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LAWS OF LARGE NUMBERS WITH INFINITE MEAN

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Abstract. In this paper, we study the weak law and strong law of large numbers based on $\tilde{\rho}$-mixing random variables with infinite mean. If the random variables satisfy the Pareto type distributions, then some weak laws of large numbers are presented. If the random variables satisfy the two tailed Pareto distribution and asymmetrical Cauchy distribution, the strong laws of large numbers are also obtained. Furthermore, we do some simulations for the laws of large numbers for two tailed Pareto distribution and asymmetrical Cauchy distribution.

1. Introduction

In this paper, we are interested in the studying the weak law and strong law of large numbers for weighted random variables with infinite mean. When the random variables have the finite means, the ordinary laws of large numbers are formulated by the sample average. We can see many books such as Chow and Teicher [12] and Gut [14]. But if the means are infinite, some devices are needed. We will consider some cases of Pareto and Cauchy distributions whose means are infinite.

Let a random variable X to be a two tailed Pareto distribution whose density is

$$f(x) = \begin{cases} \frac{q}{x^2}, \text{ if } x \leqslant -1, \\ 0, \text{ if } -1 < x < 1, \\ \frac{p}{x^2}, \text{ if } x \geqslant 1, \end{cases} \tag{1}$$

where $p+q=1$. Adler [6] considered independent and identically distributed ($i.i.d.$) random variables satisfying (1) and obtained the strong law of large numbers for them.

Base on the Pareto distribution, for $0 < \alpha \leqslant 1$, Nakata [19] considered a random variable X satisfying tail probability

$$P(|X| > x) \asymp x^{-\alpha}, \tag{2}$$

i.e.

$$0 < \liminf_{x\to\infty} x^{\alpha} P(|X| > x) \leqslant \limsup_{x\to\infty} x^{\alpha} P(|X| > x) < \infty.$$

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Then Nakata [19] obtained the weak law of large numbers for weighted independent random variables satisfying (2).

Let a random variable X to be an asymmetrical Cauchy random variables with a slight twist, i.e. the density is

$$f(x) = \begin{cases} \frac{p}{\pi(1+x^2)}, \text{ if } x \geqslant 0, \\ \frac{q}{\pi(1+x^2)}, \text{ if } x < 0, \end{cases} \tag{3}$$

where $p+q=2$. If $p=q=1$, then we get the usual Cauchy distribution. Adler [5] obtained the strong law of large numbers for the $i.i.d.$ asymmetrical Cauchy random variables satisfying (3).

If a random variable X satisfies (1), (2) or (3), then it can be checked that $E|X| = \infty$.

Inspired by the papers above, we will investigate the weak and strong law of large numbers for the weighted dependent random variables based on the Pareto random variable and asymmetrical Cauchy random variable.

Let $\{X_n, n \geqslant 1\}$ be a sequence of random variables and write $\mathscr{F}_S = \sigma(X_i, i \in S \subset \mathbb{N})$. Given σ-algebras $\mathscr{B}, \mathscr{R}$ in $\mathscr{F}$, denote

$$\rho(\mathscr{B}, \mathscr{R}) = \sup_{X \in L_2(\mathscr{B}), Y \in L_2(\mathscr{R})} \frac{|EXY - EXEY|}{(Var(X)Var(Y))^{1/2}}.$$

Define the

$$\tilde{\rho}(k) = \sup\{\rho(\mathscr{F}_S, \mathscr{F}_T)\},$$

where $S, T \subset \mathbb{N}$, are finite subsets such that $dist(S,T) \geqslant k$, $k \geqslant 0$.

DEFINITION 1.1. *A sequence of random variables $\{X_n, n \geqslant 1\}$ is said to be a $\tilde{\rho}$-mixing sequence if there exists $k \in \mathbb{N}$ such that $\tilde{\rho}(k) < 1$.*

The concept of $\tilde{\rho}$-mixing random variables dates back at least to 1972 (see Stein [20, page 398]). Bradley [9] systematically studied the properties of $\tilde{\rho}$-mixing random variables and obtained the central limit theorem. There are many examples such as moving average process and Markov chain that can structure $\tilde{\rho}$-mixing random variables (see Bradley [10]). Much more works of $\tilde{\rho}$-mixing random variables, one can refer to An and Yuan [8], Gan [13], Kuczmaszewska [15], Li et al. [16], Sung [21], Utev and Peligrad [22], Wang et al. [23, 25], Wang et al. [24] and so on. On the other hand, for the more research of laws of large numbers for $i.i.d.$ random variables with infinite mean, we can refer to works of Adler [3-7], Matsumoto and Nakata [17], Nakata [18, 19] and the references therein.

Through out the paper, denote $C, C_1, C_2, \ldots,$ to be some positive constants independent on n. Let $\log x = \log(\max(x,e))$ and $I(A)$ be the indicator function of A. For simplicity, $\rightarrow$ means convergence as $n \rightarrow \infty$, $\xrightarrow{P}$ means convergence in probability, $\xrightarrow{a.s.}$ means almost surely convergence and $X \stackrel{d}{=} Y$ means that X and Y have the same distribution.

2. Some lemmas

LEMMA 2.1 (Adler and Rosalsky [1, Lemma 1] and Adler et al. [2, Lemma 3]). *Let $\{X_n, n \geqslant 1\}$ be a sequence of random variables, which is stochastically dominated by a random variable X, i.e.*

$$\sup_{n\geqslant 1} P(|X_n| > x) \leqslant CP(|X| > x), \text{ for all } x \geqslant 0.$$

Then, for any $\alpha > 0$ and $\beta > 0$, the following two statements hold:

$$E[|X_n|^\alpha I(|X_n| \leqslant \beta)] \leqslant C_1\{E[|X|^\alpha I(|X| \leqslant \beta)] + \beta^\alpha P(|X| > \beta)\},$$
$$E[|X_n|^\alpha I(|X_n| > \beta)] \leqslant C_2 E[|X|^\alpha I(|X| > \beta)],$$

where C_1 and C_2 are positive constants independent on n.

LEMMA 2.2. *Let $0 < \alpha \leqslant 1$ and $\{X_n, n \geqslant 1\}$ be a sequence of random variables, which is stochastically dominated by a random variable X satisfying*

$$\limsup_{x\to\infty} x^\alpha P(|X| > x) < \infty. \tag{4}$$

Moreover, let $\{a_n\}$ and $\{b_n\}$ be the sequences of positive constants satisfying that

$$\sum_{j=1}^{n} a_j^\alpha = o(b_n^\alpha). \tag{5}$$

Then we have

$$\sum_{j=1}^{n} P\Big(|X_j| > \frac{b_n}{a_j}\Big) \to 0. \tag{6}$$

In addition, for $1 \leqslant j \leqslant n$, denote

$$X_{nj} = -\frac{b_n}{a_j} I\Big(X_j < -\frac{b_n}{a_j}\Big) + X_j I\Big(|X_j| \leqslant \frac{b_n}{a_j}\Big) + \frac{b_n}{a_j} I\Big(X_j > \frac{b_n}{a_j}\Big). \tag{7}$$

Then for some $0 < \alpha \leqslant 1$ and $p > 1$, there is a positive C_1 such that

$$E|X_{nj}|^p \leqslant C_1 \Big(\frac{b_n}{a_j}\Big)^{p-\alpha}, \quad 1 \leqslant j \leqslant n. \tag{8}$$

Similarly, for $0 < \alpha < 1$, there is a positive C_2 such that

$$E|X_{nj}| \leqslant C_2 \Big(\frac{b_n}{a_j}\Big)^{1-\alpha}, \quad 1 \leqslant j \leqslant n. \tag{9}$$

Proof. Obviously, for $0 < \alpha \leqslant 1$ and any $1 \leqslant j \leqslant n$, by the nonnegativity and (5), it follows

$$0 \leqslant \Big(\frac{a_j}{b_n}\Big)^\alpha \leqslant \sum_{j=1}^{n} \Big(\frac{a_j}{b_n}\Big)^\alpha \to 0,$$

which implies that for any $j = 1, 2, \ldots, n$,

$$\left(\left(\frac{a_j}{b_n}\right)^{\alpha}\right)^{1/\alpha} = \frac{a_j}{b_n} \to 0.$$

Consequently, by stochastic domination, (4) and (5), there exist a positive constant C such that

$$\sum_{j=1}^{n} P\Big(|X_j| > \frac{b_n}{a_j}\Big) \leqslant C_1 \sum_{j=1}^{n} P\Big(|X| > \frac{b_n}{a_j}\Big) \leqslant C_2 b_n^{-\alpha} \sum_{j=1}^{n} a_j^{\alpha} \to 0,$$

i.e. (6) holds true. On the other hand, for some $0 < \alpha \leqslant 1$ and $p > 1$, by Lemma 2.1 and (4), there exists a integer n_0 large enough that for all integer $n \geqslant n_0$,

$$\begin{aligned}
E|X_{nj}|^p &\leqslant C_1\Big(\Big(\frac{b_n}{a_j}\Big)^p P(|X| > \frac{b_n}{a_j}) + E\Big(|X|^p I(|X| \leqslant \frac{b_n}{a_j})\Big)\Big) \\
&\leqslant C_2\Big(\Big(\frac{b_n}{a_j}\Big)^p \Big(\frac{b_n}{a_j}\Big)^{-\alpha} + \int_0^{n_0} P(|X|^p > t)dt + \int_{n_0}^{(b_n/a_j)^p} P(|X|^p > t)dt\Big) \\
&\leqslant C_2\Big(\Big(\frac{b_n}{a_j}\Big)^p \Big(\frac{b_n}{a_j}\Big)^{-\alpha} + C_3 + C_4 \int_{n_0}^{(b_n/a_j)^p} t^{-\alpha/p} dt\Big) \\
&\leqslant C_5\Big(\frac{b_n}{a_j}\Big)^{p-\alpha}, \quad 1 \leqslant j \leqslant n.
\end{aligned}$$

So (8) holds for some $0 < \alpha \leqslant 1$ and $p > 1$.

Similarly, for $p = 1$ and $0 < \alpha < 1$, by by stochastic domination, Lemma 2.1 and (4), we obtain that

$$E|X_{nj}| \leqslant C_1\Big(\frac{b_n}{a_j} P(|X| > \frac{b_n}{a_j}) + E\Big(|X| I(|X| \leqslant \frac{b_n}{a_j})\Big)\Big) \leqslant C_2\Big(\frac{b_n}{a_j}\Big)^{1-\alpha}, \quad 1 \leqslant j \leqslant n.$$

Thus, (9) holds true. □

REMARK 2.1. Let $0 < \alpha \leqslant 1$ and $\{X_n, n \geqslant 1\}$ be a independent sequence of random variables satisfying $P(|X_n| > x) \asymp x^{-\alpha}$ for $n \geqslant 1$ and $\limsup_{x\to\infty} \sup_{n\geqslant 1} x^{\alpha} P(|X_n| > x) < \infty$. In addition, assume that (5) holds true. Then Nakata [19] obtain (6) for the independent case (see Lemma 2.2 of Nakata [19]). In order to investigate the weak law of dependent case, we combine stochastic domination with (4), and obtain (6) in this paper.

LEMMA 2.3 (Utev and Peligrad [22], Theorem 2.1). *For a positive integer $n_0 \geqslant 1$ and positive real numbers $p \geqslant 2$ and $0 \leqslant r < 1$, there is a positive constant $C = C(p, n_0, r)$ such that if $\{X_n, n \geqslant 1\}$ is a sequence of $\tilde{\rho}$-mixing random variables with $\tilde{\rho}(n_0) \leqslant r$, $EX_n = 0$ and $E|X_n|^p < \infty$, $n \geqslant 1$, then*

$$E\Big(\max_{1\leqslant k\leqslant n}\Big|\sum_{i=1}^{k} X_i\Big|^p\Big) \leqslant C\Big\{\sum_{i=1}^{n} E|X_i|^p + \Big(\sum_{i=1}^{n} EX_i^2\Big)^{p/2}\Big\}, \quad n \geqslant 1.$$

REMARK 2.2. In Lemma 2.3, the mixing coefficient $\tilde{\rho}(n)$ needs a only condition that there exist a positive integer $n_0 \geqslant 1$ such that $\tilde{\rho}(n_0) \leqslant r$, where $0 \leqslant r < 1$. It is a weak condition. In other words, we don't need the mixing coefficient $\tilde{\rho}(n)$ satisfies $\tilde{\rho}(n) \to 0$ as $n \to \infty$, which is quite different from other mixing sequences such as ρ-mixing, φ-mixing and α-mixing. For more details, one can refer to a survey of basic properties of mixing conditions by Bradley [11].

Applying Lemma 2.3, one can easily obtain the convergence theorem for $\tilde{\rho}$-mixing sequence. So we omit its proof.

COROLLARY 2.1. (Wu and Jiang [26]) *Let $\{X_n, n \geqslant 1\}$ be a sequence of $\tilde{\rho}$-mixing random variables with $n_0 \geqslant 1$, $0 \leqslant r < 1$, $\tilde{\rho}(n_0) \leqslant r$. If*

$$\sum_{n=1}^{\infty} EX_n^2 < \infty$$

then $\sum_{n=1}^{\infty}(X_n - EX_n)$ converges almost surely.

LEMMA 2.4. *(Adler* [5, Lemma 1.1]*).*

$$\lim_{x\to\infty} \frac{\pi - 2\arctan x}{x^{-1}} = 2.$$

3. Weak law of large numbers

In this section, we study the weak law of large numbers for Pareto type distributions. First, we consider the case satisfying (4) in Section 2. The similar case is discussed in Nakata [19].

THEOREM 3.1. *For $n_0 \geqslant 1$, $0 \leqslant r < 1$ and $\tilde{\rho}(n_0) \leqslant r$, let $\{X_n, n \geqslant 1\}$ be a sequence of $\tilde{\rho}$-mixing random variables, which is stochastically dominated by a random variable X satisfying (4). Suppose that $\{a_n\}$ and $\{b_n\}$ are two sequences of positive constants satisfying (5). Then we have that*

$$\frac{1}{b_n}\sum_{j=1}^{n} a_j(X_j - EX_{nj}) \xrightarrow{P} 0, \tag{10}$$

where X_{nj} is defined in (7). In particular, if there exists a constant A such that

$$\frac{1}{b_n}\sum_{j=1}^{n} a_j E\left(X_j I\left(|X_j| \leqslant \frac{b_n}{a_j}\right)\right) \to A, \tag{11}$$

then it follows

$$\frac{1}{b_n}\sum_{j=1}^{n} a_j X_j \xrightarrow{P} A. \tag{12}$$

Proof. To prove (10), we need to show that

$$\frac{1}{b_n}\sum_{j=1}^{n} a_j(X_j - X_{nj}) \xrightarrow{P} 0, \tag{13}$$

$$\frac{1}{b_n}\sum_{j=1}^{n} a_j(X_{nj} - EX_{nj}) \xrightarrow{P} 0. \tag{14}$$

For every $\varepsilon > 0$, by (6), it yields

$$P\Big(\frac{1}{b_n}\Big|\sum_{j=1}^{n} a_j(X_j - X_{nj})\Big| > \varepsilon\Big) \leqslant P\Big(\bigcup_{j=1}^{n}(X_j \neq X_{nj})\Big) \leqslant \sum_{j=1}^{n} P(|X_j| > b_n/a_j) \to 0. \tag{15}$$

So (13) is proved.

In addition, for some $p \geqslant 2$, by Markov's inequality, (5) and (8) and Lemma 2.3, it can be argued that for every $\varepsilon > 0$,

$$\begin{aligned}
P\Big(\frac{1}{b_n}\Big|\sum_{j=1}^{n} a_j(X_{nj} - EX_{nj})\Big| > \varepsilon\Big) &\leqslant \frac{1}{b_n^p\varepsilon^p} E\Big(\max_{1\leqslant k\leqslant n}\Big|\sum_{j=1}^{k} a_j(X_{nj} - EX_{nj})\Big|^p\Big) \\
&\leqslant \frac{C_1}{b_n^p\varepsilon^p}\Big[\Big(\sum_{j=1}^{n} a_j^2 E|X_{nj}|^2\Big)^{\frac{p}{2}} + \sum_{j=1}^{n} a_j^p E|X_{nj}|^p\Big] \\
&\leqslant \frac{C_2}{b_n^p\varepsilon^p}\Big[\Big(\sum_{j=1}^{n} a_j^2 (\frac{b_n}{a_j})^{2-\alpha}\Big)^{\frac{p}{2}} + \sum_{j=1}^{n} a_j^p (\frac{b_n}{a_j})^{p-\alpha}\Big] \\
&= \frac{C_2}{\varepsilon^p}\Big[\Big(\sum_{j=1}^{n} \frac{a_j^\alpha}{b_n^\alpha}\Big)^{\frac{p}{2}} + \sum_{j=1}^{n} \frac{a_j^\alpha}{b_n^\alpha}\Big] \\
&\leqslant \frac{C_3}{\varepsilon^p}\sum_{j=1}^{n} \frac{a_j^\alpha}{b_n^\alpha} \to 0,
\end{aligned} \tag{16}$$

by the fact that $\Big(\sum_{j=1}^{n} \frac{a_j^\alpha}{b_n^\alpha}\Big)^{\frac{p}{2}} \leqslant C\sum_{j=1}^{n} \frac{a_j^\alpha}{b_n^\alpha} \to 0$ for some $p \geqslant 2$. Thus, (14) follows from (16). Consequently, by (13) and (14), (10) holds true.

Furthermore, by (6) and (7), it has

$$\frac{1}{b_n}\Big|\sum_{j=1}^{n} a_j\Big[EX_{nj} - E\Big(X_j I\Big(|X_j| \leqslant \frac{b_n}{a_j}\Big)\Big)\Big]\Big| \leqslant C\sum_{j=1}^{n} P(|X_j| > b_n/a_j) \to 0. \tag{17}$$

Therefore, (12) follows from (10), (11) and (17). □

The following is a corollary of Theorem 3.1 for the case of $0 < \alpha < 1$.

COROLLARY 3.1. *For $n_0 \geqslant 1$, $0 \leqslant r < 1$ and $\tilde{\rho}(n_0) \leqslant r$, let $\{X_n, n \geqslant 1\}$ be a sequence of $\tilde{\rho}$-mixing random variables, which is stochastically dominated by a random variable X with satisfying (4) for $0 < \alpha < 1$. Suppose that $\{a_n\}$ and $\{b_n\}$ are two sequences of positive constants satisfying (5). Then it has*

$$\frac{1}{b_n}\sum_{j=1}^{n} a_j X_j \xrightarrow{P} 0. \tag{18}$$

Proof. For $0<\alpha<1$, by (5) and (9), we obtain

$$\frac{1}{b_n}\Big|\sum_{j=1}^{n} a_j EX_{nj}\Big| \leqslant \frac{1}{b_n}\sum_{j=1}^{n} a_j|EX_{nj}| \leqslant \frac{C_1}{b_n}\sum_{j=1}^{n} a_j\Big(\frac{b_n}{a_j}\Big)^{1-\alpha} = C_1\sum_{j=1}^{n}\frac{a_j^{\alpha}}{b_n^{\alpha}} \to 0.$$

Therefore, we apply Theorem 3.1 with $A=0$ and obtain (18) immediately. □

Second, we consider the Pareto-Zipf distributions for the case $\alpha=1$ in Theorem 3.1, which was discussed in Adler [7] and Nakata [19].

THEOREM 3.2. *For $n_0 \geqslant 1$, $0 \leqslant r < 1$ and $\tilde{\rho}(n_0) \leqslant r$, let $\{X_n, n \geqslant 1\}$ be a nonnegative sequence of $\tilde{\rho}$-mixing random variables whose distributions are defined by $P(X_n=0)=1-\frac{1}{c_n}$ for $n \geqslant 1$ and the tail probability*

$$P(X_n > x) = (x+c_n)^{-1} \quad \text{for } x>0 \text{ and } n \geqslant 1, \tag{19}$$

where $\{c_n\}$ is a positive constant sequence with $c_n \geqslant 1$ and

$$C_n := \sum_{j=1}^{n}\frac{1}{c_j} \to \infty. \tag{20}$$

Then we have

$$\frac{\sum_{j=1}^{n} c_j^{-1}X_j}{C_n \log C_n} \xrightarrow{P} 1. \tag{21}$$

Proof. By (19), it is easy to see that

$$\sup_{n\geqslant 1} P(X_n > x) \leqslant \frac{1}{x}, \quad \text{for all } x>0.$$

So there exists a random variable X whose distribution satisfies (4) with $\alpha=1$, and $\{X_n, n\geqslant 1\}$ is stochastically dominated by X. By taking $a_n = c_n^{-1}$ and $b_n = C_n \log C_n$, we have that (5) holds with $\alpha=1$, in view of (20). Therefore, as an application of Theorem 3.1, it is sufficient to show $A=1$ in (11). It can be checked by (19) that for any $j \geqslant 1$,

$$E\Big(X_j I\Big(X_j \leqslant b_n c_j\Big)\Big) = \int_0^{b_n c_j} P(X_j > t)dt = \int_0^{b_n c_j}\frac{1}{t+c_j}dt \sim \log b_n.$$

Moreover, by (20), it can be checked that

$$\frac{1}{b_n}\sum_{j=1}^{n} c_j^{-1} = \frac{C_n}{C_n \log C_n} \to 0,$$

and

$$\frac{1}{b_n}\sum_{j=1}^{n} c_j^{-1}\log b_n = \frac{C_n \log(C_n \log C_n)}{C_n \log C_n} \to 1,$$

which yields

$$b_n^{-1}\sum_{j=1}^{n} c_j^{-1}E\Big(X_jI\Big(X_j \leqslant b_nc_j\Big)\Big) \sim \frac{1}{b_n}\sum_{j=1}^{n} c_j^{-1}\log b_n \to 1.$$

Applying Theorem 3.1, we obtain (21) immediately. □

If we choose $c_n = n$, then $a_n = 1/n$ and $b_n = \log n \log\log n$. We have a corollary of Theorem 3.2, which was obtained by Nakata [19]) for independent case.

COROLLARY 3.2. *Let the conditions of Theorem 3.2 hold with $c_n = n,\ n \geqslant 1$. Then for all $\gamma > -1$ and real δ, we have*

$$\frac{\sum_{j=1}^{n} j^{-1}(\log j)^{\gamma}(\log\log j)^{\delta}X_j}{(\log n)^{\gamma+1}(\log\log n)^{\delta+1}} \xrightarrow{P} \frac{1}{\gamma+1}. \tag{22}$$

Proof. Combining our Theorem 3.2 with the proof of Theorem 3.2 in Nakata [19], one can easily obtain (22). □

4. Strong law of large numbers

In this section, we consider the strong law of large numbers for the cases of two tailed Pareto distribution with (1) and asymmetrical Cauchy distribution with (3).

First, we discussed the case of two tailed Pareto distribution which was discussed in Adler [6].

THEOREM 4.1. *For $n_0 \geqslant 1,\ 0 \leqslant r < 1$ and $\tilde{\rho}(n_0) \leqslant r$, let $\{X_n, n \geqslant 1\}$ be a sequence of $\tilde{\rho}$-mixing random variables with the same distributions from a two tailed Pareto distribution by (1). Then for all $\beta > 0$ we have*

$$\frac{1}{\log^{\beta} n}\sum_{j=1}^{n}\frac{\log^{\beta-2} j}{j}X_j \xrightarrow{a.s.} \frac{p-q}{\beta}. \tag{23}$$

Proof. Let $a_j = \frac{\log^{\beta-2} j}{j}$, $b_j = \log^{\beta} j$ and $c_j = \frac{b_j}{a_j} = j\log^2 j$. For $j \geqslant 1$, denote

$$\tilde{X}_j = -c_jI(X_j < -c_j) + X_jI(|X_j| \leqslant c_j) + c_jI(X_j > c_j).$$

We can make the following decomposition

$$\begin{aligned}\frac{1}{b_n}\sum_{j=1}^{n} a_jX_j \;=\; & \frac{1}{b_n}\sum_{j=1}^{n} a_j[\tilde{X}_j - E\tilde{X}_j] \\ & + \frac{1}{b_n}\sum_{j=1}^{n} a_j[c_jI(X_j < -c_j) + X_jI(|X_j| > c_j) - c_jI(X_j > c_j)] \\ & + \frac{1}{b_n}\sum_{j=1}^{n} a_j[-c_jP(X_j < -c_j) + EX_jI(|X_j| \leqslant c_j) + c_jP(X_j > c_j)] \\ =: & \ I_1 + I_2 + I_3. \end{aligned} \tag{24}$$

It can be seen that $\{[\tilde{X}_j - E\tilde{X}_j], j \geqslant 1\}$ and $\{\check{X}_j = \frac{1}{c_j}[\tilde{X}_j - E\tilde{X}_j], j \geqslant 1\}$ are also mean zero sequences of $\tilde{\rho}$-mixing random variables. Obviously, by (1), it holds

$$\begin{aligned}\sum_{j=1}^{\infty} P(|X_j| > c_j) &= \sum_{j=1}^{\infty}\Big(\int_{-\infty}^{-c_j} qx^{-2}dx + \int_{c_j}^{\infty} px^{-2}dx\Big)\\ &= \sum_{j=1}^{\infty}\frac{p+q}{c_j} = \sum_{j=1}^{\infty}\frac{1}{c_j} = \sum_{j=1}^{\infty}\frac{1}{j\log^2 j} < \infty,\end{aligned} \tag{25}$$

(or see Adler [6]). Similarly, it follows from (1), (25) and the condition of same distribution that

$$\begin{aligned}\sum_{j=1}^{\infty} E\check{X}_j^2 &\leqslant C_1\sum_{j=1}^{\infty}\frac{1}{c_j^2}EX_1^2 I(|X_1| \leqslant c_j) + C_2\sum_{j=1}^{\infty}P(|X_1| > c_j)\\ &= C_1\sum_{j=1}^{\infty}\frac{1}{c_j^2}\Big(\int_{-c_j}^{-1} qdx + \int_1^{c_j} pdx\Big) + C_2\sum_{j=1}^{\infty}P(|X_1| > c_j)\\ &= C_1\sum_{j=1}^{\infty}\frac{1}{c_j^2}[q(c_j-1) + p(c_j-1)] + C_2\sum_{j=1}^{\infty}P(|X_1| > c_j)\\ &= C_1\sum_{j=1}^{\infty}\frac{c_j-1}{c_j^2} + C_2\sum_{j=1}^{\infty}\frac{1}{c_j}\\ &\leqslant C_3\sum_{j=1}^{\infty}\frac{1}{c_j} = C_3\sum_{j=1}^{\infty}\frac{1}{j\log^2 j} < \infty.\end{aligned} \tag{26}$$

Consequently, by Corollary 2.1 and (26), we have that

$$\sum_{j=1}^{\infty}\check{X}_j = \sum_{j=1}^{\infty}\frac{a_j}{b_j}[\tilde{X}_j - E\tilde{X}_j] \text{ converges, } a.s.$$

Combining Kronecker's lemma with $b_n \to \infty$, we obtain that

$$I_1 = \frac{1}{b_n}\sum_{j=1}^{n} a_j[\tilde{X}_j - E\tilde{X}_j] \xrightarrow{a.s.} 0. \tag{27}$$

Combining (25) with Borel-Cantelli lemma, we obtain that

$$|I_2| \leqslant \frac{2}{b_n}\sum_{j=1}^{n} a_j|X_j|I(|X_j| > c_j) \xrightarrow{a.s.} 0. \tag{28}$$

For some positive integer n_0 and integer $n \geqslant n_0$, by (1), it can be seen that

$$\begin{aligned} H &:= \frac{1}{b_n}\Big|\sum_{j=1}^{n} a_j[-c_jP(X_j<-c_j)+c_jP(X_j>c_j)]\Big| \\ &\leqslant \sum_{j=1}^{n_0}\frac{b_j}{b_n}P(|X_j|>c_j)+\sum_{j=n_0+1}^{n}\frac{b_j}{b_n}P(|X_j|>c_j) \\ &= \frac{1}{\log^{\beta} n}\sum_{j=1}^{n_0}\frac{1}{j\log^{2-\beta} j}+\frac{1}{\log^{\beta} n}\sum_{j=n_0+1}^{n}\frac{1}{j\log^{2-\beta} j} \\ &=: H_1+H_2. \end{aligned} \tag{29}$$

Obviously, for any $\beta>0$,

$$H_1 \to 0. \tag{30}$$

Meanwhile, for $0<\beta<1$, it can be argued that

$$H_2=\frac{1}{\log^{\beta} n}\sum_{j=n_0+1}^{n}\frac{1}{j\log^{2-\beta} j}\leqslant\frac{1}{\log^{\beta} n}\sum_{j=n_0+1}^{\infty}\frac{1}{j\log^{2-\beta} j}\to 0. \tag{31}$$

For $\beta=1$, we have

$$H_2=\frac{1}{\log n}\sum_{j=n_0+1}^{n}\frac{1}{j\log j}\sim\frac{\log\log n}{\log n}\to 0. \tag{32}$$

Otherwise, for $\beta>1$,

$$H_2=\frac{1}{\log^{\beta} n}\sum_{j=n_0+1}^{n}\frac{1}{j\log^{2-\beta} j}\sim\frac{\log^{\beta-1} n}{\log^{\beta} n}\to 0. \tag{33}$$

Thus, by (29)-(33), we obtain that

$$H \to 0. \tag{34}$$

Moreover, it follows from (1) that

$$\begin{aligned} EX_nI(|X_n|\leqslant c_n) &= \int_{-c_n}^{-1} qx^{-1}dx+\int_{1}^{c_n} px^{-1}dx \\ &= -q\log c_n+p\log c_n=(p-q)\log c_n\sim(p-q)\log n, \end{aligned}$$

(or see Adler [6]). Thus,

$$\sum_{j=1}^{n}\frac{a_j}{b_n}EX_jI(|X_j|\leqslant c_j)\sim\frac{p-q}{\log^{\beta} n}\sum_{j=1}^{n}\frac{\log^{\beta-1} j}{j}\to\frac{p-q}{\beta}. \tag{35}$$

Consequently, together (34) with (35), it holds

$$I_3\to\frac{p-q}{\beta}. \tag{36}$$

Last, by (24), (27), (28) and (36), we immediately establish (23). □

Second, similar to Theorem 4.1, we consider the case of asymmetrical Cauchy random variables with a slight twist (3), which was discussed in Adler [5]).

THEOREM 4.2. *For $n_0 \geqslant 1$, $0 \leqslant r < 1$ and $\tilde{\rho}(n_0) \leqslant r$, let $\{X_n, n \geqslant 1\}$ be a sequence of $\tilde{\rho}$-mixing random variables with the same distributions from an asymmetrical Cauchy random variables by a slight twist (3). Then for all $\beta > 0$ we have*

$$\frac{1}{\log^{\beta} n} \sum_{j=1}^{n} \frac{\log^{\beta-2} j}{j} X_j \xrightarrow{a.s.} \frac{p-q}{\pi\beta}. \tag{37}$$

Proof. We use the same notation such as a_j, b_j, c_j, $\tilde{X}_j$, $\check{X}_j$, etc, in the proof of Theorem 4.1. Together with the proof of (25), (3) and Lemma 2.4, it yields

$$\begin{aligned}
\sum_{j=1}^{\infty} P(|X_j| > c_j) &= \sum_{j=1}^{\infty} \Big(\int_{-\infty}^{-c_j} \frac{q}{\pi(1+x^2)} dx + \int_{c_j}^{\infty} \frac{p}{\pi(1+x^2)} dx \Big) \\
&= \frac{1}{\pi} \sum_{j=1}^{\infty} \Big(-q \arctan c_j + \frac{q\pi}{2} + \frac{p\pi}{2} - p \arctan c_j \Big) \\
&\leqslant C \sum_{j=1}^{\infty} \frac{1}{c_j} = C \sum_{j=1}^{\infty} \frac{1}{j \log^2 j} < \infty,
\end{aligned} \tag{38}$$

(or see Adler [5]). Similar to the proof of (26), by (3) and (38), we establish that

$$\begin{aligned}
\sum_{j=1}^{\infty} E\check{X}_j^2 &\leqslant C_1 \sum_{j=1}^{\infty} \frac{1}{c_j^2} EX_1^2 I(|X_1| \leqslant c_j) + C_2 \sum_{j=1}^{\infty} P(|X_1| > c_j) \\
&= C_1 \sum_{j=1}^{\infty} \frac{1}{c_j^2} \Big(\int_{-c_j}^{0} \frac{qx^2}{\pi(1+x^2)} dx + \int_{0}^{c_j} \frac{px^2}{\pi(1+x^2)} dx \Big) + C_2 \sum_{j=1}^{\infty} P(|X_1| > c_j) \\
&\leqslant C_3 \sum_{j=1}^{\infty} \frac{1}{c_j^2} \Big(\int_{-c_j}^{0} dx + \int_{0}^{c_j} dx \Big) + C_2 \sum_{j=1}^{\infty} P(|X_1| > c_j) \\
&\leqslant C_4 \sum_{j=1}^{\infty} \frac{1}{c_j} = C_4 \sum_{j=1}^{\infty} \frac{1}{j \log^2 j} < \infty.
\end{aligned} \tag{39}$$

By the proofs of (29) and (38), we have

$$\begin{aligned}
H &:= \frac{1}{b_n} \Big| \sum_{j=1}^{n} a_j [-c_j P(X_j < -c_j) + c_j P(X_j > c_j)] \Big| \\
&\leqslant \frac{C}{\log^{\beta} n} \sum_{j=1}^{n_0} \frac{1}{j \log^{2-\beta} j} + \frac{C}{\log^{\beta} n} \sum_{j=n_0+1}^{n} \frac{1}{j \log^{2-\beta} j} \to 0.
\end{aligned} \tag{40}$$

Moreover, by (3), we obtain that

$$\begin{aligned}
EX_n I(|X_n| \leqslant c_n) &= \int_{-c_n}^{0} \frac{qx}{\pi(1+x^2)} dx + \int_{0}^{c_n} \frac{px}{\pi(1+x^2)} dx \\
&= \frac{1}{2\pi} [-q \log(1+c_n^2) + p \log(1+c_n^2)] \\
&= \frac{p-q}{2\pi} \log(1+c_n^2) \\
&\sim \frac{p-q}{\pi} \log c_n \sim \frac{p-q}{\pi} \log n,
\end{aligned}$$

(or see Adler [5]). Then,

$$\sum_{j=1}^{n} \frac{a_j}{b_n} EX_j I(|X_j| \leqslant c_j) \sim \frac{p-q}{\pi \log^{\beta} n} \sum_{j=1}^{n} \frac{\log^{\beta-1} j}{j} \to \frac{p-q}{\pi\beta}. \tag{41}$$

Combining the proof of Theorem 4.1 with (38)-(41), we obtain (37) immediately. □

5. Simulation

It is known that if $Y \overset{d}{=} U(0,1)$, then for any given distribution $F(x)$, $x \in R$, the random variable $X = F^{-1}(Y) \sim F(x)$, where $F^{-1}(u) = \inf\{x : F(x) \geqslant u\}$, $u \in (0,1)$. So we use this method to generate a random variable of two tailed Pareto distribution or asymmetrical Cauchy distribution. In view of the infinite mean, we use the truncated method in the practical simulation. For example, let ε be a small positive constant and M be a large positive constant. For $u \in (0,1)$, if $u < \varepsilon$, then $x = -M$; if $u \geqslant 1-\varepsilon$, then $x = M$.

By (1), it can be seen that the two tailed Pareto distribution is

$$F(x) = \begin{cases} -\frac{q}{x}, & \text{if } x \leqslant -1, \\ q, & \text{if } -1 < x < 1, \\ 1 - \frac{p}{x}, & \text{if } x \geqslant 1. \end{cases}$$

where $p+q=1$ and $p,q \geqslant 0$. Now, let us give the algorithm of generation of two tailed Pareto distribution. Let $\varepsilon = 10^{-1000}$, $M = 2^{1000}$. For given $p \geqslant 0$ and $q \geqslant 0$ with $p+q=1$, we generate a uniform random variable $U(0,1)$. If $u \leqslant \varepsilon$, then $x = -M$; if $\varepsilon < u < q$, then $x = -q/u$; if $u = q$, then $x = -1$; if $q < u < 1-\varepsilon$, then $x = p/(1-u)$; if $1-\varepsilon \leqslant u \leqslant 1$, then $x = M$. Let

$$T_n = \frac{1}{\log^{\beta} n} \sum_{j=1}^{n} \frac{\log^{\beta-2} j}{j} X_j - \frac{p-q}{\beta}. \tag{42}$$

It is difficult to plot some box plots for T_n in one frame since the variation ranges of occurrence values of random variable X are very big. So we take the following Table 1 for T_n in (42), by repeating the experiments 1000 times. Mean, Var, Min, Max, $Q(1)$, m_e and $Q(3)$ stand for mean, variance, minimal, maximal, quantile($\frac{1}{4}$), median and quantile($\frac{3}{4}$) for T_n, respectively.

Table 1: Two tailed Pareto distribution for T_n

p	q	β	n	Mean	Var	Min	Max	Q(1)	m_e	Q(3)
0.6	0.4	4	1000	0.0721	1.4685	-25.3069	17.6390	-0.0338	0.0319	0.1212
0.6	0.4	4	2000	-0.1991	217.6026	-639.2232	109.9941	-0.0386	0.0274	0.1106
0.6	0.4	4	5000	0.1452	18.5986	-165.0443	166.5787	-0.0413	0.0290	0.1163
0.6	0.4	4	10000	0.1234	22.3713	-188.4595	247.6981	-0.0376	0.0294	0.1152
0.6	0.4	6	1000	-0.0708	14.2040	-326.9453	55.4108	-0.0330	-0.0313	-0.0270
0.6	0.4	6	2000	-0.0070	0.2811	-6.1186	33.0231	-0.0330	-0.0313	-0.0268
0.6	0.4	6	5000	0.0128	7.0626	-217.9158	108.2512	-0.0328	-0.0296	0.0163
0.6	0.4	6	10000	0.0682	13.6881	-108.0921	295.1316	-0.0252	0.0154	0.0677
0.7	0.3	6	1000	0.0125	17.1299	-11.1316	405.8907	-0.0629	-0.0614	-0.0576
0.7	0.3	6	2000	-0.0209	0.4749	-26.6703	37.8729	-0.0628	-0.0610	-0.0526
0.7	0.3	6	5000	0.0436	3.3402	-112.0727	75.6017	-0.0622	-0.0540	0.0317
0.7	0.3	6	10000	0.0121	88.9769	-868.6450	235.4846	-0.0079	0.0302	0.0912
0.7	0.3	8	1000	0.0315	2.2334	-41.0813	18.1388	-0.0099	0.0187	0.0595
0.7	0.3	8	2000	0.0818	1.3361	-7.9868	45.8754	-0.0065	0.0225	0.0707
0.7	0.3	8	5000	0.0540	1.3777	-52.1305	22.5732	-0.0088	0.0200	0.0651
0.7	0.3	8	10000	0.0918	1.3389	-17.7700	53.0901	-0.0073	0.0210	0.0662

On the one hand, by the Table 1, it can be found that the mean of T_n does not decrease to zero as sample n increases by 1000, 2000, 5000 and 10000, and the variance of T_n performs an increasing trend as the sample n increases. So one should not use the mean of T_n to estimate T_n since it is not robust. On the other hand, compared to mean, by the Table 1, it can be seen that the median m_e of T_n has a good performance since it is close to zero.

Meanwhile, by (3), it can be seen that the asymmetrical Cauchy distribution is

$$F(x) = \begin{cases} \frac{q}{2} + \frac{q}{\pi}\arctan x, & \text{if } x < 0, \\ \frac{q}{2} + \frac{p}{\pi}\arctan x, & \text{if } x \geqslant 0, \end{cases}$$

where $p+q=2$ and $p,q \geqslant 0$. So, similar to the algorithm of two tailed Pareto distribution, we give the algorithm of generation of asymmetrical Cauchy distribution. Let $\varepsilon = 10^{-1000}$, $M = 2^{1000}$. For given $p \geqslant 0$ and $q \geqslant 0$ with $p+q=2$, we generate a uniform random variable $U(0,1)$. If $u \leqslant \varepsilon$, then $x = -M$; if $\varepsilon < u \leqslant q/2$, then $x = \tan((u-q/2)\pi/q)$; if $q/2 < u < 1-\varepsilon$, then $x = \tan((u-q/2)\pi/p)$; if $1-\varepsilon \leqslant u \leqslant 1$, then $x = M$. Let

$$\tilde{T}_n = \frac{1}{\log^{\beta} n}\sum_{j=1}^{n}\frac{\log^{\beta-2} j}{j}X_j - \frac{p-q}{\pi\beta} \tag{43}$$

Similar to Table 1, we obtain the following Table 2 for asymmetrical Cauchy distribution of $\tilde{T}_n$ in (43), by repeating the experiments 1000 times.

Table 2: Asymmetrical Cauchy distribution for $\tilde{T}_n$

p	q	β	n	Mean	Var	Min	Max	Q(1)	m_e	Q(3)
p	q	β	n	Mean	Var	Min	Max	Q(1)	m_e	Q(3)
1.1	0.9	4	1000	-0.1277	41.7066	-201.4230	23.1959	-0.0361	0.0083	0.0588
1.1	0.9	4	2000	0.0106	4.4502	-75.5523	44.9267	-0.0363	0.0081	0.0602
1.1	0.9	4	5000	0.0324	1.7688	-43.9488	36.9070	-0.0382	0.0082	0.0615
1.1	0.9	4	10000	-0.0545	23.6353	-297.5537	98.9471	-0.0373	0.0075	0.0597
1.1	0.9	6	1000	-0.0071	2.5193	-47.6440	10.5151	-0.0242	0.0035	0.0335
1.1	0.9	6	2000	-0.0833	6.8506	-99.9745	5.3900	-0.0231	0.0035	0.0344
1.1	0.9	6	5000	0.1140	27.4555	-18.7877	358.7516	-0.0241	0.0031	0.0339
1.1	0.9	6	10000	-0.0545	2.8957	-138.2517	48.9460	-0.0235	0.0033	0.0343
1.5	0.5	4	1000	0.0759	10.8746	-90.9387	35.8580	-0.0040	0.0394	0.1084
1.5	0.5	4	2000	0.1515	2.7080	-13.4087	63.9893	-0.0019	0.0382	0.1091
1.5	0.5	4	5000	0.1232	1.9087	-55.6143	30.1641	-0.0015	0.0380	0.1062
1.5	0.5	4	10000	0.2015	14.0169	-23.1490	300.1361	0.0006	0.0412	0.1152
1.5	0.5	6	1000	0.1741	4.1107	-6.5125	55.2070	-0.0019	0.0214	0.0658
1.5	0.5	6	2000	-0.0646	32.7001	-252.6613	16.4857	-0.0035	0.0206	0.0643
1.5	0.5	6	5000	0.1535	11.1481	-6.4231	164.8726	-0.0047	0.0199	0.0615
1.5	0.5	6	10000	0.0989	2.0673	-52.0123	81.6424	-0.0037	0.0210	0.0618

Similar to Table 1, for the case of asymmetrical Cauchy distribution, by Table 2, it also can be found that the mean and variance of $\tilde{T}_n$ in (43) are not robust, but the median m_e of $\tilde{T}_n$ has a good performance closing to zero.

6. Conclusion

In this paper, we combine the properties of $\tilde{\rho}$-mixing such as moment inequalities and convergence theorem with the properties of Pareto type distributions and asymmetrical Cauchy distribution, we obtain the weak laws and strong laws for the weighted sums of $\tilde{\rho}$-mixing random variables with infinite mean (see our results in Sections 3 and 4). So our Theorem 3.1, Corollary 3.1, Theorem 3.2 and Corollary 3.2 are extended the Theorem 2.1, Corollary 2.1, Theorem 3.1 and Theorem 3.2 of Nakata [19]

for independent case to dependent one, respectively. Meanwhile, our Theorem 4.1 and Theorem 4.2 extend Theorem 2.1 of Adler [6]) and Theorem 2.1 of Adler [5]) for independent case to dependent one, respectively. Since $\tilde{\rho}(n)$-mixing is quite different from other mixing sequences such as ρ-mixing and α-mixing (see our Remark 2.2 an a survey of basic properties of mixing conditions by Bradley [11]), the results obtained by this paper are interesting. In addition, we do some simulations of the laws of large numbers for the tailed Pareto distributions and asymmetrical Cauchy distributions. By Tables 1 and 2, it can be found that the mean and variance of sums are not robust but the median has good performance of robust. It can be explained that the occurrence values of random variable with infinite mean sometimes take some big positive or negative values, which does not lead to be robust. But the median avoids this problem and is robust.

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