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ON THE CYCLIC HOMOGENEOUS POLYNOMIAL INEQUALITIES OF DEGREE FOUR OF THREE NONNEGATIVE REAL VARIABLES

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Abstract. Let $f(x,y,z)$ is a cyclic homogeneous polynomial of degree four of three nonnegative real variables satisfying the condition $f(1,1,1)=0$. We find necessary and sufficient condition to be true the inequality $f(x,y,z)\geqslant 0$, for this aim we introduce a characteristic polynomial $J_f(t)$ and by its root $t_0>0$ we formulate the condition.

1. Introduction

Inequalities of cyclic or symmetric homogeneous polynomials of three variables are explored in numerous articles [1]–[9]. Tetsuya Ando solves this problem by finding necessary and sufficient conditions of degree three ([1], [2] and [3], Theorem 1.2). We could reformulate the condition in [3], Theorem 1.2 as we introduce an auxiliary function $h(t)=t^2-2t^{-1}$, $t\in(0,+\infty)$. This function is continuous, monotone increasing and takes values in the whole real line.

Theorem. Let a, b, c are real constants and let $t_0>0$ is a root of the characteristic equation $h(t)=t^2-2t^{-1}=a$. For arbitrary nonnegative numbers x, y and z it is true the inequality

$$x^3+y^3+z^3+a(xy^2+yz^2+zx^2)+b(x^2y+y^2z+z^2x)+cxyz\geqslant 0,$$

if and only if when $3+3a+3b+c\geqslant 0$ and $b\geqslant h(t_0^{-1})=t_0^{-2}-2t_0$.

Let a, b, c are real constants and

$$f(x,y,z)=x^4+y^4+z^4+a(x^3y+y^3z+z^3x)+b(xy^3+yz^3+zx^3)$$
$$+c(x^2y^2+y^2z^2+z^2x^2)-(1+a+b+c)xyz(x+y+z)$$

is a cyclic homogeneous polynomial of degree four, $f(1,1,1)=0$.

When x, y and z are real variables Vasile Cirtoaje proves that the necessary and sufficient condition to be true $f(x,y,z)\geqslant 0$ is $3(c+1)\geqslant a^2+ab+b^2$ ([4], Theorem 2.1). Vasile Cirtoaje finds also the necessary and sufficient condition when f is a symmetric polynomial of nonnegative real variables ([4], Theorem 2.6).

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When f is a cyclic polynomial of nonnegative real variables Tetsuya Ando proves that to be true the inequality $f(x,y,z) \geqslant 0$ is necessary and sufficient to be verified at least one of six conditions ([3], Theorem 1.3).

We introduce the characteristic polynomial $J(t) = 2t^4 + at^3 - bt - 2$ and by its root $t_0 > 0$ we formulate the necessary and sufficient condition to be true $f(x,y,z) \geqslant 0$ (Theorem 1).

Necessary and sufficient conditions for cyclic homogeneous polynomial inequalities of degree four for which $f(1,1,1) \geqslant 0$ are pointed by Cirtoaje and Zhou for real variables [5] and for nonnegative real variables ([9], Theorem 2.1 and Theorem 2.2). T. Ando explore symmetric cyclic homogeneous polynomial inequalities of degree five ([3], Theorem 1.4) while V. Cirtoaje of degree six [6], [7], [8].

2. Main results

For brevity we set

$$w_4 = x^4 + y^4 + z^4 - xyz(x+y+z), \quad w_3 = x^3y + y^3z + z^3x - xyz(x+y+z),$$

$$w_2 = x^2y^2 + y^2z^2 + z^2x^2 - xyz(x+y+z) \quad \text{and} \quad w_1 = xy^3 + yz^3 + zx^3 - xyz(x+y+z).$$

REMARK. For arbitrary real numbers x, y, z the following inequalities hold $w_4 \geqslant w_3$, $w_4 \geqslant w_1$ and $w_4 \geqslant w_2 \geqslant 0$, and for arbitrary nonnegative numbers x, y, z the following inequalities hold $w_1 \geqslant 0$, $w_3 \geqslant 0$ and $w_3 w_1 \geqslant (w_2)^2$ ([4], inequality 5.2). The last inequality follows from the identity

$$w_3 w_1 - (w_2)^2 = xyz(x+y+z)(x^2+y^2+z^2-xy-yz-zx)^2 \geqslant 0.$$

Let a, b and c be real constants. For arbitrary nonnegative numbers x, y, z we will explore the inequality $f(x,y,z) = w_4 + aw_3 + bw_1 + cw_2 \geqslant 0$.

The equation $J(t) = 2t^4 + at^3 - bt - 2 = 0$ has at least one root $t_0 \in (0, +\infty)$, because $J(0) = -2$ and $\lim\limits_{t \to +\infty} J(t) = +\infty$. Let us set

$$a_0 = t_0^{-1} - 2t_0, \ \ b_0 = t_0 - 2t_0^{-1} \quad \text{and} \ \ k = (a - a_0)t_0. \tag{1}$$

From $J(t_0) = 0$ we find $b = b_0 + kt_0$, i.e.

$$a = a_0 + kt_0^{-1} = (k+1)t_0^{-1} - 2t_0, \quad \text{and} \quad b = b_0 + kt_0 = (k+1)t_0 - 2t_0^{-1}. \tag{2}$$

REMARK. When $k \geqslant -1$ the equation $J(t) = 0$ has exactly one root $t_0 \in (0, +\infty)$, because

$$J(t) = 2t^4 + \left[(k+1)t_0^{-1} - 2t_0\right]t^3 - \left[(k+1)t_0 - 2t_0^{-1}\right]t - 2 = (t - t_0)J_1(t),$$

where

$$J_1(t) = 2t^3 + (k+1)t_0^{-1}t^2 + (k+1)t + 2t_0^{-1} > 0.$$

THEOREM 1. *Let a, b and c be real constants and $t_0 > 0$ is a root of the characteristic equation $J(t) = 2t^4 + at^3 - bt - 2 = 0$, $k = at_0 + 2t_0^2 - 1$ and $c_0 = (t_0 - t_0^{-1})^2$. For arbitrary nonnegative numbers x, y and z the inequality*

$$f(x,y,z) = w_4 + aw_3 + bw_1 + cw_2 \geqslant 0,$$

holds if and only if

$$c \geqslant \begin{cases} c_0 - 2k \text{ when } k \geqslant 0 \\ r \qquad \text{ when } k \leqslant 0, \text{ where } r = \dfrac{a^2 + ab + b^2}{3} - 1. \end{cases}$$

REMARK. It is always true that $r \geqslant c_0 - 2k$ as $3(r - c_0 + 2k) = k^2(t_0^2 + t_0^{-2} + 1) \geqslant 0$.

When $k \geqslant 0$ and $c = c_0 - 2k$ the equality holds if and only if $x = y = z$ or $\{z = 0, x = t_0 y\}$, or any cyclic permutation thereof.

When $k \geqslant 0$ and $c > c_0 - 2k$ the equality holds if and only if $x = y = z$.

When $k < 0$ and $c > r$ the equality holds if and only if $x = y = z$.

When $k < 0$ and $c = r$ the equality holds when $x = y = z$ and at least for one triplet $(x_0, y_0, 1)$ such that $x_0 > 0$, $x_0 \neq 1$ and $y_0 > 0$, or any cyclic permutation thereof. The only exception is the case $a = b = -2 \Rightarrow c = r = 3$, the equality holds if and only if $x = y = z$.

For completeness we will also add the following theorem.

THEOREM 2. *Let x, y, z be arbitrary nonnegative numbers and a, b, c, d be real constants. A necessary and sufficient condition the inequality*

$$f(x,y,z) = aw_3 + bw_1 + cw_2 + dxyz(x + y + z) \geqslant 0$$

to be true is $a \geqslant 0$, $b \geqslant 0$, $d \geqslant 0$ and $c + 2\sqrt{ab} \geqslant 0$.

Proof Theorem 1. We will divide the proof into several lemmas.

$$\text{We set } \quad p = -\frac{1}{3}(2a + b), \;\; q = -\frac{1}{3}(a + 2b),$$

$$u = x^2 - z^2 - pxy + (p - q)xz + qyz \quad \text{and} \quad v = y^2 - z^2 - qxy + (q - p)yz + pxz.$$

The following identity holds ([4], 3.2)

$$f = w_4 + aw_3 + bw_1 + cw_2 = u^2 - uv + v^2 + (c - r)w_2. \quad \square \tag{3}$$

LEMMA 1.1. ([4], 3.2) *If $c \geqslant r$ then for arbitrary real numbers x, y, z the following inequality holds*

$$f(x,y,z) = w_4 + aw_3 + bw_1 + cw_2 \geqslant 0.$$

Proof. According to identity (3) $f = u^2 - uv + v^2 + (c - r)w_2 \geqslant 0$, because

$$2(u^2 - uv + v^2) = u^2 + v^2 + (u - v)^2 \geqslant 0 \text{ and } w_2 \geqslant 0. \quad \square$$

LEMMA 1.2. *Let x, y, z be arbitrary nonnegative numbers. If $a \geqslant 0$ and $b \geqslant 0$, then the inequality $aw_3 + bw_1 \geqslant 2\sqrt{ab}w_2$ holds.*

The equality holds if and only if $x = y = z$ or $x = y = 0$, or $x : y : z = 0 : \sqrt{a} : \sqrt{b}$, or any cyclic permutation thereof.

Proof. As $a \geqslant 0$, $b \geqslant 0$, $w_3 \geqslant 0$, $w_1 \geqslant 0$ and $w_2 \geqslant 0$, then from $w_3w_1 \geqslant (w_2)^2$ it follows the inequality $aw_3 + bw_1 \geqslant 2\sqrt{abw_3w_1} \geqslant 2\sqrt{ab}w_2$.

LEMMA 1.3. *When $p > 0$, $q > 0$, $pq > 1$ and $(p,q) \neq (2,2)$ the system*

$$\begin{cases} u = x^2 - z^2 - pxy + (p-q)xz + qyz = 0 \\ v = y^2 - z^2 - qxy + (q-p)yz + pxz = 0 \end{cases} \tag{4}$$

has at least one solution $(x_0, y_0, 1)$ such that $x_0 > 0$, $x_0 \neq 1$ and $y_0 > 0$.

Proof. (Similarly to lemma 3.1 from [4]).

If $p = q \neq 2$ then from $p = q > 0$ and $pq > 1$ it follows that $p > 1$, i.e. $(x_0, y_0, z_0) = (p-1, 1, 1) \neq (1,1,1)$ is a solution of the system (4).

Let $p \neq q$ and $z = 1$. From the first equation $u = 0$ we find

$$y_0 = [x^2 + (p-q)x - 1](px-q)^{-1}$$

and we substitute into the second equation $(px-q)^2 v(x, y_0, 1) = (x-1)l(x)$ where

$$\begin{aligned} l(x) = {} & (pq-1)(1-x^3) - [(p-1)^2 + (p^2+2)q - (p+1)q^2 + q^3]x \\ & + [(q-1)^2 + (q^2+2)p - (q+1)p^2 + p^3]x^2. \end{aligned}$$

We find $l(1) = (p-q)[(p-2)^2 + (q-2)^2 - (p-2)(q-2)]$.

Case 1. Let $p > q$. From $l(1) > 0$ and $\lim\limits_{x \to +\infty} l(x) = -\infty$ follows that it exists a root $x_0 \in (1; +\infty)$.

From $x_0 > 1 \Rightarrow px_0 - q > p - q > 0$ and $x_0^2 + (p-q)x_0 - 1 > 1 + (p-q) - 1 > 0$, i.e. $y_0 = [x_0^2 + (p-q)x_0 - 1](px_0 - q)^{-1} > 0$.

Case 2. Let $p < q$. From $l(1) < 0$ and $l(0) = pq - 1 > 0$ follows that it exists a root $x_0 \in (0; 1)$.

From $x_0 < 1 \Rightarrow px_0 - q < p - q < 0$ and from $0 < x_0 < 1 \Rightarrow x_0^2 + (p-q)x_0 - 1 = (x_0^2 - 1) + (p-q)x_0 < 0$, i.e. again $y_0 > 0$. □

Let us proceed to the proof of the theorem.

When $k = 0$ then we have $3c_0 = a_0^2 + a_0b_0 + b_0^2 - 3$ and according to Lemma 1.1 it follows that $w_4 + a_0w_3 + b_0w_1 + c_0w_2 \geqslant 0$. In this case $p = t_0$, $q = t_0^{-1}$ and the solution of system (4) are: $\{x = y = z\}$, $\{x = 0, x = t_0 z\}$, $\{y = 0, z = t_0 x\}$ and $\{z = 0, x = t_0 y\}$, i.e. the equality holds if only if $x = y = z$ or $\{z = 0, x = t_0 y\}$, or any cyclic permutation thereof.

When $t > 0$ according to Lemma 1.2. we have $t^{-1}w_3 + tw_1 \geqslant 2w_2$. The equality holds if and only if $x = y = z$ or $x = y = 0$, or $\{z = 0, x = ty\}$, or any cyclic permutation thereof.

Let $k \geqslant 0$ and $c \geqslant c_0 - 2k$. Then

$$\begin{aligned} f &= w_4 + aw_3 + bw_1 + cw_2 \\ &= w_4 + (a_0 + kt_0^{-1})w_3 + (b_0 + kt_0)w_1 + (c_0 + c - c_0)w_2 \\ &= w_4 + a_0w_3 + b_0w_1 + c_0w_2 + k(t_0^{-1}w_3 + t_0w_1) + (c - c_0)w_2 \\ &\geqslant 0 + 2kw_2 + (c - c_0)w_2 = (c - c_0 + 2k)w_2 \geqslant 0 \end{aligned} \tag{5}$$

When $k \geqslant 0$ and $c = c_0 - 2k$ the equality holds if only if $x = y = z$ or $\{z = 0, x = t_0y\}$, or any cyclic permutation thereof.

When $k \geqslant 0$ and $c > c_0 - 2k$ the equality holds if only if $x = y = z$.

When $k \geqslant 0$ and $c < c_0 - 2k$ the inequality is not true. From the expression $f = w_4 + aw_3 + bw_1 + (c_0 - 2k)w_2 + (c - c_0 + 2k)w_2$ we obtain $f(t_0, 1, 0) = 0 + (c - c_0 + 2k)t_0^2 < 0$.

When $k < 0$ and $c \geqslant r$ the inequality holds according to Lemma 1.1.

Let $k < 0$ and $c < r$. From $t_0 > 0 \Rightarrow p = -\frac{1}{3}(2a + b) = t_0 - \frac{k}{3}(t_0 + 2t_0^{-1}) > t_0 > 0$, $q = -\frac{1}{3}(a + 2b) = t_0^{-1} - \frac{k}{3}(t_0^{-1} + 2t_0) > t_0^{-1} > 0$ and $pq > t_0t_0^{-1} = 1$.

According to Lemma 1.3 when $(p, q) \neq (2, 2)$ it exists positive numbers $(x_0, y_0, 1) \neq (1, 1, 1)$ such that $u(x_0, y_0, 1) = 0$ and $v(x_0, y_0, 1) = 0$. According to the identity (3) we have $f = u^2 - uv + v^2 + (c - r)w_2$ and $f(x_0, y_0, 1) = 0 + (c - r)w_2(x_0, y_0, 1) < 0$ because $c - r < 0$ and $w_2(x_0, y_0, 1) > 0$.

When $p = q = 2$ then $u(x, 1, 1) = (x - 1)^2$, $v(x, 1, 1) = 0$, $w_2(x, 1, 1) = (x - 1)^2$ and for $x_1 = 1 + \sqrt{0.5(r - c)}$ then

$$f(x_1, 1, 1) = [0.5(r - c)]^2 + (c - r)[0.5(r - c)] = -0.25(r - c)^2 < 0.$$

When $k < 0$ and $c > r$ the equality holds if only if $x = y = z$. When $k < 0$ and $c = r$ the equality holds if only when $x = y = z$ and at least for one triple $(x_0, y_0, 1)$ such that $x_0 > 0$, $x_0 \neq 1$ and $y_0 > 0$, or any cyclic permutation thereof, with the only exception of the case $a = b = -2 \Rightarrow c = r = 3$. When $a = b = -2$ the equality holds if and only if $x = y = z$. $\square$

Proof of Theorem 2. Necessity. From $0 \leqslant f(1, 1, 1) = 3d \Rightarrow d \geqslant 0$. Let n be an arbitrary natural number. From $0 \leqslant f(n, n^{-3}, 0) = a + bn^{-8} + cn^{-4}$ follows that $\lim\limits_{n \to +\infty}(a + bn^{-8} + cn^{-4}) = a \geqslant 0$. Analogously, from $f(n^{-3}, n, 0) \geqslant 0$ follows that $b \geqslant 0$. From $f(\sqrt{b}, \sqrt{a}, 0) = ab(c + 2\sqrt{ab}) \geqslant 0$. If $a > 0$, $b > 0 \Rightarrow c + 2\sqrt{ab} \geqslant 0$. If $a = 0$, then from $0 \leqslant f(n, n^{-1}, 0) = c + bn^{-2}$ and $\lim\limits_{n \to +\infty}(c + bn^{-2}) = c$ it follows that $c \geqslant 0$. Analogously, from $b = 0 \Rightarrow c \geqslant 0$, i.e. always $c + 2\sqrt{ab} \geqslant 0$.

Sufficiency. We apply Lemma 1.2 and we obtain

$$f = aw_3 + bw_1 + cw_2 + dxyz(x + y + z) \geqslant (2\sqrt{ab} + c)w_2 + 0 \geqslant 0. \quad \square$$

REMARK. All identities are verified via the Maplesoft platform.

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