \left(\frac{1}{q(n-2)+2} \right)^{\frac{1}{q}} \\ & \quad \times \left(\frac{n^{p+2} - (n+2p+2)(n-2)^{p+1}}{4(p+1)(p+2)} \right)^{\frac{1}{p}} [\max \{ |f^{(n)}(a)|^q, |f^{(n)}(b)|^q \}]^{\frac{1}{q}} \end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $p > 1$.

Proof. From Lemma 1 and the Hölder integral inequality, we have

$$\begin{aligned} & \left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx - \sum_{k=1}^{n-1} \frac{k-1}{2(k+1)!} (a-b)^k f^{(k)}(b) \right| \\ & \leq \frac{(b-a)^n}{2n!} \int_0^1 (1-t)^{n-1} (2t+n-2) |f^{(n)}(ta+(1-t)b)| dt \\ & \leq \frac{(b-a)^n}{2n!} \left(\int_0^1 (1-t)(2t+n-2)^p dt \right)^{\frac{1}{p}} \left(\int_0^1 (1-t)^{nq-2q+1} |f^{(n)}(ta+(1-t)b)|^q dt \right)^{\frac{1}{q}}. \end{aligned}$$

Since $|f^{(n)}|^q$ is quasicconvex on $[a, b]$, for $q > 1$, then we get

$$\begin{aligned} & \left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx - \sum_{k=1}^{n-1} \frac{k-1}{2(k+1)!} (a-b)^k f^{(k)}(b) \right| \\ & \leq \frac{(b-a)^n}{2n!} \left(\int_0^1 (1-t)(2t+n-2)^p dt \right)^{\frac{1}{p}} \\ & \quad \times \left(\int_0^1 (1-t)^{nq-2q+1} dt \right)^{\frac{1}{q}} [\max \{ |f^{(n)}(a)|^q, |f^{(n)}(b)|^q \}]^{\frac{1}{q}} \end{aligned}$$

$$\begin{aligned}
 &= \frac{(b-a)^n}{2n!} \left(\frac{1}{q(n-2)+2} \right)^{\frac{1}{q}} \\
 &\quad \times \left(\frac{n^{p+2} - (n+2p+2)(n-2)^{p+1}}{4(p+1)(p+2)} \right)^{\frac{1}{q}} \left[\max \left\{ \left| f^{(n)}(a) \right|^q, \left| f^{(n)}(b) \right|^q \right\} \right]^{\frac{1}{q}}.
 \end{aligned}$$

This completes the proof. \square

THEOREM 6. For $n \geq 2$, let $f : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$ be n -time differentiable and $0 \leq a < b$. If $f^{(n)} \in L[a, b]$ and $\left| f^{(n)} \right|^q$ is quasiconvex on $[a, b]$, for $q \geq 1$, then the following inequality holds:

$$\begin{aligned}
 &\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx - \sum_{k=1}^{n-1} \frac{k-1}{2(k+1)!} (a-b)^k f^{(k)}(b) \right| \tag{2.5} \\
 &\leq \frac{(b-a)^n}{2n!} \left(\frac{n-1}{n+1} \right) \left[\max \left\{ \left| f^{(n)}(a) \right|^q, \left| f^{(n)}(b) \right|^q \right\} \right]^{\frac{1}{q}}.
 \end{aligned}$$

Proof. From Lemma 1 and using the well known power-mean integral inequality, we have

$$\begin{aligned}
 &\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx - \sum_{k=1}^{n-1} \frac{k-1}{2(k+1)!} (a-b)^k f^{(k)}(b) \right| \\
 &\leq \frac{(b-a)^n}{2n!} \int_0^1 (1-t)^{n-1} (2t+n-2) \left| f^{(n)}(ta+(1-t)b) \right| dt \\
 &\leq \frac{(b-a)^n}{2n!} \left(\int_0^1 (1-t)^{n-1} (2t+n-2) dt \right)^{1-\frac{1}{q}} \\
 &\quad \times \left(\int_0^1 (1-t)^{n-1} (2t+n-2) \left| f^{(n)}(ta+(1-t)b) \right|^q dt \right)^{\frac{1}{q}}.
 \end{aligned}$$

Since $\left| f^{(n)} \right|^q$ is quasiconvex on $[a, b]$, for $q \geq 1$, then we obtain

$$\begin{aligned}
 &\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx - \sum_{k=1}^{n-1} \frac{k-1}{2(k+1)!} (a-b)^k f^{(k)}(b) \right| \\
 &\leq \frac{(b-a)^n}{2n!} \left(\int_0^1 (1-t)^{n-1} (2t+n-2) dt \right)^{1-\frac{1}{q}} \\
 &\quad \times \left(\int_0^1 (1-t)^{n-1} (2t+n-2) dt \right)^{\frac{1}{q}} \left[\max \left\{ \left| f^{(n)}(a) \right|^q, \left| f^{(n)}(b) \right|^q \right\} \right]^{\frac{1}{q}} \\
 &= \frac{(b-a)^n}{2n!} \left(\frac{n-1}{n+1} \right) \left[\max \left\{ \left| f^{(n)}(a) \right|^q, \left| f^{(n)}(b) \right|^q \right\} \right]^{\frac{1}{q}}
 \end{aligned}$$

which completes the proof. \square

COROLLARY 4. Under conditions of Theorem 6, if we choose $q = 1$, then we obtain

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx - \sum_{k=1}^{n-1} \frac{k-1}{2(k+1)!} (a-b)^k f^{(k)}(b) \right| \\ & \leq \frac{(b-a)^n}{2n!} \binom{n-1}{n+1} \left[\max \left\{ \left| f^{(n)}(a) \right|, \left| f^{(n)}(b) \right| \right\} \right]. \end{aligned}$$

REMARK 1. Under conditions of Theorem 6, if we choose $n = 2$, then we obtain inequality (1.2).

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