

MORE REFINEMENTS OF THE OPERATOR REVERSE AM-GM INEQUALITY FOR POSITIVE LINEAR MAPS

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Abstract. This paper aims to present some operator inequalities for positive linear maps. These inequalities are refinements of the results presented by Xue in [J. Inequal. Appl. 2017:283, 2017].

1. Introduction

Throughout this article, let us reserve M and m for scalars and I for the identity operator. Other capital letters are used to denote the general elements of the C^* -algebra $\mathcal{B}(\mathcal{H})$ (with units) of all bounded linear operators acting on Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$. Let $\|\cdot\|$ denote operator norm. We write $A \geq 0$ to mean that the operator A is positive. A positive invertible operator A is denoted as $A > 0$. A linear map $\Phi : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ is called positive if for $A \geq 0$ we have $\Phi(A) \geq 0$ and it is called unital if $\Phi(I) = I$. Also, if $A - B \geq 0$, we mean $A \geq B$. For positive operators A and B , the geometric mean $A\sharp B$ is defined by $A\sharp B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}}A^{\frac{1}{2}}$.

For $0 < m \leq A, B \leq M$, Tominaga [1] proved that the following operator inequality holds:

$$\frac{A+B}{2} \leq S(h)A\sharp B, \tag{1.1}$$

where $S(h) = \frac{h^{\frac{1}{h-1}}}{e \log h^{\frac{1}{h-1}}}$ is called specht's ratio with $h = \frac{M}{m}$.

The inequality (1.1) can be regarded as a counterpart of the following well-known AM-GM inequality

$$\frac{A+B}{2} \geq A\sharp B. \tag{1.2}$$

Lin [2, (3.3)] observed that

$$S(h) \leq \left(\frac{M+m}{2\sqrt{Mm}} \right)^2 \leq S^2(h) \quad (h \geq 1). \tag{1.3}$$

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By inequalities (1.1) and (1.3), we can easily obtained the following inequality

$$\frac{A+B}{2} \leq \left(\frac{M+m}{2\sqrt{Mm}} \right)^2 A\sharp B. \quad (1.4)$$

Because Φ is order preserving then (1.4) implies that

$$\Phi\left(\frac{A+B}{2}\right) \leq \left(\frac{M+m}{2\sqrt{Mm}}\right)^2 \Phi(A\sharp B). \quad (1.5)$$

For positive linear map Φ and $A, B \geq 0$. Ando [3] has proved the following inequality

$$\Phi(A\sharp B) \leq (\Phi(A)\sharp\Phi(B)). \quad (1.6)$$

Then, by (1.5) and (1.6), we have

$$\Phi\left(\frac{A+B}{2}\right) \leq \left(\frac{M+m}{2\sqrt{Mm}}\right)^2 (\Phi(A)\sharp\Phi(B)). \quad (1.7)$$

Here, we want to mention that the generalized forms and refinements of some related inequalities one can see in [4].

It is well known that t^s is operator monotone function for $0 \leq s \leq 1$ and not so is t^2 , see [5]. Lin [2] observed that the inequalities (1.5) and (1.7) can be squared as follows:

$$\Phi^2\left(\frac{A+B}{2}\right) \leq \left(\left(\frac{M+m}{2\sqrt{Mm}}\right)^2\right)^2 \Phi^2(A\sharp B), \quad (1.8)$$

and

$$\Phi^2\left(\frac{A+B}{2}\right) \leq \left(\left(\frac{M+m}{2\sqrt{Mm}}\right)^2\right)^2 (\Phi(A)\sharp\Phi(B))^2. \quad (1.9)$$

Recently, Xue [6] proved that if $\sqrt{\frac{M}{m}} \leq 2.314$, then the following refinement of the inequality (1.4) holds:

$$\frac{A+B}{2} \leq \frac{M+m}{2\sqrt{Mm}} A\sharp B. \quad (1.10)$$

Inspired by Lin idea [2] of squaring, Xue [6] also proved that if $\sqrt{\frac{M}{m}} \leq 2.314$, then

$$\left(\frac{A+B}{2}\right)^2 \leq \left(\frac{M+m}{2\sqrt{Mm}}\right)^2 (A\sharp B)^2, \quad (1.11)$$

$$\Phi^2 \left(\frac{A+B}{2} \right) \leq \left(\frac{M+m}{2\sqrt{Mm}} \right)^2 \Phi^2 (A\sharp B), \tag{1.12}$$

and

$$\Phi^2 \left(\frac{A+B}{2} \right) \leq \left(\frac{M+m}{2\sqrt{Mm}} \right)^2 (\Phi(A)\sharp\Phi(B))^2. \tag{1.13}$$

Inequalities (1.12) and (1.13) are refinements of (1.8) and (1.9) respectively.

Moreover, Xue [6] solved Lin conjectured [2]. she proved that if $0 < m \leq A, B \leq M$ and $\sqrt{\frac{M}{m}} \leq 2.314$, then

$$\Phi^2 \left(\frac{A+B}{2} \right) \leq S^2(h)\Phi^2 (A\sharp B), \tag{1.14}$$

and

$$\Phi^2 \left(\frac{A+B}{2} \right) \leq S^2(h) (\Phi(A)\sharp\Phi(B))^2. \tag{1.15}$$

Currently, Xue obtained more generalized and sharper forms of the reverse AM-GM inequality, for comprehensive study, the reader is referred to [7].

In this paper, in section 2, we will further refine the inequalities (1.10)-(1.15) for the condition number $\sqrt{\frac{M}{m}} \leq 2.314$.

2. Main results

We need some useful lemmas to prove our main results of this paper.

LEMMA 2.1.[8]. *Let $A, B > 0$, then the following norm inequality holds:*

$$\|AB\| \leq \frac{1}{4}\|A+B\|^2. \tag{2.1}$$

LEMMA 2.2.[9]. *If $A > 0$ and Φ be a positive unital linear map, then*

$$\Phi^{-1}(A) \leq \Phi(A^{-1}). \tag{2.2}$$

Now, we prove the first main result of this paper in the following Theorem.

THEOREM 2.3. *Let $0 < m \leq M$ and $\sqrt{\frac{M}{m}} \leq 2.314$, we have*

(1) *If $m \leq A, B \leq \frac{M+m}{2}$, then*

$$\begin{aligned} & \left(\frac{A+B}{2} + \frac{M+m}{2}m \left(\frac{A^{-1}+B^{-1}}{2} - (A^{-1}\sharp B^{-1}) \right) \right)^2 \\ & \leq \left(\frac{M+m}{2\sqrt{mM}} \right)^2 (A\sharp B)^2, \end{aligned} \tag{2.3}$$

(2) If $\frac{M+m}{2} \leq A, B \leq M$, then

$$\begin{aligned} & \left(\frac{A+B}{2} + \frac{M+m}{2} M \left(\frac{A^{-1}+B^{-1}}{2} - (A^{-1} \sharp B^{-1}) \right) \right)^2 \\ & \leq \left(\frac{M+m}{2\sqrt{mM}} \right)^2 (A \sharp B)^2, \end{aligned} \quad (2.4)$$

(3) If $m \leq A \leq \frac{M+m}{2} \leq B \leq M$, then

$$\begin{aligned} & \left(\frac{A+B}{2} + \frac{M+m}{2} \left(\frac{mA^{-1}+MB^{-1}}{2} - (mA^{-1} \sharp MB^{-1}) \right) \right)^2 \\ & \leq \left(\frac{M+m}{2\sqrt{mM}} \right)^2 (A \sharp B)^2, \end{aligned} \quad (2.5)$$

(4) If $m \leq B \leq \frac{M+m}{2} \leq A \leq M$, then

$$\begin{aligned} & \left(\frac{A+B}{2} + \frac{M+m}{2} \left(\frac{MA^{-1}+mB^{-1}}{2} - (MA^{-1} \sharp mB^{-1}) \right) \right)^2 \\ & \leq \left(\frac{M+m}{2\sqrt{mM}} \right)^2 (A \sharp B)^2. \end{aligned} \quad (2.6)$$

Proof. The operator inequality (2.3) is equivalent to the following

$$\begin{aligned} & \left\| \left(\frac{A+B}{2} + \frac{M+m}{2} m \left(\frac{A^{-1}+B^{-1}}{2} - (A^{-1} \sharp B^{-1}) \right) \right) (A \sharp B)^{-1} \right\| \\ & \leq \frac{M+m}{2\sqrt{mM}}. \end{aligned} \quad (2.7)$$

By first case $m \leq A, B \leq \frac{M+m}{2}$, we have

$$A + \frac{M+m}{2} mA^{-1} \leq \frac{M+m}{2} + m, \quad (2.8)$$

and

$$B + \frac{M+m}{2} mB^{-1} \leq \frac{M+m}{2} + m. \quad (2.9)$$

Compute

$$\left\| \left(\frac{A+B}{2} + \frac{M+m}{2} m \left(\frac{A^{-1}+B^{-1}}{2} - (A^{-1} \sharp B^{-1}) \right) \right) \frac{M+m}{2} m (A \sharp B)^{-1} \right\|$$

$$\begin{aligned}
 &\leq \frac{1}{4} \left\| \left(\frac{A+B}{2} + \frac{M+m}{2} m \left(\frac{A^{-1}+B^{-1}}{2} - (A^{-1}\sharp B^{-1}) \right) \right) \right. \\
 &\quad \left. + \frac{M+m}{2} m (A\sharp B)^{-1} \right\|^2 \quad (\text{by (2.1)}) \\
 &= \frac{1}{4} \left\| \left(\frac{A+B}{2} + \frac{M+m}{2} m \left(\frac{A^{-1}+B^{-1}}{2} - (A^{-1}\sharp B^{-1}) \right) \right) \right. \\
 &\quad \left. + \frac{M+m}{2} m (A^{-1}\sharp B^{-1}) \right\|^2 \\
 &= \frac{1}{4} \left\| \frac{A+B}{2} + \frac{M+m}{2} m \frac{A^{-1}+B^{-1}}{2} \right\|^2 \\
 &\leq \frac{1}{4} \left(\frac{M+m}{2} + m \right)^2 \quad (\text{by (2.8), (2.9)}).
 \end{aligned}$$

That is,

$$\begin{aligned}
 &\left\| \left(\frac{A+B}{2} + \frac{M+m}{2} m \left(\frac{A^{-1}+B^{-1}}{2} - (A^{-1}\sharp B^{-1}) \right) \right) (A\sharp B)^{-1} \right\| \\
 &\leq \frac{\left(\frac{M+m}{2} + m \right)^2}{4 \frac{M+m}{2} m}. \tag{2.10}
 \end{aligned}$$

Since $1 \leq \sqrt{\frac{M}{m}} \leq 2.314$, it follows that

$$\left(\sqrt{\frac{M}{m}} - 1 \right)^2 \left[\left(\sqrt{\frac{M}{m}} \right)^3 - \frac{2M}{m} + \sqrt{\frac{M}{m}} - 4 \right] \leq 0. \tag{2.11}$$

It is easy to show that $\frac{\left(\frac{M+m}{2} + m \right)^2}{4 \frac{M+m}{2} m} \leq \frac{M+m}{2\sqrt{Mm}}$ is equivalent to (2.11).

Thus, from inequality (2.10), we obtain

$$\begin{aligned}
 &\left\| \left(\frac{A+B}{2} + \frac{M+m}{2} m \left(\frac{A^{-1}+B^{-1}}{2} - (A^{-1}\sharp B^{-1}) \right) \right) (A\sharp B)^{-1} \right\| \\
 &\leq \frac{M+m}{2\sqrt{Mm}}.
 \end{aligned}$$

By 2nd case $\frac{M+m}{2} \leq A, B \leq M$, we have

$$A + \frac{M+m}{2} MA^{-1} \leq \frac{M+m}{2} + M, \tag{2.12}$$

and

$$B + \frac{M+m}{2} MB^{-1} \leq \frac{M+m}{2} + M. \tag{2.13}$$

Similarly, we obtain

$$\left\| \left(\frac{A+B}{2} + \frac{M+m}{2} M \left(\frac{A^{-1}+B^{-1}}{2} - (A^{-1}\sharp B^{-1}) \right) \right) (A\sharp B)^{-1} \right\|$$

$$\leq \frac{\left(\frac{M+m}{2} + M\right)^2}{4\frac{M+m}{2}M} \leq \frac{\left(\frac{M+m}{2} + m\right)^2}{4\frac{M+m}{2}m} \leq \frac{M+m}{2\sqrt{Mm}}.$$

By third case $m \leq A \leq \frac{M+m}{2} \leq B \leq M$, we have

$$\begin{aligned} & \left\| \left(\frac{A+B}{2} + \frac{M+m}{2} \left(\frac{mA^{-1} + MB^{-1}}{2} - (mA^{-1} \sharp MB^{-1}) \right) \right) \frac{M+m}{2} \sqrt{mM} (A \sharp B)^{-1} \right\| \\ & \leq \frac{1}{4} \left\| \left(\frac{A+B}{2} + \frac{M+m}{2} \left(\frac{mA^{-1} + MB^{-1}}{2} - (mA^{-1} \sharp MB^{-1}) \right) \right) \right. \\ & \quad \left. + \frac{M+m}{2} \sqrt{mM} (A \sharp B)^{-1} \right\|^2 \quad (\text{by (2.1)}) \\ & = \frac{1}{4} \left\| \left(\frac{A+B}{2} + \frac{M+m}{2} \left(\frac{mA^{-1} + MB^{-1}}{2} - (mA^{-1} \sharp MB^{-1}) \right) \right) \right. \\ & \quad \left. + \frac{M+m}{2} (mA^{-1} \sharp MB^{-1}) \right\|^2 \\ & = \frac{1}{4} \left\| \frac{A+B}{2} + \frac{M+m}{2} \left(\frac{mA^{-1} + MB^{-1}}{2} \right) \right\| \\ & \leq \frac{(M+m)^2}{4} \quad (\text{by (2.8), (2.13)}). \end{aligned} \tag{2.14}$$

That is,

$$\begin{aligned} & \left\| \left(\frac{A+B}{2} + \frac{M+m}{2} \left(\frac{mA^{-1} + MB^{-1}}{2} - (mA^{-1} \sharp MB^{-1}) \right) \right) (A \sharp B)^{-1} \right\| \\ & \leq \frac{(M+m)^2}{4\frac{M+m}{2}\sqrt{mM}} = \frac{M+m}{2\sqrt{mM}}. \end{aligned}$$

Similarly, by last case $m \leq B \leq \frac{M+m}{2} \leq A \leq M$ and by the inequalities (2.1), (2.9) and (2.12), we have

$$\begin{aligned} & \left\| \left(\frac{A+B}{2} + \frac{M+m}{2} \left(\frac{MA^{-1} + mB^{-1}}{2} - (MA^{-1} \sharp mB^{-1}) \right) \right) (A \sharp B)^{-1} \right\| \\ & \leq \frac{M+m}{2\sqrt{mM}}. \end{aligned}$$

This completes the proof. \square

REMARK 2.4. By inequality (1.2), it is clear that Theorem 2.3 is a refinement of (1.11).

REMARK 2.5. Since t^s is operator monotone function for $0 \leq s \leq 1$, so, we can easily obtain refinement of (1.10) by taking power $\frac{1}{2}$ both sides of (2.3), (2.4), (2.5) and (2.6) respectively.

THEOREM 2.6. Let Φ be a positive linear map, $0 < m \leq M$ and $\sqrt{\frac{M}{m}} \leq 2.314$, we have

(1) If $m \leq A, B \leq \frac{M+m}{2}$, then

$$\begin{aligned} & \Phi^2 \left(\frac{A+B}{2} + \frac{M+m}{2} m \left(\frac{A^{-1}+B^{-1}}{2} - (A^{-1} \sharp B^{-1}) \right) \right) \\ & \leq \left(\frac{M+m}{2\sqrt{mM}} \right)^2 \Phi^2 (A \sharp B), \end{aligned} \tag{2.15}$$

(2) If $\frac{M+m}{2} \leq A, B \leq M$, then

$$\begin{aligned} & \Phi^2 \left(\frac{A+B}{2} + \frac{M+m}{2} M \left(\frac{A^{-1}+B^{-1}}{2} - (A^{-1} \sharp B^{-1}) \right) \right) \\ & \leq \left(\frac{M+m}{2\sqrt{mM}} \right)^2 \Phi^2 (A \sharp B), \end{aligned} \tag{2.16}$$

(3) If $m \leq A \leq \frac{M+m}{2} \leq B \leq M$, then

$$\begin{aligned} & \Phi^2 \left(\frac{A+B}{2} + \frac{M+m}{2} \left(\frac{mA^{-1}+MB^{-1}}{2} - (mA^{-1} \sharp MB^{-1}) \right) \right) \\ & \leq \left(\frac{M+m}{2\sqrt{mM}} \right)^2 \Phi^2 (A \sharp B), \end{aligned} \tag{2.17}$$

(4) If $m \leq B \leq \frac{M+m}{2} \leq A \leq M$, then

$$\begin{aligned} & \Phi^2 \left(\frac{A+B}{2} + \frac{M+m}{2} \left(\frac{MA^{-1}+mB^{-1}}{2} - (MA^{-1} \sharp mB^{-1}) \right) \right) \\ & \leq \left(\frac{M+m}{2\sqrt{mM}} \right)^2 \Phi^2 (A \sharp B). \end{aligned} \tag{2.18}$$

Proof. Inequality (2.15) is equivalent to the following

$$\begin{aligned} & \left\| \Phi \left(\frac{A+B}{2} + \frac{M+m}{2} m \left(\frac{A^{-1}+B^{-1}}{2} - (A^{-1} \sharp B^{-1}) \right) \right) \Phi^{-1} (A \sharp B) \right\| \\ & \leq \left(\frac{M+m}{2\sqrt{mM}} \right). \end{aligned} \tag{2.19}$$

First we consider the case $m \leq A, B \leq \frac{M+m}{2}$ and compute

$$\begin{aligned} & \left\| \Phi \left(\frac{A+B}{2} + \frac{M+m}{2} m \left(\frac{A^{-1}+B^{-1}}{2} - (A^{-1} \sharp B^{-1}) \right) \right) \frac{M+m}{2} m \Phi^{-1} (A \sharp B) \right\| \\ & \leq \frac{1}{4} \left\| \Phi \left(\frac{A+B}{2} + \frac{M+m}{2} m \left(\frac{A^{-1}+B^{-1}}{2} - (A^{-1} \sharp B^{-1}) \right) \right) \right\|^2 \quad (\text{by (2.1)}) \\ & \quad + \frac{M+m}{2} m \Phi^{-1} (A \sharp B) \end{aligned}$$

$$\begin{aligned}
 &\leq \frac{1}{4} \left\| \Phi \left(\frac{A+B}{2} + \frac{M+m}{2} m \left(\frac{A^{-1}+B^{-1}}{2} - (A^{-1}\sharp B^{-1}) \right) \right) \right\|^2 \quad (\text{by (2.2)}) \\
 &\quad + \frac{M+m}{2} m \Phi((A\sharp B)^{-1}) \\
 &= \frac{1}{4} \left\| \Phi \left(\left(\frac{A+B}{2} + \frac{M+m}{2} m \left(\frac{A^{-1}+B^{-1}}{2} - (A^{-1}\sharp B^{-1}) \right) \right) \right) \right\|^2 \\
 &\quad + \frac{M+m}{2} m (A^{-1}\sharp B^{-1}) \\
 &= \frac{1}{4} \left\| \Phi \left(\frac{A+B}{2} + \frac{M+m}{2} m \left(\frac{A^{-1}+B^{-1}}{2} \right) \right) \right\|^2 \\
 &\leq \frac{1}{4} \left(\frac{M+m}{2} + m \right)^2 \quad (\text{by (2.8), (2.9)}). \tag{2.20}
 \end{aligned}$$

That is,

$$\begin{aligned}
 &\left\| \Phi \left(\frac{A+B}{2} + \frac{M+m}{2} m \left(\frac{A^{-1}+B^{-1}}{2} - (A^{-1}\sharp B^{-1}) \right) \right) \Phi^{-1}(A\sharp B) \right\| \\
 &\leq \frac{\left(\frac{M+m}{2} + m \right)^2}{4 \frac{M+m}{2} m}.
 \end{aligned}$$

So, by condition 1 $\leq \sqrt{\frac{M}{m}} \leq 2.314$ and (2.11), we have

$$\begin{aligned}
 &\left\| \Phi \left(\frac{A+B}{2} + \frac{M+m}{2} m \left(\frac{A^{-1}+B^{-1}}{2} - A^{-1}\sharp B^{-1} \right) \right) \Phi^{-1}(A\sharp B) \right\| \\
 &\leq \left(\frac{M+m}{2\sqrt{mM}} \right).
 \end{aligned}$$

Similarly, by 2nd case $\frac{M+m}{2} \leq A, B \leq M$ and by the inequalities (2.1),(2.2),(2.12),

(2.13), $\frac{\left(\frac{M+m}{2} + M \right)^2}{4 \frac{M+m}{2} M} \leq \frac{\left(\frac{M+m}{2} + m \right)^2}{4 \frac{M+m}{2} m}$ and (2.11), we have

$$\begin{aligned}
 &\left\| \Phi \left(\frac{A+B}{2} + \frac{M+m}{2} M \left(\frac{A^{-1}+B^{-1}}{2} - A^{-1}\sharp B^{-1} \right) \right) \Phi^{-1}(A\sharp B) \right\| \\
 &\leq \left(\frac{M+m}{2\sqrt{mM}} \right).
 \end{aligned}$$

Now, consider third case $m \leq A \leq \frac{M+m}{2} \leq B \leq M$ and compute

$$\begin{aligned}
 &\left\| \Phi \left(\frac{A+B}{2} + \frac{M+m}{2} \left(\frac{mA^{-1}+MB^{-1}}{2} - (mA^{-1}\sharp MB^{-1}) \right) \right) \right\| \\
 &\quad \frac{M+m}{2} \sqrt{mM} \Phi^{-1}(A\sharp B) \\
 &\leq \frac{1}{4} \left\| \Phi \left(\frac{A+B}{2} + \frac{M+m}{2} \left(\frac{mA^{-1}+MB^{-1}}{2} - (mA^{-1}\sharp MB^{-1}) \right) \right) \right\|^2 \quad (\text{by (2.1)}) \\
 &\quad + \frac{M+m}{2} \sqrt{mM} \Phi^{-1}(A\sharp B)
 \end{aligned}$$

$$\begin{aligned}
 &\leq \frac{1}{4} \left\| \left\| \Phi \left(\frac{A+B}{2} + \frac{M+m}{2} \left(\frac{mA^{-1}+MB^{-1}}{2} - (mA^{-1}\sharp MB^{-1}) \right) \right) \right\|^2 \right. \\
 &\quad \left. + \frac{M+m}{2} \sqrt{mM} \Phi((A\sharp B)^{-1}) \right\|^2 \quad (\text{by (2.2)}) \\
 &= \frac{1}{4} \left\| \left\| \Phi \left(\frac{A+B}{2} + \frac{M+m}{2} \left(\frac{mA^{-1}+MB^{-1}}{2} - (mA^{-1}\sharp MB^{-1}) \right) \right) \right\|^2 \right. \\
 &\quad \left. + \frac{M+m}{2} (mA^{-1}\sharp MB^{-1}) \right\|^2 \\
 &= \frac{1}{4} \left\| \left\| \Phi \left(\frac{A+B}{2} + \frac{M+m}{2} \left(\frac{mA^{-1}+MB^{-1}}{2} \right) \right) \right\|^2 \right. \\
 &\quad \left. \leq \frac{(M+m)^2}{4} \quad (\text{by (2.8), (2.13)}). \right. \tag{2.21}
 \end{aligned}$$

That is,

$$\begin{aligned}
 &\left\| \Phi \left(\frac{A+B}{2} + \frac{M+m}{2} \left(\frac{mA^{-1}+MB^{-1}}{2} - (mA^{-1}\sharp MB^{-1}) \right) \right) \Phi^{-1}(A\sharp B) \right\| \\
 &\leq \frac{(M+m)^2}{4 \frac{M+m}{2} \sqrt{mM}} = \frac{M+m}{2\sqrt{mM}}.
 \end{aligned}$$

Similarly, by last case $m \leq B \leq \frac{M+m}{2} \leq A \leq M$ and by the inequalities (2.1),(2.2), (2.9) and (2.12), we have

$$\begin{aligned}
 &\left\| \Phi \left(\frac{A+B}{2} + \frac{M+m}{2} \left(\frac{MA^{-1}+mB^{-1}}{2} - (MA^{-1}\sharp mB^{-1}) \right) \right) \Phi^{-1}(A\sharp B) \right\| \\
 &\leq \frac{M+m}{2\sqrt{mM}}.
 \end{aligned}$$

This completes the proof. \square

REMARK 2.7. Obviously, Theorem 2.6 is refinement of (1.12). By (1.3) and Theorem 2.6, we obtain the following refinement of the inequality (1.14)

COROLLARY 2.8. Let Φ be a positive linear map, $0 < m \leq M$ and $\sqrt{\frac{M}{m}} \leq 2.314$, we have

(1) If $m \leq A, B \leq \frac{M+m}{2}$, then

$$\begin{aligned}
 &\Phi^2 \left(\frac{A+B}{2} + \frac{M+m}{2} m \left(\frac{A^{-1}+B^{-1}}{2} - (A^{-1}\sharp B^{-1}) \right) \right) \\
 &\leq S^2(h) \Phi^2(A\sharp B),
 \end{aligned}$$

(2) If $\frac{M+m}{2} \leq A, B \leq M$, then

$$\begin{aligned}
 &\Phi^2 \left(\frac{A+B}{2} + \frac{M+m}{2} M \left(\frac{A^{-1}+B^{-1}}{2} - (A^{-1}\sharp B^{-1}) \right) \right) \\
 &\leq S^2(h) \Phi^2(A\sharp B),
 \end{aligned}$$

(3) If $m \leq A \leq \frac{M+m}{2} \leq B \leq M$, then

$$\begin{aligned} & \Phi^2 \left(\frac{A+B}{2} + \frac{M+m}{2} \left(\frac{mA^{-1} + MB^{-1}}{2} - (mA^{-1} \# MB^{-1}) \right) \right) \\ & \leq S^2(h) \Phi^2(A \# B), \end{aligned}$$

(4) If $m \leq B \leq \frac{M+m}{2} \leq A \leq M$, then

$$\begin{aligned} & \Phi^2 \left(\frac{A+B}{2} + \frac{M+m}{2} \left(\frac{MA^{-1} + mB^{-1}}{2} - (MA^{-1} \# mB^{-1}) \right) \right) \\ & \leq S^2(h) \Phi^2(A \# B), \end{aligned}$$

where $S(h) = \frac{h^{\frac{1}{h-1}}}{e \log h^{\frac{1}{h-1}}}$, $h = \frac{M}{m}$.

THEOREM 2.9. Let Φ be a positive linear map, $0 < m \leq M$ and $\sqrt{\frac{M}{m}} \leq 2.314$, we have

(1) If $m \leq A, B \leq \frac{M+m}{2}$, then

$$\begin{aligned} & \Phi^2 \left(\frac{A+B}{2} + \frac{M+m}{2} m \left(\frac{A^{-1} + B^{-1}}{2} - (A^{-1} \# B^{-1}) \right) \right) \\ & \leq \left(\frac{M+m}{2\sqrt{mM}} \right)^2 (\Phi(A) \# \Phi(B))^2, \end{aligned} \tag{2.22}$$

(2) If $\frac{M+m}{2} \leq A, B \leq M$, then

$$\begin{aligned} & \Phi^2 \left(\frac{A+B}{2} + \frac{M+m}{2} M \left(\frac{A^{-1} + B^{-1}}{2} - (A^{-1} \# B^{-1}) \right) \right) \\ & \leq \left(\frac{M+m}{2\sqrt{mM}} \right)^2 (\Phi(A) \# \Phi(B))^2, \end{aligned} \tag{2.23}$$

(3) If $m \leq A \leq \frac{M+m}{2} \leq B \leq M$, then

$$\begin{aligned} & \Phi^2 \left(\frac{A+B}{2} + \frac{M+m}{2} \left(\frac{mA^{-1} + MB^{-1}}{2} - (mA^{-1} \# MB^{-1}) \right) \right) \\ & \leq \left(\frac{M+m}{2\sqrt{mM}} \right)^2 (\Phi(A) \# \Phi(B))^2, \end{aligned} \tag{2.24}$$

(4) If $m \leq B \leq \frac{M+m}{2} \leq A \leq M$, then

$$\begin{aligned} & \Phi^2 \left(\frac{A+B}{2} + \frac{M+m}{2} \left(\frac{MA^{-1} + mB^{-1}}{2} - (MA^{-1} \# mB^{-1}) \right) \right) \\ & \leq \left(\frac{M+m}{2\sqrt{mM}} \right)^2 (\Phi(A) \# \Phi(B))^2. \end{aligned} \tag{2.25}$$

Proof. Inequality (2.22) is equivalent to the following

$$\begin{aligned} & \left\| \Phi \left(\frac{A+B}{2} + \frac{M+m}{2} m \left(\frac{A^{-1}+B^{-1}}{2} - (A^{-1}\sharp B^{-1}) \right) \right) (\Phi(A)\sharp\Phi(B))^{-1} \right\| \\ & \leq \left(\frac{M+m}{2\sqrt{mM}} \right). \end{aligned} \tag{2.26}$$

Compute

$$\begin{aligned} & \left\| \Phi \left(\frac{A+B}{2} + \frac{M+m}{2} m \left(\frac{A^{-1}+B^{-1}}{2} - (A^{-1}\sharp B^{-1}) \right) \right) \frac{M+m}{2} m (\Phi(A)\sharp\Phi(B))^{-1} \right\| \\ & \leq \frac{1}{4} \left\| \Phi \left(\frac{A+B}{2} + \frac{M+m}{2} m \left(\frac{A^{-1}+B^{-1}}{2} - (A^{-1}\sharp B^{-1}) \right) \right) \right. \\ & \quad \left. + \frac{M+m}{2} m (\Phi(A)\sharp\Phi(B))^{-1} \right\|^2 \\ & \leq \frac{1}{4} \left\| \Phi \left(\frac{A+B}{2} + \frac{M+m}{2} m \left(\frac{A^{-1}+B^{-1}}{2} - (A^{-1}\sharp B^{-1}) \right) \right) \right. \\ & \quad \left. + \frac{M+m}{2} m \Phi^{-1}(A\sharp B) \right\|^2 \quad (\text{by (1.6)}) \\ & \leq \frac{1}{4} \left(\frac{M+m}{2} + m \right)^2 \quad (\text{by (2.20)}) \end{aligned}$$

That is,

$$\begin{aligned} & \left\| \Phi \left(\frac{A+B}{2} + \frac{M+m}{2} m \left(\frac{A^{-1}+B^{-1}}{2} - (A^{-1}\sharp B^{-1}) \right) \right) (\Phi(A)\sharp\Phi(B))^{-1} \right\| \\ & \leq \frac{\left(\frac{M+m}{2} + m \right)^2}{4 \frac{M+m}{2} m}. \end{aligned} \tag{2.27}$$

By condition 1 $\leq \sqrt{\frac{M}{m}} \leq 2.314$ and (2.11), from (2.27), we have

$$\begin{aligned} & \left\| \Phi \left(\frac{A+B}{2} + \frac{M+m}{2} m \left(\frac{A^{-1}+B^{-1}}{2} - (A^{-1}\sharp B^{-1}) \right) \right) (\Phi(A)\sharp\Phi(B))^{-1} \right\| \\ & \leq \left(\frac{M+m}{2\sqrt{mM}} \right). \end{aligned}$$

Since $\frac{\left(\frac{M+m}{2} + M \right)^2}{4 \frac{M+m}{2} M} \leq \frac{\left(\frac{M+m}{2} + m \right)^2}{4 \frac{M+m}{2} m}$, similarly, we can easily prove the inequality (2.23).

Inequality (2.24) is equivalent to the following

$$\begin{aligned} & \left\| \Phi \left(\frac{A+B}{2} + \frac{M+m}{2} \left(\frac{mA^{-1}+MB^{-1}}{2} - (mA^{-1}\sharp MB^{-1}) \right) \right) \right. \\ & \quad \left. (\Phi(A)\sharp\Phi(B))^{-1} \right\| \\ & \leq \frac{M+m}{2\sqrt{mM}}. \end{aligned} \tag{2.28}$$

compute

$$\begin{aligned}
 & \left\| \Phi \left(\frac{A+B}{2} + \frac{M+m}{2} \left(\frac{mA^{-1}+MB^{-1}}{2} - (mA^{-1}\sharp MB^{-1}) \right) \right) \right\| \\
 & \quad \frac{M+m}{2} \sqrt{mM} (\Phi(A)\sharp\Phi(B))^{-1} \\
 & \leq \frac{1}{4} \left\| \Phi \left(\frac{A+B}{2} + \frac{M+m}{2} \left(\frac{mA^{-1}+MB^{-1}}{2} - (mA^{-1}\sharp MB^{-1}) \right) \right) \right\|^2 \quad (\text{by (2.1)}) \\
 & \quad + \frac{M+m}{2} \sqrt{mM} (\Phi(A)\sharp\Phi(B))^{-1} \\
 & \leq \frac{1}{4} \left\| \Phi \left(\frac{A+B}{2} + \frac{M+m}{2} \left(\frac{mA^{-1}+MB^{-1}}{2} - (mA^{-1}\sharp MB^{-1}) \right) \right) \right\|^2 \quad (\text{by (1.6)}) \\
 & \quad + \frac{M+m}{2} \sqrt{mM} \Phi^{-1}(A\sharp B) \\
 & \leq \frac{(M+m)^2}{4} \quad (\text{by (2.21)})
 \end{aligned}$$

That is,

$$\begin{aligned}
 & \left\| \Phi \left(\frac{A+B}{2} + \frac{M+m}{2} \left(\frac{mA^{-1}+MB^{-1}}{2} - (mA^{-1}\sharp MB^{-1}) \right) \right) \right\| \\
 & \quad (\Phi(A)\sharp\Phi(B))^{-1} \\
 & \leq \frac{M+m}{2\sqrt{mM}}.
 \end{aligned}$$

So, (2.24) proved.

The process of the proof of inequality (2.25) is similar to that of inequality (2.24).

This completes the proof. \square

REMARK 2.10. Obviously, Theorem 2.9 is refinement of (1.13).

By (1.3) and Theorem 2.9, we obtain the following refinement of the inequality (1.15).

COROLLARY 2.11. Let Φ be a positive linear map, $0 < m \leq M$ and $\sqrt{\frac{M}{m}} \leq 2.314$, we have

(1) If $m \leq A, B \leq \frac{M+m}{2}$, then

$$\begin{aligned}
 & \Phi^2 \left(\frac{A+B}{2} + \frac{M+m}{2} m \left(\frac{A^{-1}+B^{-1}}{2} - (A^{-1}\sharp B^{-1}) \right) \right) \\
 & \leq S^2(h) (\Phi(A)\sharp\Phi(B))^2,
 \end{aligned}$$

(2) If $\frac{M+m}{2} \leq A, B \leq M$, then

$$\begin{aligned}
 & \Phi^2 \left(\frac{A+B}{2} + \frac{M+m}{2} M \left(\frac{A^{-1}+B^{-1}}{2} - (A^{-1}\sharp B^{-1}) \right) \right) \\
 & \leq S^2(h) (\Phi(A)\sharp\Phi(B))^2,
 \end{aligned}$$

(3) If $m \leq A \leq \frac{M+m}{2} \leq B \leq M$, then

$$\begin{aligned} & \Phi^2 \left(\frac{A+B}{2} + \frac{M+m}{2} \left(\frac{mA^{-1} + MB^{-1}}{2} - (mA^{-1} \sharp MB^{-1}) \right) \right) \\ & \leq S^2(h) (\Phi(A) \sharp \Phi(B))^2, \end{aligned}$$

(4) If $m \leq B \leq \frac{M+m}{2} \leq A \leq M$, then

$$\begin{aligned} & \Phi^2 \left(\frac{A+B}{2} + \frac{M+m}{2} \left(\frac{MA^{-1} + mB^{-1}}{2} - (MA^{-1} \sharp mB^{-1}) \right) \right) \\ & \leq S^2(h) (\Phi(A) \sharp \Phi(B))^2, \end{aligned}$$

where $S(h) = \frac{h^{\frac{1}{h-1}}}{\text{el} \log h^{\frac{1}{h-1}}}$, $h = \frac{M}{m}$.

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