

A GENERALIZATION OF THE CAUCHY–SCHWARZ INEQUALITY

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Abstract. N. Harvey generalizes the Cauchy-Schwarz inequality to an inequality involving four vectors. Here we show a stronger result than the inequality. Moreover we generalize the result to an inequality involving any number of real or complex vectors.

1. Introduction

Throughout the paper, $\|\cdot\|$ will denote the Euclidean norm (*i.e.* the 2-norm) for vectors. In [1], the author generalizes the well-known Cauchy-Schwarz inequality

$$\|a\|^2\|b\|^2 \geq (a^T b)^2, \forall a, b \in \mathbb{R}^n,$$

into

$$\begin{aligned} & \|a\|^2\|b\|^2 + \|c\|^2\|d\|^2 \\ & \geq 2a^T c b^T d + (a^T b)^2 + (c^T d)^2 - (a^T d)^2 - (b^T c)^2, \forall a, b, c, d \in \mathbb{R}^n \end{aligned} \quad (1.1)$$

In this paper, we will show the following stronger result

$$\|a\|^2\|b\|^2 + \|c\|^2\|d\|^2 \geq (a^T b)^2 + (c^T d)^2 + 2|a^T c b^T d - a^T d b^T c|, \forall a, b, c, d \in \mathbb{R}^n$$

and furthermore we will generalize the inequality as follows:

$$\begin{aligned} \sum_{k=1}^m \|a_{(k)}\|^2 \|b_{(k)}\|^2 & \geq \sum_{k=1}^m (a_{(k)}^T b_{(k)})^2 \\ & + \frac{2}{m-1} \sum_{1 \leq k < l \leq m} |a_{(k)}^T a_{(l)} b_{(k)}^T b_{(l)} - a_{(k)}^T b_{(l)} b_{(k)}^T a_{(l)}| \end{aligned}$$

for any $a_{(k)}, b_{(k)} \in \mathbb{R}^n, k = 1, \dots, m$ and

$$\begin{aligned} \sum_{k=1}^m \|a_{(k)}\|^2 \|b_{(k)}\|^2 & \geq \sum_{k=1}^m |a_{(k)}^T b_{(k)}|^2 \\ & + \frac{2}{m-1} \sum_{1 \leq k < l \leq m} |\operatorname{Re}(\bar{a}_{(k)}^T a_{(l)} b_{(k)}^T \bar{b}_{(l)} - \bar{a}_{(k)}^T \bar{b}_{(l)} b_{(k)}^T a_{(l)})| \end{aligned}$$

for any $a_{(k)}, b_{(k)} \in \mathbb{C}^n, k = 1, \dots, m$.

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2. Proofs of the inequalities

THEOREM 1. *Let a, b, c and d be vectors in \mathbb{R}^n . Then*

$$\|a\|^2\|b\|^2 + \|c\|^2\|d\|^2 \geq (a^T b)^2 + (c^T d)^2 + 2|a^T c b^T d - a^T d b^T c|, \tag{2.1}$$

where the equality holds if and only if $a_i b_j - a_j b_i = c_i d_j - c_j d_i$ for all $i = 1, \dots, n$.

Proof. The proof is direct as in [1, Theorem 1]. Using the Lagrange identity [2]

$$\|a\|^2\|b\|^2 - (a^T b)^2 = \frac{1}{2} \sum_{i,j=1}^n (a_i b_j - a_j b_i)^2,$$

we have

$$\begin{aligned} & \|a\|^2\|b\|^2 - (a^T b)^2 + \|c\|^2\|d\|^2 - (c^T d)^2 \\ &= \frac{1}{2} \sum_{i,j=1}^n [(a_i b_j - a_j b_i)^2 + (c_i d_j - c_j d_i)^2] \\ &= \frac{1}{2} \sum_{i,j=1}^n [(a_i b_j - a_j b_i - c_i d_j + c_j d_i)^2 + 2(a_i b_j - a_j b_i)(c_i d_j - c_j d_i)] \\ &= \frac{1}{2} \sum_{i,j=1}^n (a_i b_j - a_j b_i - c_i d_j + c_j d_i)^2 + 2 \sum_{i=1}^n a_i c_i \sum_{j=1}^n b_j d_j - 2 \sum_{i=1}^n a_i d_i \sum_{j=1}^n b_j c_j \\ &\geq 2a^T c b^T d - 2a^T d b^T c. \end{aligned}$$

Replacing c by $-c$ in the above argument, we get the desired inequality. The condition for equality is obvious. \square

Since $(a^T d)^2 + (b^T c)^2 \geq 2a^T d b^T c$, (2.1) is stronger than (1.1). Now we extend Theorem 1 into an inequality involving any number of real vectors.

THEOREM 2. *Let $a_{(1)}, \dots, a_{(m)}, b_{(1)}, \dots, b_{(m)}$, $m \geq 2$, be vectors in \mathbb{R}^n . Then*

$$\begin{aligned} \sum_{k=1}^m \|a_{(k)}\|^2 \|b_{(k)}\|^2 &\geq \sum_{k=1}^m (a_{(k)}^T b_{(k)})^2 \\ &+ \frac{2}{m-1} \sum_{1 \leq k < l \leq m} |a_{(k)}^T a_{(l)} b_{(k)}^T b_{(l)} - a_{(k)}^T b_{(l)} b_{(k)}^T a_{(l)}|, \end{aligned}$$

where the equality holds if and only if $a_{(k),i} b_{(k),j} - a_{(k),j} b_{(k),i} = a_{(l),i} b_{(l),j} - a_{(l),j} b_{(l),i}$ for all $k, l = 1, \dots, m$ and $i, j = 1, \dots, n$.

Proof. By Theorem 1, we have

$$\begin{aligned} & \|a_{(k)}\|^2 \|b_{(k)}\|^2 - (a_{(k)}^T b_{(k)})^2 + \|a_{(l)}\|^2 \|b_{(l)}\|^2 - (a_{(l)}^T b_{(l)})^2 \\ &\geq 2|a_{(k)}^T a_{(l)} b_{(k)}^T b_{(l)} - a_{(k)}^T b_{(l)} b_{(k)}^T a_{(l)}| \end{aligned}$$

for $1 \leq k < l \leq m$. Therefore

$$\begin{aligned} & \sum_{k=1}^m (||a_{(k)}||^2 ||b_{(k)}||^2 - (a_{(k)}^T b_{(k)})^2) \\ &= \frac{1}{m-1} \sum_{1 \leq k < l \leq m} (||a_{(k)}||^2 ||b_{(k)}||^2 - (a_{(k)}^T b_{(k)})^2 + ||a_{(l)}||^2 ||b_{(l)}||^2 - (a_{(l)}^T b_{(l)})^2) \\ &\geq \frac{2}{m-1} \sum_{1 \leq k < l \leq m} |a_{(k)}^T a_{(l)} b_{(k)}^T b_{(l)} - a_{(k)}^T b_{(l)} b_{(k)}^T a_{(l)}|. \quad \square \end{aligned}$$

A similar inequality exists for complex vectors.

THEOREM 3. *Let a, b, c and d are vectors in \mathbb{C}^n . Then*

$$||a||^2 ||b||^2 + ||c||^2 ||d||^2 \geq |a^T b| + |c^T d| + 2|\text{Re}(\bar{a}^T c b^T \bar{d} - \bar{a}^T \bar{d} b^T c)|,$$

where the equality holds if and only if $\bar{a}_i b_j - \bar{a}_j b_i = \bar{c}_i d_j - \bar{c}_j d_i$ for all $i, j = 1, \dots, n$.

Proof. Using the Lagrange identity for complex numbers [2]

$$||a||^2 ||b||^2 - |a^T b|^2 = \frac{1}{2} \sum_{i,j=1}^n |\bar{a}_i b_j - \bar{a}_j b_i|^2,$$

we have

$$\begin{aligned} & ||a||^2 ||b||^2 - |a^T b|^2 + ||c||^2 ||d||^2 - |c^T d|^2 \\ &= \frac{1}{2} \sum_{i,j=1}^n [|\bar{a}_i b_j - \bar{a}_j b_i|^2 + |\bar{c}_i d_j - \bar{c}_j d_i|^2] \\ &= \frac{1}{2} \sum_{i,j=1}^n [|\bar{a}_i b_j - \bar{a}_j b_i - \bar{c}_i d_j + \bar{c}_j d_i|^2 + 2\text{Re}(\bar{a}_i b_j - \bar{a}_j b_i)(\bar{c}_i d_j - \bar{c}_j d_i)] \\ &= \frac{1}{2} \sum_{i,j=1}^n |\bar{a}_i b_j - \bar{a}_j b_i - \bar{c}_i d_j + \bar{c}_j d_i|^2 + 2\text{Re}(\bar{a}^T c b^T \bar{d} - \bar{a}^T \bar{d} b^T c) \\ &\geq 2\text{Re}(\bar{a}^T c b^T \bar{d} - \bar{a}^T \bar{d} b^T c). \end{aligned}$$

Replacing c by $-c$ in the above argument, we have the desired inequality. \square

Using Theorem 3 and the same argument in the proof of Theorem 2, we can show the following.

THEOREM 4. *Let $a_{(1)}, \dots, a_{(m)}, b_{(1)}, \dots, b_{(m)}$, $m \geq 2$, be vectors in \mathbb{C}^n . Then,*

$$\begin{aligned} \sum_{k=1}^m ||a_{(k)}||^2 ||b_{(k)}||^2 &\geq \sum_{k=1}^m |a_{(k)}^T b_{(k)}|^2 \\ &+ \frac{2}{m-1} \sum_{1 \leq k < l \leq m} |\text{Re}(\bar{a}_{(k)}^T a_{(l)} b_{(k)}^T \bar{b}_{(l)} - \bar{a}_{(k)}^T \bar{b}_{(l)} b_{(k)}^T a_{(l)})|, \end{aligned}$$

where the equality holds if and only if $\bar{a}_{(k),i} b_{(k),j} - \bar{a}_{(k),j} b_{(k),i} = \bar{a}_{(l),i} b_{(l),j} - \bar{a}_{(l),j} b_{(l),i}$ for all $k, l = 1, \dots, m$ and $i, j = 1, \dots, n$.

REFERENCES

- [1] N. HARVEY, *A generalization of the Cauchy-Schwarz inequality involving four vectors*, J. Math. Inequal. **9** (2), 2015.
- [2] S. DRAGOMIR, *A survey on Cauchy-Buniakowsky-Schwartz type discrete inequalities*, J. Inequal. Pure & Appl. Math., **4** (3), 2003.

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