

## SOME NEW INEQUALITIES FOR $K$ -FRAMES

ZHONG-QI XIANG

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*Abstract.* In this paper, we establish some inequalities for dual  $K$ -frames from the point of view of operator theory. We also present a new inequality for Parseval  $K$ -frames associated with a scalar  $\lambda \in [0, 1]$ , which is more general and covers one existing corresponding result recently obtained by F. Arabyani Neyshaburi et al.

### 1. Introduction

Frames (classical frames), appeared first in the early 1950's, offer us an important tool in dozens of fields because of their flexibility and redundancy, see [4, 5, 6, 7, 8, 14, 15] for more information on frame theory and its applications.

The concept of  $K$ -frames was introduced by L. Găvruta in [10] to investigate the atomic systems associated with a linear and bounded operator  $K$ . A  $K$ -frame is a generalization of a frame, which allows an atomic decomposition of elements from the range of  $K$  and, in general, the range may not be closed. When  $K$  is an orthogonal projection, a  $K$ -frame turns to be an atom system for subspace which was proposed by H. G. Feichtinger and T. Werther in [9]. It should be remarked that in many ways  $K$ -frames behave completely differently from frames as shown in [1, 12, 16, 17], though the definition of a  $K$ -frame is similar to a frame in form.

We need to recall some notations and basic definitions.

Throughout this paper, we use  $\mathcal{H}$ ,  $\mathbb{R}$ , and  $\mathbb{J}$  respectively to denote a separable Hilbert space, the field of real numbers, and a countable index set. The notation  $B(\mathcal{H})$  is reserved for the set of all linear bounded operators on  $\mathcal{H}$ .

**DEFINITION 1.1.** Suppose  $K \in B(\mathcal{H})$ . A sequence of vectors  $\{f_j\}_{j \in \mathbb{J}}$  is said to be a  $K$ -frame for  $\mathcal{H}$ , if there exist two constants  $0 < C \leq D < \infty$  such that

$$C\|K^*f\|^2 \leq \sum_{j \in \mathbb{J}} |\langle f, f_j \rangle|^2 \leq D\|f\|^2, \quad \forall f \in \mathcal{H}. \quad (1.1)$$

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If only the right-hand inequality of (1.1) is satisfied, then we call  $\{f_j\}_{j \in \mathbb{J}}$  a Bessel sequence for  $\mathcal{H}$ . A  $K$ -frame  $\{f_j\}_{j \in \mathbb{J}}$  for  $\mathcal{H}$  is said to be Parseval, if

$$\|K^*f\|^2 = \sum_{j \in \mathbb{J}} |\langle f, f_j \rangle|^2, \quad \forall f \in \mathcal{H}. \tag{1.2}$$

Let  $\{f_j\}_{j \in \mathbb{J}}$  be a Parseval  $K$ -frame for  $\mathcal{H}$ . Then it is easy to check that

$$KK^*f = \sum_{j \in \mathbb{J}} \langle f, f_j \rangle f_j, \quad \forall f \in \mathcal{H}. \tag{1.3}$$

DEFINITION 1.2. Suppose  $K \in B(\mathcal{H})$  and  $\{f_j\}_{j \in \mathbb{J}}$  is a  $K$ -frame for  $\mathcal{H}$ . A Bessel sequence  $\{g_j\}_{j \in \mathbb{J}}$  for  $\mathcal{H}$  is called a dual  $K$ -frame of  $\{f_j\}_{j \in \mathbb{J}}$  if

$$Kf = \sum_{j \in \mathbb{J}} \langle f, g_j \rangle f_j, \quad \forall f \in \mathcal{H}. \tag{1.4}$$

Let  $\{f_j\}_{j \in \mathbb{J}}$  be a Bessel sequence for  $\mathcal{H}$ . For any  $\mathbb{I} \subset \mathbb{J}$ , we let  $\mathbb{I}^c = \mathbb{J} \setminus \mathbb{I}$  and define the operator  $S_{\mathbb{I}}: \mathcal{H} \rightarrow \mathcal{H}$  by

$$S_{\mathbb{I}}f = \sum_{j \in \mathbb{I}} \langle f, f_j \rangle f_j, \tag{1.5}$$

which is positive, linear bounded and self-adjoint.

R. Balan et al. [3] found a surprising identity for Parseval frames when they devoted to the study of efficient algorithms for signal reconstruction. Moreover, in [3] the following inequality was given:

THEOREM 1.3. *If  $\{f_j\}_{j \in \mathbb{J}}$  is a Parseval frame for  $\mathcal{H}$ , then for every  $\mathbb{I} \subset \mathbb{J}$  and every  $f \in \mathcal{H}$ , we have*

$$\sum_{j \in \mathbb{I}} |\langle f, f_j \rangle|^2 + \left\| \sum_{j \in \mathbb{I}^c} \langle f, f_j \rangle f_j \right\|^2 = \sum_{j \in \mathbb{I}^c} |\langle f, f_j \rangle|^2 + \left\| \sum_{j \in \mathbb{I}} \langle f, f_j \rangle f_j \right\|^2 \geq \frac{3}{4} \|f\|^2. \tag{1.6}$$

Later on, P. Găvruta in [11] extended inequality (1.6) to general frames and dual frames. Recently, F. Arabyani Neyshaburi et al. [2] obtained the following inequalities for Parseval  $K$ -frames on the basis of the work in [3, 11].

THEOREM 1.4. *Suppose  $K \in B(\mathcal{H})$  and  $\{f_j\}_{j \in \mathbb{J}}$  is a Parseval  $K$ -frame for  $\mathcal{H}$ . Then for every  $\mathbb{I} \subset \mathbb{J}$  and every  $f \in \mathcal{H}$ ,*

$$\begin{aligned} & \operatorname{Re} \left( \sum_{j \in \mathbb{I}^c} \langle f, f_j \rangle \overline{\langle KK^*f, f_j \rangle} \right) + \left\| \sum_{j \in \mathbb{I}} \langle f, f_j \rangle f_j \right\|^2 \\ &= \operatorname{Re} \left( \sum_{j \in \mathbb{I}} \langle f, f_j \rangle \overline{\langle KK^*f, f_j \rangle} \right) + \left\| \sum_{j \in \mathbb{I}^c} \langle f, f_j \rangle f_j \right\|^2 \geq \frac{3}{4} \|KK^*f\|^2. \end{aligned} \tag{1.7}$$

**THEOREM 1.5.** *Suppose  $K \in B(\mathcal{H})$  and  $\{f_j\}_{j \in \mathbb{J}}$  is a Parseval  $K$ -frame for  $\mathcal{H}$ . Then for every  $\mathbb{I} \subset \mathbb{J}$  and every  $f \in \mathcal{H}$ , we obtain*

$$\begin{aligned} \frac{1}{2} \|KK^*f\|^2 &\leq \left\| \sum_{j \in \mathbb{I}} \langle f, f_j \rangle f_j \right\|^2 + \left\| \sum_{j \in \mathbb{I}^c} \langle f, f_j \rangle f_j \right\|^2 \\ &\leq 2\|K\|^2\|K^*f\|^2 - \frac{1}{2}\|KK^*f\|^2. \end{aligned} \tag{1.8}$$

In this paper, we establish some inequalities for dual  $K$ -frames from the point of view of operator theory. We also present a new inequality for Parseval  $K$ -frames associated with a scalar  $\lambda \in [0, 1]$  and show that Theorem 1.4 is a particular case of our result when taking  $\lambda = \frac{1}{2}$ . Finally, we point out that the upper bound condition of the middle term in inequality (1.8) can be replaced with a better one under the condition that  $S_{\mathbb{I}}$  commutes with  $S_{\mathbb{I}^c}$ .

### 2. Main results and their proofs

To prove the main results, we need the following lemmas.

**LEMMA 2.1.** (see [6]) *Suppose that  $\mathcal{T} \in B(\mathcal{H})$  has closed range, then there exists a unique operator  $\mathcal{T}^\dagger \in B(\mathcal{H})$ , called the pseudo-inverse of  $\mathcal{T}$ , satisfying*

$$\mathcal{T}\mathcal{T}^\dagger\mathcal{T} = \mathcal{T}, \quad \mathcal{T}^\dagger\mathcal{T}\mathcal{T}^\dagger = \mathcal{T}^\dagger, \quad (\mathcal{T}^\dagger)^* = (\mathcal{T}^*)^\dagger.$$

**LEMMA 2.2.** (see [13]) *Suppose that  $U, V, \mathcal{T} \in B(\mathcal{H})$ , that  $U + V = \mathcal{T}$ , and that  $\mathcal{T}$  has closed range. Then we have*

$$\mathcal{T}^*\mathcal{T}^\dagger U + V^*\mathcal{T}^\dagger V = V^*\mathcal{T}^\dagger\mathcal{T} + U^*\mathcal{T}^\dagger U.$$

**LEMMA 2.3.** *If  $U, V, K \in B(\mathcal{H})$  satisfy  $U + V = K$ , then*

$$U^*U + \frac{1}{2}(V^*K + K^*V) = V^*V + \frac{1}{2}(U^*K + K^*U) \geq \frac{3}{4}K^*K.$$

*Proof.* We have

$$\begin{aligned} U^*U + \frac{1}{2}(V^*K + K^*V) &= U^*U + \frac{1}{2}((K^* - U^*)K + K^*(K - U)) \\ &= U^*U - \frac{1}{2}(U^*K + K^*U) + K^*K \end{aligned}$$

and

$$\begin{aligned} V^*V + \frac{1}{2}(U^*K + K^*U) &= (K^* - U^*)(K - U) + \frac{1}{2}(U^*K + K^*U) \\ &= U^*U - (K^*U + U^*K) + K^*K + \frac{1}{2}(U^*K + K^*U) \end{aligned}$$

$$\begin{aligned}
&= U^*U - \frac{1}{2}(U^*K + K^*U) + K^*K \\
&= \left(U - \frac{1}{2}K\right)^* \left(U - \frac{1}{2}K\right) + \frac{3}{4}K^*K \\
&\geq \frac{3}{4}K^*K. \quad \square
\end{aligned}$$

LEMMA 2.4. *If  $U, V, K \in B(\mathcal{H})$  satisfy  $U + V = KK^*$ , then for any  $\lambda \in [0, 1]$  we have*

$$\begin{aligned}
U^*U + \lambda(V^*KK^* + KK^*V) &= V^*V + (1 - \lambda)(U^*KK^* + KK^*U) + (2\lambda - 1)(KK^*)^2 \\
&\geq (2\lambda - \lambda^2)(KK^*)^2. \tag{2.1}
\end{aligned}$$

*Proof.* For any  $\lambda \in [0, 1]$  we obtain

$$\begin{aligned}
U^*U + \lambda(V^*KK^* + KK^*V) &= U^*U + \lambda((KK^* - U^*)KK^* + KK^*(KK^* - U)) \\
&= U^*U + \lambda((KK^*)^2 - U^*KK^* + (KK^*)^2 - KK^*U) \\
&= U^*U - \lambda(U^*KK^* + KK^*U) + 2\lambda(KK^*)^2. \tag{2.2}
\end{aligned}$$

We also have

$$\begin{aligned}
&V^*V + (1 - \lambda)(U^*KK^* + KK^*U) + (2\lambda - 1)(KK^*)^2 \\
&= (KK^* - U^*)(KK^* - U) + (1 - \lambda)(U^*KK^* + K^*KU) + (2\lambda - 1)(KK^*)^2 \\
&= U^*U - (U^*KK^* + KK^*U) + (1 - \lambda)(U^*KK^* + KK^*U) + 2\lambda(KK^*)^2 \\
&= U^*U - \lambda(U^*KK^* + KK^*U) + 2\lambda(KK^*)^2 \\
&= (U - \lambda KK^*)^*(U - \lambda KK^*) + (2\lambda - \lambda^2)(KK^*)^2 \\
&\geq (2\lambda - \lambda^2)(KK^*)^2. \tag{2.3}
\end{aligned}$$

Combination of (2.2) and (2.3) yields (2.1).  $\square$

We first establish some inequalities for dual  $K$ -frames.

THEOREM 2.5. *Suppose  $K \in B(\mathcal{H})$ . Let  $\{f_j\}_{j \in \mathbb{J}}$  be a  $K$ -frame for  $\mathcal{H}$  and  $\{g_j\}_{j \in \mathbb{J}}$  be a dual  $K$ -frame of  $\{f_j\}_{j \in \mathbb{J}}$ . Then for every  $\mathbb{I} \subset \mathbb{J}$  and every  $f \in \mathcal{H}$ , we have*

$$\begin{aligned}
\frac{3}{4}\|Kf\|^2 &\leq \operatorname{Re} \sum_{j \in \mathbb{I}} \langle f, g_j \rangle \overline{\langle Kf, f_j \rangle} + \left\| \sum_{j \in \mathbb{I}^c} \langle f, g_j \rangle f_j \right\|^2 \\
&= \operatorname{Re} \sum_{j \in \mathbb{I}^c} \langle f, g_j \rangle \overline{\langle Kf, f_j \rangle} + \left\| \sum_{j \in \mathbb{I}} \langle f, g_j \rangle f_j \right\|^2 \\
&\leq \frac{3\|K\|^2 + \|F_{\mathbb{I}} - F_{\mathbb{I}^c}\|^2}{4} \|f\|^2, \tag{2.4}
\end{aligned}$$

where the operator  $F_{\mathbb{I}} : \mathcal{H} \rightarrow \mathcal{H}$  is defined by  $F_{\mathbb{I}} = \sum_{j \in \mathbb{I}} \langle f, g_j \rangle f_j$ .

*Proof.* For any  $\mathbb{I} \subset \mathbb{J}$  we have  $F_{\mathbb{I}} + F_{\mathbb{I}^c} = K$ . Noting that

$$\langle K^* F_{\mathbb{I}} f, f \rangle = \langle F_{\mathbb{I}} f, K f \rangle = \sum_{j \in \mathbb{I}} \langle f, g_j \rangle \langle f_j, K f \rangle = \sum_{j \in \mathbb{I}} \langle f, g_j \rangle \overline{\langle K f, f_j \rangle}$$

for each  $f \in \mathcal{H}$ , by Lemma 2.3 we obtain

$$\begin{aligned} & \operatorname{Re} \sum_{j \in \mathbb{I}} \langle f, g_j \rangle \overline{\langle K f, f_j \rangle} + \left\| \sum_{j \in \mathbb{I}^c} \langle f, g_j \rangle f_j \right\|^2 \\ &= \frac{1}{2} (\langle F_{\mathbb{I}}^* K f, f \rangle + \langle K^* F_{\mathbb{I}} f, f \rangle) + \|F_{\mathbb{I}^c} f\|^2 \\ &= \frac{1}{2} (\langle F_{\mathbb{I}^c}^* K f, f \rangle + \langle K^* F_{\mathbb{I}^c} f, f \rangle) + \|F_{\mathbb{I}} f\|^2 \\ &= \operatorname{Re} \sum_{j \in \mathbb{I}^c} \langle f, g_j \rangle \overline{\langle K f, f_j \rangle} + \left\| \sum_{j \in \mathbb{I}} \langle f, g_j \rangle f_j \right\|^2 \\ &\geq \frac{3}{4} \|K f\|^2. \end{aligned}$$

We now prove the right-hand inequality of (2.4). For any  $f \in \mathcal{H}$  we get

$$\begin{aligned} & \operatorname{Re} \sum_{j \in \mathbb{I}^c} \langle f, g_j \rangle \overline{\langle K f, f_j \rangle} + \left\| \sum_{j \in \mathbb{I}} \langle f, g_j \rangle f_j \right\|^2 \\ &= \operatorname{Re} \langle F_{\mathbb{I}^c}^* K f, f \rangle + \langle F_{\mathbb{I}} f, F_{\mathbb{I}} f \rangle \\ &= \operatorname{Re} \langle K f, (K - F_{\mathbb{I}}) f \rangle + \langle F_{\mathbb{I}} f, F_{\mathbb{I}} f \rangle \\ &= \operatorname{Re} (\langle K f, K f \rangle - \langle K f, F_{\mathbb{I}} f \rangle) + \langle F_{\mathbb{I}} f, F_{\mathbb{I}} f \rangle \\ &= \langle K f, K f \rangle - \operatorname{Re} \langle K f, F_{\mathbb{I}} f \rangle + \langle F_{\mathbb{I}} f, F_{\mathbb{I}} f \rangle \\ &= \langle K f, K f \rangle - \operatorname{Re} (\langle (K - F_{\mathbb{I}}) f, F_{\mathbb{I}} f \rangle) \\ &= \langle K f, K f \rangle - \operatorname{Re} \langle F_{\mathbb{I}^c} f, F_{\mathbb{I}} f \rangle \\ &= \langle K f, K f \rangle - \frac{1}{2} \langle F_{\mathbb{I}} f, F_{\mathbb{I}^c} f \rangle - \frac{1}{2} \langle F_{\mathbb{I}^c} f, F_{\mathbb{I}} f \rangle \\ &= \frac{3}{4} \|K f\|^2 + \frac{1}{4} \langle F_{\mathbb{I}} f + F_{\mathbb{I}^c} f, F_{\mathbb{I}} f + F_{\mathbb{I}^c} f \rangle - \frac{1}{2} \langle F_{\mathbb{I}} f, F_{\mathbb{I}^c} f \rangle - \frac{1}{2} \langle F_{\mathbb{I}^c} f, F_{\mathbb{I}} f \rangle \\ &= \frac{3}{4} \|K f\|^2 + \frac{1}{4} \langle (F_{\mathbb{I}} - F_{\mathbb{I}^c}) f, (F_{\mathbb{I}} - F_{\mathbb{I}^c}) f \rangle \\ &\leq \frac{3}{4} \|K\|^2 \|f\|^2 + \frac{1}{4} \|F_{\mathbb{I}} - F_{\mathbb{I}^c}\|^2 \|f\|^2 = \frac{3\|K\|^2 + \|F_{\mathbb{I}} - F_{\mathbb{I}^c}\|^2}{4} \|f\|^2. \end{aligned}$$

This completes the proof.  $\square$

Let  $\{v_j\}_{j \in \mathbb{J}}$  be a bounded sequence of complex numbers. If in Lemma 2.3 we take

$$U f = \sum_{j \in \mathbb{J}} v_j \langle f, g_j \rangle f_j, \quad V f = \sum_{j \in \mathbb{J}} (1 - v_j) \langle f, g_j \rangle f_j,$$

then we have

**THEOREM 2.6.** *Suppose  $K \in B(\mathcal{H})$ . Let  $\{f_j\}_{j \in \mathbb{J}}$  be a  $K$ -frame for  $\mathcal{H}$  and  $\{g_j\}_{j \in \mathbb{J}}$  be a dual  $K$ -frame of  $\{f_j\}_{j \in \mathbb{J}}$ . Then for all bounded sequence  $\{v_j\}_{j \in \mathbb{J}}$  and all  $f \in \mathcal{H}$ , we have*

$$\begin{aligned} & \operatorname{Re} \sum_{j \in \mathbb{J}} v_j \langle f, g_j \rangle \overline{\langle Kf, f_j \rangle} + \left\| \sum_{j \in \mathbb{J}} (1 - v_j) \langle f, g_j \rangle f_j \right\|^2 \\ &= \operatorname{Re} \sum_{j \in \mathbb{J}} (1 - v_j) \langle f, g_j \rangle \overline{\langle Kf, f_j \rangle} + \left\| \sum_{j \in \mathbb{J}} v_j \langle f, g_j \rangle f_j \right\|^2 \geq \frac{3}{4} \|Kf\|^2. \end{aligned}$$

*Proof.* The result follows immediately from the left-hand inequality of (2.4) if we take  $\mathbb{I} \subset \mathbb{J}$  and

$$v_j = \begin{cases} 1, & j \in \mathbb{I}, \\ 0, & j \in \mathbb{I}^c. \end{cases} \quad \square$$

**THEOREM 2.7.** *Suppose that  $K \in B(\mathcal{H})$  is positive and that it has closed range. Let  $\{f_j\}_{j \in \mathbb{J}}$  be a  $K$ -frame for  $\mathcal{H}$  and  $\{g_j\}_{j \in \mathbb{J}}$  be a dual  $K$ -frame of  $\{f_j\}_{j \in \mathbb{J}}$ . Then for every  $\mathbb{I} \subset \mathbb{J}$  and every  $f \in \mathcal{H}$ , we have*

$$\begin{aligned} & \operatorname{Re} \sum_{j \in \mathbb{I}} \langle f, g_j \rangle \langle f_j, K^\dagger Kf \rangle + \left\langle \sum_{j \in \mathbb{I}^c} \langle f, g_j \rangle K^\dagger f_j, \sum_{j \in \mathbb{I}^c} \langle f, g_j \rangle f_j \right\rangle \\ &= \operatorname{Re} \sum_{j \in \mathbb{I}^c} \langle K^\dagger Kf, f_j \rangle \langle g_j, f \rangle + \left\langle \sum_{j \in \mathbb{I}} \langle f, g_j \rangle K^\dagger f_j, \sum_{j \in \mathbb{I}^c} \langle f, g_j \rangle f_j \right\rangle \\ &\geq \frac{3}{4} \|K^{\frac{1}{2}} f\|^2, \end{aligned} \tag{2.5}$$

where  $K^\dagger$  denotes the pseudo-inverse of  $K$ .

*Proof.* Since  $K$  is positive, it is self-adjoint and thus by Lemma 2.1,  $(K^\dagger)^* = (K^*)^\dagger = K^\dagger$ . Hence,  $\langle K^\dagger F_{\mathbb{I}} f, F_{\mathbb{I}} f \rangle, \langle K^\dagger F_{\mathbb{I}^c} f, F_{\mathbb{I}^c} f \rangle \in \mathbb{R}$  for each  $f \in \mathcal{H}$ . By Lemma 2.2 we get

$$\begin{aligned} & \operatorname{Re} \sum_{j \in \mathbb{I}} \langle f, g_j \rangle \langle f_j, K^\dagger Kf \rangle + \left\langle \sum_{j \in \mathbb{I}^c} \langle f, g_j \rangle K^\dagger f_j, \sum_{j \in \mathbb{I}^c} \langle f, g_j \rangle f_j \right\rangle \\ &= \operatorname{Re} \langle F_{\mathbb{I}} f, K^\dagger Kf \rangle + \langle K^\dagger F_{\mathbb{I}^c} f, F_{\mathbb{I}^c} f \rangle \\ &= \operatorname{Re} \langle KK^\dagger F_{\mathbb{I}} f, f \rangle + \langle F_{\mathbb{I}^c}^* K^\dagger F_{\mathbb{I}^c} f, f \rangle \\ &= \operatorname{Re} \langle (KK^\dagger F_{\mathbb{I}} + F_{\mathbb{I}^c}^* K^\dagger F_{\mathbb{I}^c}) f, f \rangle \\ &= \operatorname{Re} \langle (F_{\mathbb{I}^c}^* K^\dagger K + F_{\mathbb{I}}^* K^\dagger F_{\mathbb{I}}) f, f \rangle \\ &= \operatorname{Re} \langle (F_{\mathbb{I}^c}^* K^\dagger Kf, f) + \langle F_{\mathbb{I}}^* K^\dagger F_{\mathbb{I}} f, f \rangle \rangle \\ &= \operatorname{Re} \langle \langle K^\dagger Kf, F_{\mathbb{I}^c} f \rangle + \langle K^\dagger F_{\mathbb{I}} f, F_{\mathbb{I}} f \rangle \rangle \\ &= \operatorname{Re} \langle K^\dagger Kf, F_{\mathbb{I}^c} f \rangle + \operatorname{Re} \langle K^\dagger F_{\mathbb{I}} f, F_{\mathbb{I}} f \rangle \end{aligned}$$

$$\begin{aligned} &= \operatorname{Re} \overline{\langle F_{\mathbb{I}^c} f, K^\dagger K f \rangle} + \langle K^\dagger F_{\mathbb{I}} f, F_{\mathbb{I}} f \rangle \\ &= \operatorname{Re} \sum_{j \in \mathbb{I}^c} \langle K^\dagger K f, f_j \rangle \langle g_j, f \rangle + \left\langle \sum_{j \in \mathbb{I}} \langle f, g_j \rangle K^\dagger f_j, \sum_{j \in \mathbb{I}^c} \langle f, g_j \rangle f_j \right\rangle. \end{aligned}$$

Now, combining Lemmas 2.1 and 2.2 we have

$$\begin{aligned} &\operatorname{Re} \sum_{j \in \mathbb{I}} \langle f, g_j \rangle \langle f_j, K^\dagger K f \rangle + \left\langle \sum_{j \in \mathbb{I}^c} \langle f, g_j \rangle K^\dagger f_j, \sum_{j \in \mathbb{I}^c} \langle f, g_j \rangle f_j \right\rangle \\ &= \operatorname{Re} \langle (K K^\dagger F_{\mathbb{I}} + F_{\mathbb{I}^c}^* K^\dagger F_{\mathbb{I}^c}) f, f \rangle \\ &= \operatorname{Re} \langle (K K^\dagger (K - F_{\mathbb{I}^c}) + F_{\mathbb{I}^c}^* K^\dagger F_{\mathbb{I}^c}) f, f \rangle \\ &= \langle K f, f \rangle - \operatorname{Re} \langle K K^\dagger F_{\mathbb{I}^c} f, f \rangle + \langle F_{\mathbb{I}^c}^* K^\dagger F_{\mathbb{I}^c} f, f \rangle \\ &= \langle K^{\frac{1}{2}} f, K^{\frac{1}{2}} f \rangle - \operatorname{Re} \langle K^{\frac{1}{2}} K^{\frac{1}{2}} K^\dagger F_{\mathbb{I}^c} f, f \rangle + \langle (K^{\frac{1}{2}} K^\dagger F_{\mathbb{I}^c})^* (K^{\frac{1}{2}} K^\dagger F_{\mathbb{I}^c}) f, f \rangle \\ &= \frac{3}{4} \|K^{\frac{1}{2}} f\|^2 + \left\langle \frac{1}{2} K^{\frac{1}{2}} f - K^{\frac{1}{2}} K^\dagger S_{\mathbb{I}^c} f, \frac{1}{2} K^{\frac{1}{2}} f - K^{\frac{1}{2}} K^\dagger F_{\mathbb{I}^c} f \right\rangle \\ &\geq \frac{3}{4} \|K^{\frac{1}{2}} f\|^2 \end{aligned}$$

for every  $f \in \mathcal{H}$ . This completes the proof.  $\square$

In the following we give an inequality for Parseval  $K$ -frames, where a scalar  $\lambda \in [0, 1]$  is involved.

**THEOREM 2.8.** *Suppose  $K \in B(\mathcal{H})$  and  $\{f_j\}_{j \in \mathbb{J}}$  is a Parseval  $K$ -frame for  $\mathcal{H}$ . Then for any  $\lambda \in [0, 1]$ , for all  $\mathbb{I} \subset \mathbb{J}$  and all  $f \in \mathcal{H}$ , we have*

$$\begin{aligned} &2\lambda \left( \operatorname{Re} \sum_{j \in \mathbb{I}^c} \langle f, f_j \rangle \overline{\langle K K^* f, f_j \rangle} \right) + \left\| \sum_{j \in \mathbb{I}} \langle f, g_j \rangle f_j \right\|^2 \\ &= 2(1 - \lambda) \left( \operatorname{Re} \sum_{j \in \mathbb{I}} \langle f, g_j \rangle \overline{\langle K K^* f, f_j \rangle} \right) + \left\| \sum_{j \in \mathbb{I}^c} \langle f, f_j \rangle f_j \right\|^2 + (2\lambda - 1) \|K K^* f\|^2 \\ &\geq (2\lambda - \lambda^2) \|K K^* f\|^2. \end{aligned}$$

*Proof.* For every  $\mathbb{I} \subset \mathbb{J}$ , by (1.3) we have  $S_{\mathbb{I}} + S_{\mathbb{I}^c} = K K^*$ . Taking  $S_{\mathbb{I}}$  and  $S_{\mathbb{I}^c}$  instead of  $U$  and  $V$  respectively in Lemma 2.4 yields

$$\begin{aligned} &2\lambda \left( \operatorname{Re} \sum_{j \in \mathbb{I}^c} \langle f, f_j \rangle \overline{\langle K K^* f, f_j \rangle} \right) + \left\| \sum_{j \in \mathbb{I}} \langle f, g_j \rangle f_j \right\|^2 \\ &= \lambda (\langle S_{\mathbb{I}^c} K K^* f, f \rangle + \langle S_{\mathbb{I}^c} f, K K^* f \rangle) + \|S_{\mathbb{I}} f\|^2 \\ &= \lambda (\langle S_{\mathbb{I}^c} K K^* f, f \rangle + \langle K K^* S_{\mathbb{I}^c} f, f \rangle) + \langle S_{\mathbb{I}} S_{\mathbb{I}} f, f \rangle \\ &= \langle S_{\mathbb{I}^c} S_{\mathbb{I}^c} f, f \rangle + (1 - \lambda) (\langle K K^* S_{\mathbb{I}} f, f \rangle + \langle S_{\mathbb{I}} K K^* f, f \rangle) + (2\lambda - 1) \|K K^* f\|^2 \end{aligned}$$

$$\begin{aligned}
 &= 2(1 - \lambda) \left( \operatorname{Re} \sum_{j \in \mathbb{I}} \langle f, g_j \rangle \overline{\langle KK^* f, f_j \rangle} \right) + \left\| \sum_{j \in \mathbb{I}^c} \langle f, f_j \rangle f_j \right\|^2 + (2\lambda - 1) \|KK^* f\|^2 \\
 &\geq (2\lambda - \lambda^2) \|KK^* f\|^2
 \end{aligned}$$

for every  $f \in \mathcal{H}$  and the proof is finished.  $\square$

REMARK 2.9. If taking  $\lambda = \frac{1}{2}$  in Theorem 2.8, then we get the inequality in Theorem 1.4.

At the end of the paper, we make a remark on the right-hand inequality of (1.8) shown in Theorem 1.5.

Suppose  $K \in B(\mathcal{H})$  and  $\{f_j\}_{j \in \mathbb{J}}$  is a Parseval  $K$ -frame for  $\mathcal{H}$ . If  $S_{\mathbb{I}}$  commutes with  $S_{\mathbb{I}^c}$  for every  $\mathbb{I} \subset \mathbb{J}$ , then  $S_{\mathbb{I}^c} S_{\mathbb{I}} \geq 0$  and

$$0 \leq S_{\mathbb{I}} S_{\mathbb{I}^c} = S_{\mathbb{I}} (KK^* - S_{\mathbb{I}}) = S_{\mathbb{I}} KK^* - S_{\mathbb{I}}^2.$$

Therefore,

$$\begin{aligned}
 S_{\mathbb{I}}^2 + S_{\mathbb{I}^c}^2 &= S_{\mathbb{I}}^2 + (KK^* - S_{\mathbb{I}})^2 \\
 &= S_{\mathbb{I}}^2 + (KK^*)^2 - KK^* S_{\mathbb{I}} - S_{\mathbb{I}} KK^* + S_{\mathbb{I}}^2 \\
 &= (KK^*)^2 + (S_{\mathbb{I}}^2 - S_{\mathbb{I}} KK^*) + (S_{\mathbb{I}}^2 - KK^* S_{\mathbb{I}}) \\
 &= (KK^*)^2 - (S_{\mathbb{I}} KK^* - S_{\mathbb{I}}^2) - S_{\mathbb{I}^c} S_{\mathbb{I}} \leq (KK^*)^2.
 \end{aligned}$$

Hence for each  $f \in \mathcal{H}$  we have

$$\begin{aligned}
 \left\| \sum_{j \in \mathbb{I}} \langle f, f_j \rangle f_j \right\|^2 + \left\| \sum_{j \in \mathbb{I}^c} \langle f, f_j \rangle f_j \right\|^2 &= \langle S_{\mathbb{I}} f, S_{\mathbb{I}} f \rangle + \langle S_{\mathbb{I}^c} f, S_{\mathbb{I}^c} f \rangle \\
 &= \langle (S_{\mathbb{I}}^2 + S_{\mathbb{I}^c}^2) f, f \rangle \\
 &\leq \langle (KK^*)^2 f, f \rangle = \|KK^* f\|^2.
 \end{aligned}$$

For every  $f \in \mathcal{H}$ , since

$$\begin{aligned}
 \|KK^* f\|^2 - \left( 2\|K\|^2 \|K^* f\|^2 - \frac{1}{2} \|KK^* f\|^2 \right) &= \frac{3}{2} \|KK^* f\|^2 - 2\|K\|^2 \|K^* f\|^2 \\
 &\leq \frac{3}{2} \|K\|^2 \|K^* f\|^2 - 2\|K\|^2 \|K^* f\|^2 \\
 &= -\frac{1}{2} \|K\|^2 \|K^* f\|^2 \leq 0,
 \end{aligned}$$

showing that the upper bound condition for the middle term of (1.8) obtained in this case is better than the original one.

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Zhong-Qi Xiang  
 College of Mathematics and Computer Science  
 Shangrao Normal University  
 Shangrao, Jiangxi 334001, P. R. China  
 e-mail: lxsy20110927@163.com