

ON SOME EQUALITIES AND INEQUALITIES OF FUSION FRAME IN HILBERT C^* -MODULES

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Abstract. In this paper, we establish some new identities and inequalities for fusion frames with a scalar in Hilbert C^* -modules. Our results are more general than those previously obtained by Balan et al. for Hilbert space frames. It is shown that the results we obtained can immediately lead to the existing corresponding results when taking suitable scalar. Moreover, we give some double inequalities for fusion frames in Hilbert C^* -modules.

1. Introduction

Frames for Hilbert spaces were introduced by Duffin and Schaeffer [7] in the context of nonharmonic Fourier series. In contrast to orthonormal bases, frames form redundant systems, thereby allowing non-unique, but stable decompositions and expansions. Due to this property, frames have been widely applied in numerous applications such as filter bank theory [14], signal and image processing [11], coding and communication [15, 16], compressed sensing [5], etc. For more details about the theory and applications of frames, we refer the reader to [6].

Hilbert C^* -modules are generalizations of Hilbert spaces by allowing the inner product to take values in a C^* -algebra rather than in the field of complex numbers.

Frames and fusion frames for Hilbert spaces have natural analogues for Hilbert C^* -modules [8, 13]. Due to the complex structure of C^* -algebras embedded in the Hilbert C^* -modules, the generalization of frame theory from Hilbert spaces to Hilbert C^* -modules is not a trivial task. Frames and especial fusion frames for Hilbert C^* -modules have been studied intensively, for more details see [1, 2, 12].

We need to recall some notations and basic definitions.

Throughout this paper, the symbols \mathbb{I} and \mathcal{A} are reserved for a finite or countable index set and a unital C^* -algebra, respectively. \mathcal{H} and \mathcal{K} are Hilbert C^* -modules over \mathcal{A} . We denote by $End_{\mathcal{A}}^*(\mathcal{H}, \mathcal{M})$ the set of all adjointable operators from \mathcal{H} to \mathcal{K} . If W is a closed submodule of \mathcal{H} , we denote π_W is the orthogonal projection of \mathcal{H} onto W . We say W is orthogonally complemented if $\mathcal{H} = W \oplus W^\perp$ and in this case $\pi_W \in End_{\mathcal{A}}^*(\mathcal{H}, W)$.

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DEFINITION 1. Let \mathcal{H} be a Hilbert C^* -modules over \mathcal{A} and $\{w_i\}_{i \in \mathbb{I}}$ be a family of weights in \mathcal{H} , i.e., each w_i is a positive invertible element from the center of the C^* -algebra \mathcal{A} . A sequence of closed submodules $\{(W_i, w_i)\}_{i \in \mathbb{I}}$ is a fusion frame for \mathcal{H} if every W_i is orthogonally complemented and there exist real constants $0 < C \leq D < \infty$ such that

$$C \langle x, x \rangle \leq \sum_{i \in \mathbb{I}} w_i^2 \langle \pi_{W_i}(x), \pi_{W_i}(x) \rangle \leq D \langle x, x \rangle, \quad \forall x \in \mathcal{H}. \tag{1}$$

We call C and D the lower and upper bounds of the fusion frames. The fusion frame $\{(W_i, w_i)\}_{i \in \mathbb{I}}$ is said to be C -tight if $C = D$, and said to be Parseval if $C = D = 1$. If the right-handed inequality of (1) holds, then we say that $\{(W_i, w_i)\}_{i \in \mathbb{I}}$ is a Bessel fusion sequence for \mathcal{H} with Bessel fusion bound D .

Let $\{(W_i, w_i)\}_{i \in \mathbb{I}}$ be a fusion frame for \mathcal{H} , then the fusion operator $S : \mathcal{H} \rightarrow \mathcal{H}$ defined by

$$Sx = \sum_{i \in \mathbb{I}} w_i^2 \pi_{M_i}(x), \quad \forall x \in \mathcal{H},$$

is a positive, self-adjoint and invertible operator. These properties imply that $\{(W_i, w_i)\}_{i \in \mathbb{I}}$ admits the reconstruction formular

$$x = \sum_{i \in \mathbb{I}} w_i^2 S^{-1} \pi_{W_i}(x) = \sum_{i \in \mathbb{I}} w_i^2 \pi_{W_i}(S^{-1}x), \quad \forall x \in \mathcal{H}.$$

this relation proves that the family of operators $\{w_i^2 S^{-1} \pi_{W_i}(x)\}_{i \in \mathbb{I}}$ is a resolution of identity. The family $\{(S^{-1}W_i, w_i)\}_{i \in \mathbb{I}}$ is called the dual fusion frame of $\{(W_i, w_i)\}_{i \in \mathbb{I}}$. The alternative dual of $\{(W_i, w_i)\}_{i \in \mathbb{I}}$ in Hilbert C^* -modules is similar to that in Hilbert spaces [10], which is given by

$$x = \sum_{i \in \mathbb{I}} v_i w_i \pi_{V_i} S^{-1} \pi_{W_i}(x), \quad \forall x \in \mathcal{H},$$

where $\{(V_i, v_i)\}_{i \in \mathbb{I}}$ is a Bessel fusion sequence for \mathcal{H} . In this case, $\{(V_i, v_i)\}_{i \in \mathbb{I}}$ is also a fusion frame and call the alternative dual of $\{(W_i, w_i)\}_{i \in \mathbb{I}}$.

For any $\mathbb{J} \subset \mathbb{I}$, we let $\mathbb{J}^c = \mathbb{I} \setminus \mathbb{J}$, and define the adjointable operator

$$S_{\mathbb{J}}x = \sum_{i \in \mathbb{J}} w_i^2 \pi_{W_i}x, \quad \forall x \in \mathcal{H}.$$

When Balan et al. [3] study that a signal vector can be reconstructed without a noisy phase or its estimation, they had proved a longstanding conjecture by constructing new classes of Parseval frames. While working on efficient algorithms for signal reconstruction, the authors of [4] found a new identity for Parseval frames. The authors in [9] generalized these identities to alternate dual frames and got some general results.

Some authors have extended the equalities and inequalities for g-frames [23], probabilistic frames [19], fusion frames [17], c-fusion frames [18] and g-frames in Hilbert C^* -modules [21, 22]. In order to compare with the main results in Section 2, we list the important equalities and inequalities in [9] for Hilbert spaces as follows.

THEOREM 1. *Let $\{x_i\}_{i \in \mathbb{I}}$ be a frame for \mathcal{H} with frame operator S . Then for all $\mathbb{J} \subset \mathbb{I}$ and all $x \in \mathcal{H}$, we have*

$$\begin{aligned} \sum_{i \in \mathbb{J}} |\langle x, x_i \rangle|^2 + \sum_{i \in \mathbb{I}} |\langle S^{-1} S_{\mathbb{J}^c} x, x_i \rangle|^2 &= \sum_{i \in \mathbb{J}^c} |\langle x, x_i \rangle|^2 + \sum_{i \in \mathbb{I}} |\langle S^{-1} S_{\mathbb{J}} x, x_i \rangle|^2 \\ &\geq \frac{3}{4} \sum_{i \in \mathbb{I}} |\langle x, x_i \rangle|^2. \end{aligned} \tag{2}$$

THEOREM 2. *Let $\{x_i\}_{i \in \mathbb{I}}$ be a frame for \mathcal{H} and let $\{y_i\}_{i \in \mathbb{I}}$ be an alternate dual frame of $\{x_i\}_{i \in \mathbb{I}}$. Then for all $\mathbb{J} \subset \mathbb{I}$ and all $x \in \mathcal{H}$, we have*

$$\begin{aligned} \operatorname{Re} \sum_{i \in \mathbb{J}} \langle x, y_i \rangle \overline{\langle x, x_i \rangle} + \sum_{i \in \mathbb{J}^c} \|\langle x, y_i \rangle x_i\|^2 &= \operatorname{Re} \sum_{i \in \mathbb{J}^c} \langle x, y_i \rangle \overline{\langle x, x_i \rangle} + \sum_{i \in \mathbb{J}} \|\langle x, y_i \rangle x_i\|^2 \\ &\geq \frac{3}{4} \|x\|^2. \end{aligned} \tag{3}$$

In this paper, we give some equalities and inequalities for fusion frames in Hilbert C^* -modules with a scalar $\lambda \in [0, 2]$ and also generalize the inequality to a more general form.

2. The main results and their proofs

In order to derive our main results, we first give a simple lemma.

LEMMA 1. *If $P, Q \in \operatorname{End}_{\mathcal{A}}^*(\mathcal{H})$ are self-adjoint operators satisfying $P + Q = I_{\mathcal{H}}$, then for any $\lambda \in [0, 2]$ and all $x \in \mathcal{H}$ we have*

$$\langle Px, Px \rangle + \lambda \langle Qx, x \rangle = \langle Qx, Qx \rangle + (2 - \lambda) \langle Px, x \rangle + (\lambda - 1) \langle x, x \rangle \geq (\lambda - \frac{\lambda^2}{4}) \langle x, x \rangle.$$

Proof. Since $P + Q = I_{\mathcal{H}}$, we have

$$\langle Px, Px \rangle + \lambda \langle Qx, x \rangle = \langle P^2 x, x \rangle + \lambda \langle (I_{\mathcal{H}} - P)x, x \rangle = \langle (P^2 - \lambda P + \lambda I_{\mathcal{H}})x, x \rangle,$$

and

$$\begin{aligned} &\langle Qx, Qx \rangle + (2 - \lambda) \langle Px, x \rangle + (\lambda - 1) \langle x, x \rangle \\ &= \langle (I_{\mathcal{H}} - P)^2 x, x \rangle + (2 - \lambda) \langle Px, x \rangle + (\lambda - 1) \langle x, x \rangle \\ &= \langle (P^2 - \lambda P + \lambda I_{\mathcal{H}})x, x \rangle. \end{aligned}$$

We also have

$$\begin{aligned} \langle (P^2 - \lambda P + \lambda I_{\mathcal{H}})x, x \rangle &= \left\langle \left(P^2 - \lambda P + \frac{\lambda^2}{4} + \lambda I_{\mathcal{H}} - \frac{\lambda^2}{4} I_{\mathcal{H}} \right) x, x \right\rangle \\ &= \left\langle \left(\left(P - \frac{\lambda}{2} I_{\mathcal{H}} \right)^2 + \lambda I_{\mathcal{H}} - \frac{\lambda^2}{4} I_{\mathcal{H}} \right) x, x \right\rangle \\ &= \left\langle \left(P - \frac{\lambda}{2} I_{\mathcal{H}} \right) x, \left(P - \frac{\lambda}{2} I_{\mathcal{H}} \right) x \right\rangle + (\lambda - \frac{\lambda^2}{4}) \langle x, x \rangle \end{aligned}$$

$$\geq (\lambda - \frac{\lambda^2}{4}) \langle x, x \rangle. \quad \square$$

We find that a similar identity was established in [20] as well, but the inequality in our result is a new result.

THEOREM 3. *Let $\{(W_i, w_i)\}_{i \in \mathbb{I}}$ be a fusion frame for \mathcal{H} with the fusion frame operator S . Then for any $\lambda \in [0, 2]$, for all $\mathbb{J} \subset \mathbb{I}$ and any $x \in \mathcal{H}$, we have*

$$\begin{aligned} \sum_{i \in \mathbb{I}} w_i^2 \langle \pi_{W_i} x, \pi_{W_i} x \rangle &\geq \sum_{i \in \mathbb{J}^c} w_i^2 \langle \pi_{W_i} x, \pi_{W_i} x \rangle + \sum_{i \in \mathbb{I}} w_i^2 \langle \pi_{W_i} S^{-1} S_{\mathbb{J}} x, \pi_{W_i} S^{-1} S_{\mathbb{J}} x \rangle \\ &= \sum_{i \in \mathbb{J}} w_i^2 \langle \pi_{W_i} x, \pi_{W_i} x \rangle + \sum_{i \in \mathbb{I}} w_i^2 \langle \pi_{W_i} S^{-1} S_{\mathbb{J}^c} x, \pi_{W_i} S^{-1} S_{\mathbb{J}^c} x \rangle \\ &\geq (\lambda - \frac{\lambda^2}{4}) \sum_{i \in \mathbb{J}} w_i^2 \langle \pi_{W_i} x, \pi_{W_i} x \rangle + (1 - \frac{\lambda^2}{4}) \sum_{i \in \mathbb{J}^c} w_i^2 \langle \pi_{W_i} x, \pi_{W_i} x \rangle. \end{aligned} \quad (4)$$

Proof. Since $S = S_{\mathbb{J}} + S_{\mathbb{J}^c}$, it follows that

$$I_{\mathcal{H}} = S^{-1/2} S_{\mathbb{J}} S^{-1/2} + S^{-1/2} S_{\mathbb{J}^c} S^{-1/2}.$$

Let $P = S^{-1/2} S_{\mathbb{J}} S^{-1/2}$, $Q = S^{-1/2} S_{\mathbb{J}^c} S^{-1/2}$ and let $S^{1/2} x$ instead of $x \in \mathcal{H}$ in Lemma 1, we have

$$\langle PS^{1/2} x, PS^{1/2} x \rangle + \lambda \langle QS^{1/2} x, S^{1/2} x \rangle = \langle S^{-1} S_{\mathbb{J}} x, S_{\mathbb{J}} x \rangle + \lambda \langle S_{\mathbb{J}^c} x, x \rangle. \quad (5)$$

And

$$\begin{aligned} &\langle QS^{1/2} x, QS^{1/2} x \rangle + (2 - \lambda) \langle PS^{1/2} x, S^{1/2} x \rangle + (\lambda - 1) \langle S^{1/2} x, S^{1/2} x \rangle \\ &= \langle S^{-1} S_{\mathbb{J}^c} x, S_{\mathbb{J}^c} x \rangle + (2 - \lambda) \langle S_{\mathbb{J}} x, x \rangle + (\lambda - 1) \langle Sx, x \rangle. \end{aligned} \quad (6)$$

By (5) and (6), we have

$$\langle S^{-1} S_{\mathbb{J}} x, S_{\mathbb{J}} x \rangle + \lambda \langle S_{\mathbb{J}^c} x, x \rangle = \langle S^{-1} S_{\mathbb{J}^c} x, S_{\mathbb{J}^c} x \rangle + (2 - \lambda) \langle S_{\mathbb{J}} x, x \rangle + (\lambda - 1) \langle Sx, x \rangle.$$

After subtracting both sides by $\lambda \langle S_{\mathbb{J}^c} x, x \rangle$, we obtain

$$\begin{aligned} \langle S^{-1} S_{\mathbb{J}} x, S_{\mathbb{J}} x \rangle &= \langle S^{-1} S_{\mathbb{J}^c} x, S_{\mathbb{J}^c} x \rangle + (2 - \lambda) \langle S_{\mathbb{J}} x, x \rangle + (\lambda - 1) \langle Sx, x \rangle - \lambda \langle S_{\mathbb{J}^c} x, x \rangle \\ &= \langle S^{-1} S_{\mathbb{J}^c} x, S_{\mathbb{J}^c} x \rangle + 2 \langle S_{\mathbb{J}} x, x \rangle - \lambda \langle (S_{\mathbb{J}^c} + S_{\mathbb{J}}) x, x \rangle + (\lambda - 1) \langle Sf, f \rangle \\ &= \langle S^{-1} S_{\mathbb{J}^c} x, S_{\mathbb{J}^c} x \rangle + 2 \langle S_{\mathbb{J}} x, x \rangle - \langle Sx, x \rangle \\ &= \langle S^{-1} S_{\mathbb{J}^c} x, S_{\mathbb{J}^c} x \rangle + 2 \langle S_{\mathbb{J}} x, x \rangle - \langle (S_{\mathbb{J}} + S_{\mathbb{J}^c}) x, x \rangle \\ &= \langle S^{-1} S_{\mathbb{J}^c} x, S_{\mathbb{J}^c} x \rangle + \langle S_{\mathbb{J}} x, x \rangle - \langle S_{\mathbb{J}^c} x, x \rangle. \end{aligned}$$

Thus,

$$\langle S^{-1} S_{\mathbb{J}} x, S_{\mathbb{J}} x \rangle + \langle S_{\mathbb{J}^c} x, x \rangle = \langle S^{-1} S_{\mathbb{J}^c} x, S_{\mathbb{J}^c} x \rangle + \langle S_{\mathbb{J}} x, x \rangle. \quad (7)$$

On the other hand, we have

$$\langle S^{-1} S_{\mathbb{J}} x, S_{\mathbb{J}} x \rangle = \langle SS^{-1} S_{\mathbb{J}} x, S^{-1} S_{\mathbb{J}} x \rangle = \sum_{i \in \mathbb{I}} w_i^2 \langle \pi_{W_i} S^{-1} S_{\mathbb{J}} x, \pi_{W_i} S^{-1} S_{\mathbb{J}} x \rangle. \quad (8)$$

Similarly, we obtain

$$\langle S_{\mathbb{J}^c}x, x \rangle = \sum_{i \in \mathbb{J}^c} w_i^2 \langle \pi_{W_i}x, \pi_{W_i}x \rangle, \quad \langle S_{\mathbb{J}}x, x \rangle = \sum_{i \in \mathbb{J}} w_i^2 \langle \pi_{W_i}x, \pi_{W_i}x \rangle. \quad (9)$$

$$\langle S^{-1}S_{\mathbb{J}^c}x, S_{\mathbb{J}^c}x \rangle = \sum_{i \in \mathbb{I}} w_i^2 \langle \pi_{W_i}S^{-1}S_{\mathbb{J}^c}x, \pi_{W_i}S^{-1}S_{\mathbb{J}^c}x \rangle. \quad (10)$$

Using (7)-(10), we proof the equality of (4). Next, we proof the first inequality of (4). Since $P = S^{-1/2}S_{\mathbb{J}}S^{-1/2}$, $Q = S^{-1/2}S_{\mathbb{J}^c}S^{-1/2}$ are positive operators, then

$$0 \leq PQ = P(I_{\mathcal{H}} - P) = P - P^2 = S^{-1/2}(S_{\mathbb{J}} - S_{\mathbb{J}}S^{-1}S_{\mathbb{J}})S^{-1/2},$$

from which we conclude that $S_{\mathbb{J}} - S_{\mathbb{J}}S^{-1}S_{\mathbb{J}} \geq 0$. Therefore, By (8) and (9), we have

$$\begin{aligned} & \sum_{i \in \mathbb{I}} w_i^2 \langle \pi_{W_i}S^{-1}S_{\mathbb{J}}x, \pi_{W_i}S^{-1}S_{\mathbb{J}}x \rangle + \sum_{i \in \mathbb{J}^c} w_i^2 \langle \pi_{W_i}x, \pi_{W_i}x \rangle \\ &= \langle S^{-1}S_{\mathbb{J}}x, S_{\mathbb{J}}x \rangle + \langle S_{\mathbb{J}^c}x, x \rangle = \langle S_{\mathbb{J}}S^{-1}S_{\mathbb{J}}x, x \rangle + \langle S_{\mathbb{J}^c}x, x \rangle \\ &\leq \langle S_{\mathbb{J}}x, x \rangle + \langle S_{\mathbb{J}^c}x, x \rangle = \langle Sx, x \rangle \\ &= \sum_{i \in \mathbb{I}} w_i^2 \langle \pi_{W_i}x, \pi_{W_i}x \rangle. \end{aligned}$$

We now proof the last inequality. By Lemma 1 and (5), we have

$$\langle S^{-1}S_{\mathbb{J}}x, S_{\mathbb{J}}x \rangle + \lambda \langle S_{\mathbb{J}^c}x, x \rangle \geq (\lambda - \lambda^2/4) \langle Sx, x \rangle.$$

And then,

$$\begin{aligned} \langle S^{-1}S_{\mathbb{J}}x, S_{\mathbb{J}}x \rangle &\geq (\lambda - \lambda^2/4) \langle Sx, x \rangle - \lambda \langle S_{\mathbb{J}^c}x, x \rangle \\ &= \lambda \langle Sx, x \rangle - \lambda \langle S_{\mathbb{J}^c}x, x \rangle - \frac{\lambda^2}{4} \langle Sx, x \rangle \\ &= \lambda \langle S_{\mathbb{J}}x, x \rangle - \frac{\lambda^2}{4} \langle S_{\mathbb{J}}x, x \rangle - \frac{\lambda^2}{4} \langle S_{\mathbb{J}^c}x, x \rangle \\ &= (\lambda - \frac{\lambda^2}{4}) \langle S_{\mathbb{J}}x, x \rangle + (1 - \frac{\lambda^2}{4}) \langle S_{\mathbb{J}^c}x, x \rangle - \langle S_{\mathbb{J}^c}x, x \rangle. \end{aligned} \quad (11)$$

Hence,

$$\langle S^{-1}S_{\mathbb{J}}x, S_{\mathbb{J}}x \rangle + \langle S_{\mathbb{J}^c}x, x \rangle \geq (\lambda - \frac{\lambda^2}{4}) \langle S_{\mathbb{J}}x, x \rangle + (1 - \frac{\lambda^2}{4}) \langle S_{\mathbb{J}^c}x, x \rangle.$$

Therefore the proof is completed. \square

Theorem 3 leads to a direct consequences as follows.

COROLLARY 1. *Let $\{(W_i, w_i)\}_{i \in \mathbb{I}}$ be a Parseval fusion frame for \mathcal{H} . Then for any $\lambda \in [0, 2]$, for all $\mathbb{J} \subset \mathbb{I}$ and any $x \in \mathcal{H}$, we have*

$$\begin{aligned} \langle x, x \rangle &\geq \sum_{i \in \mathbb{J}^c} w_i^2 \langle \pi_{W_i}x, \pi_{W_i}x \rangle + \sum_{i \in \mathbb{I}} w_i^2 \langle \pi_{W_i}S_{\mathbb{J}}x, \pi_{W_i}S_{\mathbb{J}}x \rangle \\ &\geq (\lambda - \frac{\lambda^2}{4}) \sum_{i \in \mathbb{J}} w_i^2 \langle \pi_{W_i}x, \pi_{W_i}x \rangle + (1 - \frac{\lambda^2}{4}) \sum_{i \in \mathbb{J}^c} w_i^2 \langle \pi_{W_i}x, \pi_{W_i}x \rangle. \end{aligned}$$

In Corollary 1, if we set $\langle x, x \rangle = |x|^2$ for all $x \in \mathcal{H}$, then we have

$$\sum_{i \in \mathbb{I}} w_i^2 \langle \pi_{W_i} S_{\mathbb{J}} x, \pi_{W_i} S_{\mathbb{J}} x \rangle = |S_{\mathbb{J}} x|^2 = \left| \sum_{i \in \mathbb{J}} w_i^2 \pi_{W_i} x \right|^2.$$

This leads to the following result.

COROLLARY 2. *Let $\{(W_i, w_i)\}_{i \in \mathbb{I}}$ be a C -tight fusion frame for \mathcal{H} . If we set $\langle x, x \rangle = |x|^2$ for all $x \in \mathcal{H}$, then for any $\lambda \in [0, 2]$, for all $\mathbb{J} \subset \mathbb{I}$ and any $x \in \mathcal{H}$, we have*

$$\begin{aligned} |x|^2 &\geq \sum_{i \in \mathbb{J}^c} w_i^2 |\pi_{W_i} x|^2 + \frac{1}{C} \left| \sum_{i \in \mathbb{J}} w_i^2 \pi_{W_i} x \right|^2 = \sum_{i \in \mathbb{J}} w_i^2 |\pi_{W_i} x|^2 + \frac{1}{C} \left| \sum_{i \in \mathbb{J}^c} w_i^2 \pi_{W_i} x \right|^2 \\ &\geq \left(\lambda - \frac{\lambda^2}{4} \right) \sum_{i \in \mathbb{J}} w_i^2 |\pi_{W_i} x|^2 + \left(1 - \frac{\lambda^2}{4} \right) \sum_{i \in \mathbb{J}^c} w_i^2 |\pi_{W_i} x|^2. \end{aligned}$$

COROLLARY 3. *Let $\{(W_i, w_i)\}_{i \in \mathbb{I}}$ be a Parseval fusion frame for \mathcal{H} . If we set $\langle x, x \rangle = |x|^2$ for all $x \in \mathcal{H}$, then for all $\mathbb{K} \subset \mathbb{J} \subset \mathbb{I}$ and any $x \in \mathcal{H}$, we have*

$$\begin{aligned} &\left| \sum_{i \in \mathbb{J} \setminus \mathbb{K}} w_i^2 \pi_{W_i} x \right|^2 - \left| \sum_{i \in \mathbb{J}^c \cup \mathbb{K}} w_i^2 \pi_{W_i} x \right|^2 \\ &= \left| \sum_{i \in \mathbb{J}} w_i^2 \pi_{W_i} x \right|^2 - \left| \sum_{i \in \mathbb{J}^c} w_i^2 \pi_{W_i} x \right|^2 + 2 \sum_{i \in \mathbb{K}} w_i^2 |\pi_{W_i} x|^2. \end{aligned}$$

Proof. By taking $C = 1$ in Corollary 2, for all $x \in \mathcal{H}$, we have

$$\begin{aligned} &\left| \sum_{i \in \mathbb{J} \setminus \mathbb{K}} w_i^2 \pi_{W_i} x \right|^2 - \left| \sum_{i \in \mathbb{J}^c \cup \mathbb{K}} w_i^2 \pi_{W_i} x \right|^2 = \sum_{i \in \mathbb{J} \setminus \mathbb{K}} w_i^2 |\pi_{W_i} x|^2 - \sum_{i \in \mathbb{J}^c \cup \mathbb{K}} w_i^2 |\pi_{W_i} x|^2 \\ &= \sum_{i \in \mathbb{J}} w_i^2 |\pi_{W_i} x|^2 - \sum_{i \in \mathbb{K}} w_i^2 |\pi_{W_i} x|^2 - \left(\sum_{i \in \mathbb{J}^c} w_i^2 |\pi_{W_i} x|^2 + \sum_{i \in \mathbb{K}} w_i^2 |\pi_{W_i} x|^2 \right) \\ &= \left| \sum_{i \in \mathbb{J}} w_i^2 \pi_{W_i} x \right|^2 - \left| \sum_{i \in \mathbb{J}^c} w_i^2 \pi_{W_i} x \right|^2 + 2 \sum_{i \in \mathbb{K}} w_i^2 |\pi_{W_i} x|^2. \quad \square \end{aligned}$$

REMARK 1. If we take $\lambda = 1$ in Theorem 3, then our result can immediately lead to the inequality in Theorem 1.

Next, We consider scalar $\lambda \in [0, 1]$ and we give the version of fusion frame in Hilbert C^* -modules for Theorem 2. We need the following lemma.

LEMMA 2. *If $P, Q \in \text{End}_{\mathcal{A}}^*(\mathcal{H})$ are self-adjoint operators satisfying $P + Q = I_{\mathcal{H}}$, then for any $\lambda \in [0, 1]$ we have*

$$P^*P + \lambda(Q^* + Q) = Q^*Q + (1 - \lambda)(P^* + P) + (2\lambda - 1)I_{\mathcal{H}} \geq \lambda(2 - \lambda)I_{\mathcal{H}}.$$

Proof. Since $P + Q = I_{\mathcal{H}}$, we have

$$P^*P + \lambda(Q^* + Q) = P^*P + \lambda(I_{\mathcal{H}} - P^* + I_{\mathcal{H}} - P) = P^*P - \lambda(P^* + P) + 2\lambda I_{\mathcal{H}},$$

and

$$\begin{aligned} Q^*Q + (1 - \lambda)(P^* + P) + (2\lambda - 1)I_{\mathcal{H}} &= (I_{\mathcal{H}} - P^*)(I_{\mathcal{H}} - P) + (1 - \lambda)(P^* + P) \\ &\quad + (2\lambda - 1)I_{\mathcal{H}} \\ &= P^*P - \lambda(P^* + P) + 2\lambda I_{\mathcal{H}}. \end{aligned}$$

We also have

$$\begin{aligned} P^*P - \lambda(P^* + P) + 2\lambda I_{\mathcal{H}} &= P^*P - \lambda(P^* + P) + 2\lambda I_{\mathcal{H}} + \lambda^2 I_{\mathcal{H}} - \lambda^2 I_{\mathcal{H}} \\ &= (P - \lambda I_{\mathcal{H}})^*(P - \lambda I_{\mathcal{H}}) + \lambda(2 - \lambda)I_{\mathcal{H}} \\ &\geq \lambda(2 - \lambda)I_{\mathcal{H}}. \quad \square \end{aligned}$$

THEOREM 4. Let $\{(W_i, w_i)\}_{i \in \mathbb{I}}$ be a fusion frame for \mathcal{H} with the fusion frame operator S and let $\{(V_i, v_i)\}_{i \in \mathbb{I}}$ be the alternate dual fusion frame of $\{(W_i, w_i)\}_{i \in \mathbb{I}}$. If we set $\langle x, x \rangle = |x|^2$ for all $x \in \mathcal{H}$, then for any $\lambda \in [0, 1]$, for all bounded sequence $\{a_i\}_{i \in \mathbb{I}}$ and any $x \in \mathcal{H}$, we have

$$\begin{aligned} &\operatorname{Re} \sum_{i \in \mathbb{I}} a_i v_i w_i \langle S^{-1} \pi_{W_i} x, \pi_{V_i} x \rangle + \left| \sum_{i \in \mathbb{I}} (1 - a_i) v_i w_i \pi_{V_i} S^{-1} \pi_{W_i} x \right|^2 \\ &= \operatorname{Re} \sum_{i \in \mathbb{I}} (1 - a_i) v_i w_i \langle S^{-1} \pi_{W_i} x, \pi_{V_i} x \rangle + \left| \sum_{i \in \mathbb{I}} a_i v_i w_i \pi_{V_i} S^{-1} \pi_{W_i} x \right|^2 \\ &\geq (2\lambda - \lambda^2) \operatorname{Re} \sum_{i \in \mathbb{I}} a_i v_i w_i \langle S^{-1} \pi_{W_i} x, \pi_{V_i} x \rangle + (1 - \lambda^2) \operatorname{Re} \sum_{i \in \mathbb{I}} (1 - a_i) v_i w_i \langle S^{-1} \pi_{W_i} x, \pi_{V_i} x \rangle. \end{aligned} \tag{12}$$

Proof. We define a bounded linear operators

$$Ex = \sum_{i \in \mathbb{I}} a_i v_i w_i \pi_{V_i} S^{-1} \pi_{W_i} x, \quad Fx = \sum_{i \in \mathbb{I}} (1 - a_i) v_i w_i \pi_{V_i} S^{-1} \pi_{W_i} x.$$

Clearly, $E + F = I_{\mathcal{H}}$. By lemma 2, we have

$$\begin{aligned} &\langle E^* E x, x \rangle + \lambda \langle (F^* + F)x, x \rangle \\ &= \langle E^* E x, x \rangle + \lambda \overline{\langle Fx, x \rangle} + \lambda \langle Fx, x \rangle \end{aligned} \tag{13}$$

$$\begin{aligned} &= \langle F^* F x, x \rangle + (1 - \lambda) \langle (E^* + E)x, x \rangle + (2\lambda - 1) \langle x, x \rangle \\ &= \langle F^* F x, x \rangle + (1 - \lambda) (\overline{\langle Ex, x \rangle} + \langle Ex, x \rangle) + (2\lambda - 1) \langle I_{\mathcal{H}} x, x \rangle. \end{aligned} \tag{14}$$

Taking real part of (15) and (16), we have

$$|Ex|^2 + 2\lambda \operatorname{Re} \langle Fx, x \rangle = |Fx|^2 + 2(1 - \lambda) \operatorname{Re} \langle Ex, x \rangle + (2\lambda - 1) \operatorname{Re} \langle I_{\mathcal{H}} x, x \rangle.$$

Thus,

$$|Ex|^2 = |Fx|^2 + 2(1 - \lambda) \operatorname{Re} \langle Ex, x \rangle - 2\lambda \operatorname{Re} \langle Fx, x \rangle + (2\lambda - 1) \operatorname{Re} \langle I_{\mathcal{H}} x, x \rangle$$

$$\begin{aligned}
 &= |Fx|^2 + 2\operatorname{Re} \langle Ex, x \rangle - 2\lambda \operatorname{Re} \langle (E + F)x, x \rangle + (2\lambda - 1)\operatorname{Re} \langle I_{\mathcal{H}}x, x \rangle \\
 &= |Fx|^2 + 2\operatorname{Re} \langle Ex, x \rangle - \operatorname{Re} \langle I_{\mathcal{H}}x, x \rangle \\
 &= |Fx|^2 + 2\operatorname{Re} \langle Ex, x \rangle - \operatorname{Re} \langle (E + F)x, x \rangle \\
 &= |Fx|^2 + \operatorname{Re} \langle Ex, x \rangle - \operatorname{Re} \langle Fx, x \rangle.
 \end{aligned}$$

Hence,

$$|Ex|^2 + \operatorname{Re} \langle Fx, x \rangle = |Fx|^2 + \operatorname{Re} \langle Ex, x \rangle. \tag{15}$$

By (15), we have

$$\begin{aligned}
 &\left| \sum_{i \in \mathbb{I}} a_i v_i w_i \pi_{V_i} S^{-1} \pi_{W_i} x \right|^2 + \operatorname{Re} \sum_{i \in \mathbb{I}} (1 - a_i) v_i w_i \langle S^{-1} \pi_{W_i} x, \pi_{V_i} x \rangle \\
 &= \left| \sum_{i \in \mathbb{I}} a_i v_i w_i \pi_{V_i} S^{-1} \pi_{W_i} x \right|^2 + \operatorname{Re} \left\langle \sum_{i \in \mathbb{I}} (1 - a_i) v_i w_i \pi_{V_i} S^{-1} \pi_{W_i} x, x \right\rangle \\
 &= |Ex|^2 + \operatorname{Re} \langle Fx, x \rangle = |Fx|^2 + \operatorname{Re} \langle Ex, x \rangle \\
 &= \left| \sum_{i \in \mathbb{I}} a_i v_i w_i \pi_{V_i} S^{-1} \pi_{W_i} x \right|^2 + \operatorname{Re} \left\langle \sum_{i \in \mathbb{I}} (1 - a_i) v_i w_i \pi_{V_i} S^{-1} \pi_{W_i} x, x \right\rangle \\
 &= \left| \sum_{i \in \mathbb{I}} (1 - a_i) v_i w_i \pi_{V_i} S^{-1} \pi_{W_i} x \right|^2 + \operatorname{Re} \sum_{i \in \mathbb{I}} a_i v_i w_i \langle S^{-1} \pi_{W_i} x, \pi_{V_i} x \rangle.
 \end{aligned}$$

We now proof the inequality of (12). By Lemma 2, we have

$$\langle E^* Ex, x \rangle + \lambda \overline{\langle Fx, x \rangle} + \lambda \langle Fx, x \rangle \geq (2\lambda - \lambda^2) \langle I_{\mathcal{H}}x, x \rangle. \tag{16}$$

Taking real part of (16), we obtain

$$|Ex|^2 + 2\lambda \operatorname{Re} \langle Fx, x \rangle \geq (2\lambda - \lambda^2) \operatorname{Re} \langle I_{\mathcal{H}}x, x \rangle,$$

then

$$\begin{aligned}
 |Ex|^2 &\geq (2\lambda - \lambda^2) \operatorname{Re} \langle I_{\mathcal{H}}x, x \rangle - 2\lambda \operatorname{Re} \langle Fx, x \rangle \\
 &= (2\lambda - \lambda^2) \operatorname{Re} \langle (E + F)x, x \rangle - 2\lambda \operatorname{Re} \langle Fx, x \rangle \\
 &= (2\lambda - \lambda^2) \operatorname{Re} \langle Ex, x \rangle - \lambda^2 \operatorname{Re} \langle Fx, x \rangle \\
 &= (2\lambda - \lambda^2) \operatorname{Re} \langle Ex, x \rangle + (1 - \lambda^2) \operatorname{Re} \langle Fx, x \rangle - \operatorname{Re} \langle Fx, x \rangle.
 \end{aligned}$$

Hence

$$|Ex|^2 + \langle Fx, x \rangle \geq (2\lambda - \lambda^2) \operatorname{Re} \langle Ex, x \rangle + (1 - \lambda^2) \operatorname{Re} \langle Fx, x \rangle.$$

The proof is completed. \square

In Theorem 4, if we take $\mathbb{J} \subset \mathbb{I}$ and

$$a_i = \begin{cases} 1, & i \in \mathbb{J} \\ 0, & i \in \mathbb{J}^c. \end{cases}$$

Theorem 4 can lead to a direct consequence as follows.

COROLLARY 4. Let $\{(W_i, w_i)\}_{i \in \mathbb{I}}$ be a fusion frame for \mathcal{H} with the fusion frame operator S and let $\{(V_i, v_i)\}_{i \in \mathbb{I}}$ be the alternate dual fusion frame of $\{(W_i, w_i)\}_{i \in \mathbb{I}}$. If we set $\langle x, x \rangle = |x|^2$ for all $x \in \mathcal{H}$, then for any $\lambda \in [0, 1]$, for all bounded sequence $\{a_i\}_{i \in \mathbb{I}}$ and any $x \in \mathcal{H}$, we have

$$\begin{aligned} & \operatorname{Re} \sum_{i \in \mathbb{J}} v_i w_i \langle S^{-1} \pi_{W_i} x, \pi_{V_i} x \rangle + \left| \sum_{i \in \mathbb{J}^c} v_i w_i \pi_{V_i} S^{-1} \pi_{W_i} x \right|^2 \\ &= \operatorname{Re} \sum_{i \in \mathbb{J}^c} v_i w_i \langle S^{-1} \pi_{W_i} x, \pi_{V_i} x \rangle + \left| \sum_{i \in \mathbb{J}} v_i w_i \pi_{V_i} S^{-1} \pi_{W_i} x \right|^2 \\ &\geq (2\lambda - \lambda^2) \operatorname{Re} \sum_{i \in \mathbb{J}} v_i w_i \langle S^{-1} \pi_{W_i} x, \pi_{V_i} x \rangle + (1 - \lambda^2) \operatorname{Re} \sum_{i \in \mathbb{J}^c} v_i w_i \langle S^{-1} \pi_{W_i} x, \pi_{V_i} x \rangle. \end{aligned}$$

REMARK 2. If we take $\lambda = \frac{1}{2}$ in Corollary 4, then we can obtain the inequalities in Theorem 2.

In [21] the author presented some inequalities for g-frames in Hilbert C^* -modules. Next, we will generalize the version of fusion frame in Hilbert C^* -modules for Theorem 2.4 in [21].

Now, we consider scalar $\lambda \in [1, 2]$ and give some exciting inequalities for fusion frames in Hilbert C^* -modules. We first give a simple lemma.

LEMMA 3. If $P, Q \in \operatorname{End}_{\mathcal{H}}^*(\mathcal{H})$ are self-adjoint and positive operators satisfying $P + Q = I_{\mathcal{H}}$, then for any $\lambda \in [1, 2]$ and all $x \in \mathcal{H}$ we have

$$\langle P^2 x, x \rangle \leq \lambda \langle P x, x \rangle, \quad \langle Q^2 x, x \rangle \leq \lambda \langle Q x, x \rangle.$$

Proof. Since P and Q are positive operators, we have

$$0 \leq PQ = P(I_{\mathcal{H}} - P) = P - P^2.$$

Then, for any $\lambda \in [1, 2]$ and any $x \in \mathcal{H}$ we obtain

$$\begin{aligned} \langle P^2 x, x \rangle + \lambda \langle Q x, x \rangle &\leq \langle P x, x \rangle + \lambda \langle (I_{\mathcal{H}} - P) x, x \rangle \\ &= (1 - \lambda) \langle P x, x \rangle + \lambda \langle x, x \rangle \\ &\leq \lambda \langle x, x \rangle, \end{aligned}$$

it follows that

$$\langle P^2 x, x \rangle \leq \lambda \langle x, x \rangle - \lambda \langle Q x, x \rangle = \lambda \langle P x, x \rangle.$$

Similarly, we can obtain $\langle Q^2 x, x \rangle \leq \lambda \langle Q x, x \rangle$. \square

THEOREM 5. Let $\{(W_i, w_i)\}_{i \in \mathbb{I}}$ be a fusion frame for \mathcal{H} with the fusion frame operator S . Then for any $\lambda \in [1, 2]$, for all $\mathbb{J} \subset \mathbb{I}$ and any $x \in \mathcal{H}$, we have

$$\begin{aligned} 0 &\leq \sum_{i \in \mathbb{J}} w_i^2 \langle \pi_{W_i} x, \pi_{W_i} x \rangle - \sum_{i \in \mathbb{I}} w_i^2 \langle \pi_{W_i} S^{-1} S_{\mathbb{J}} x, \pi_{W_i} S^{-1} S_{\mathbb{J}} x \rangle \\ &\leq (\lambda - 1) \sum_{i \in \mathbb{J}^c} w_i^2 \langle \pi_{W_i} x, \pi_{W_i} x \rangle + (1 - \frac{\lambda}{2})^2 \sum_{i \in \mathbb{I}} \langle \pi_{W_i} x, \pi_{W_i} x \rangle. \end{aligned} \tag{17}$$

Proof. As mentioned in the proof of Theorem 3, we have $S_{\mathbb{J}} - S_{\mathbb{J}}S^{-1}S_{\mathbb{J}} \geq 0$, thus, for all $x \in \mathcal{H}$ we have

$$\begin{aligned} \sum_{i \in \mathbb{J}} w_i^2 \langle \pi_{W_i}x, \pi_{W_i}x \rangle - \sum_{i \in \mathbb{I}} w_i^2 \langle \pi_{W_i}S^{-1}S_{\mathbb{J}}x, \pi_{W_i}S^{-1}S_{\mathbb{J}}x \rangle &= \langle S_{\mathbb{J}}x, x \rangle - \langle SS^{-1}S_{\mathbb{J}}x, S^{-1}S_{\mathbb{J}}x \rangle \\ &= \langle S_{\mathbb{J}}x, x \rangle - \langle S^{-1}S_{\mathbb{J}}x, S_{\mathbb{J}}x \rangle \\ &= \langle S_{\mathbb{J}}x, x \rangle - \langle S_{\mathbb{J}}S^{-1}S_{\mathbb{J}}x, x \rangle \\ &\geq 0. \end{aligned} \tag{18}$$

On the other hand, by (11) we have

$$\begin{aligned} &\sum_{i \in \mathbb{J}} w_i^2 \langle \pi_{W_i}x, \pi_{W_i}x \rangle - \sum_{i \in \mathbb{I}} w_i^2 \langle \pi_{W_i}S^{-1}S_{\mathbb{J}}x, \pi_{W_i}S^{-1}S_{\mathbb{J}}x \rangle \\ &= \langle S_{\mathbb{J}}x, x \rangle - \langle SS^{-1}S_{\mathbb{J}}x, S^{-1}S_{\mathbb{J}}x \rangle = \langle S_{\mathbb{J}}x, x \rangle - \langle S^{-1}S_{\mathbb{J}}x, S_{\mathbb{J}}x \rangle \\ &\leq \langle S_{\mathbb{J}}x, x \rangle - \lambda \langle S_{\mathbb{J}}x, x \rangle + \frac{\lambda^2}{4} \langle Sx, x \rangle \\ &= (1 - \lambda) \langle S_{\mathbb{J}}x, x \rangle + \frac{\lambda^2}{4} \langle Sx, x \rangle = (1 - \lambda) \langle (S - S_{\mathbb{J}^c})x, x \rangle + \frac{\lambda^2}{4} \langle Sx, x \rangle \\ &= (\lambda - 1) \langle S_{\mathbb{J}^c}x, x \rangle + (1 - \frac{\lambda}{2})^2 \langle Sx, x \rangle \\ &= (\lambda - 1) \sum_{i \in \mathbb{J}^c} w_i^2 \langle \pi_{W_i}x, \pi_{W_i}x \rangle + (1 - \frac{\lambda}{2})^2 \sum_{i \in \mathbb{I}} \langle \pi_{W_i}x, \pi_{W_i}x \rangle. \end{aligned} \tag{19}$$

From (18) and (19), the conclusion holds. \square

THEOREM 6. *Let $\{(W_i, w_i)\}_{i \in I}$ be a fusion frame for \mathcal{H} with the fusion frame operator S . Then for any $\lambda \in [1, 2]$, for all $\mathbb{J} \subset \mathbb{I}$ and any $x \in \mathcal{H}$, we have*

$$\begin{aligned} &(2\lambda - \frac{\lambda^2}{2} - 1) \sum_{i \in \mathbb{J}} w_i^2 \langle \pi_{W_i}x, \pi_{W_i}x \rangle + (1 - \frac{\lambda^2}{2}) \sum_{i \in \mathbb{J}^c} w_i^2 \langle \pi_{W_i}x, \pi_{W_i}x \rangle \\ &\leq \sum_{i \in \mathbb{I}} w_i^2 \langle \pi_{W_i}S^{-1}S_{\mathbb{J}}x, \pi_{W_i}S^{-1}S_{\mathbb{J}}x \rangle + \sum_{i \in \mathbb{I}} w_i^2 \langle \pi_{W_i}S^{-1}S_{\mathbb{J}}x, \pi_{W_i}S^{-1}S_{\mathbb{J}^c}x \rangle \\ &\leq \lambda \sum_{i \in \mathbb{I}} w_i^2 \langle \pi_{W_i}x, \pi_{W_i}x \rangle. \end{aligned} \tag{20}$$

Proof. As mentioned in the proof of Theorem 3, from (7) and (11), we have

$$\langle S^{-1}S_{\mathbb{J}^c}x, S_{\mathbb{J}^c}x \rangle + \langle S_{\mathbb{J}}x, x \rangle \geq (\lambda - \frac{\lambda^2}{4}) \langle S_{\mathbb{J}}x, x \rangle + (1 - \frac{\lambda^2}{4}) \langle S_{\mathbb{J}^c}x, x \rangle,$$

and then

$$\langle S^{-1}S_{\mathbb{J}^c}x, S_{\mathbb{J}^c}x \rangle \geq (\lambda - \frac{\lambda^2}{4} - 1) \langle S_{\mathbb{J}}x, x \rangle + (1 - \frac{\lambda^2}{4}) \langle S_{\mathbb{J}^c}x, x \rangle. \tag{21}$$

By Using (11) and (21), we obtain

$$\sum_{i \in \mathbb{I}} w_i^2 \langle \pi_{W_i}S^{-1}S_{\mathbb{J}}x, \pi_{W_i}S^{-1}S_{\mathbb{J}}x \rangle + \sum_{i \in \mathbb{I}} w_i^2 \langle \pi_{W_i}S^{-1}S_{\mathbb{J}}x, \pi_{W_i}S^{-1}S_{\mathbb{J}^c}x \rangle$$

$$\begin{aligned}
&= \langle SS^{-1}S_{\mathbb{J}}x, S^{-1}S_{\mathbb{J}}x \rangle + \langle SS^{-1}S_{\mathbb{J}^c}x, S^{-1}S_{\mathbb{J}^c}x \rangle = \langle S^{-1}S_{\mathbb{J}}x, S_{\mathbb{J}}x \rangle + \langle S^{-1}S_{\mathbb{J}^c}x, S_{\mathbb{J}^c}x \rangle \\
&\geq \lambda \langle S_{\mathbb{J}}x, x \rangle - \frac{\lambda^2}{4} \langle Sx, x \rangle + \left(\lambda - \frac{\lambda^2}{4} - 1\right) \langle S_{\mathbb{J}}x, x \rangle + \left(1 - \frac{\lambda^2}{4}\right) \langle S_{\mathbb{J}^c}x, x \rangle \\
&= \left(2\lambda - \frac{\lambda^2}{2} - 1\right) \langle S_{\mathbb{J}}x, x \rangle + \left(1 - \frac{\lambda^2}{2}\right) \langle S_{\mathbb{J}^c}x, x \rangle \\
&= \left(2\lambda - \frac{\lambda^2}{2} - 1\right) \sum_{i \in \mathbb{J}} w_i^2 \langle \pi_{W_i}x, \pi_{W_i}x \rangle + \left(1 - \frac{\lambda^2}{2}\right) \sum_{i \in \mathbb{J}^c} w_i^2 \langle \pi_{W_i}x, \pi_{W_i}x \rangle.
\end{aligned}$$

Next, we proof the last inequality of (20). Since $P = S^{-1/2}S_{\mathbb{J}}S^{-1/2}$, $Q = S^{-1/2}S_{\mathbb{J}^c}S^{-1/2}$ are positive and self-adjoint operators, by Lemma 3, we have

$$\begin{aligned}
&\sum_{i \in \mathbb{I}} w_i^2 \langle \pi_{W_i}S^{-1}S_{\mathbb{J}}x, \pi_{W_i}S^{-1}S_{\mathbb{J}}x \rangle + \sum_{i \in \mathbb{I}} w_i^2 \langle \pi_{W_i}S^{-1}S_{\mathbb{J}^c}x, \pi_{W_i}S^{-1}S_{\mathbb{J}^c}x \rangle \\
&= \langle SS^{-1}S_{\mathbb{J}}x, S^{-1}S_{\mathbb{J}}x \rangle + \langle SS^{-1}S_{\mathbb{J}^c}x, S^{-1}S_{\mathbb{J}^c}x \rangle = \langle S^{-1}S_{\mathbb{J}}x, S_{\mathbb{J}}x \rangle + \langle S^{-1}S_{\mathbb{J}^c}x, S_{\mathbb{J}^c}x \rangle \\
&= \langle S^{-1/2}S_{\mathbb{J}}x, S^{-1/2}S_{\mathbb{J}}x \rangle + \langle S^{-1/2}S_{\mathbb{J}^c}x, S^{-1/2}S_{\mathbb{J}^c}x \rangle \\
&= \left\langle S^{-1/2}S_{\mathbb{J}}S^{-1/2}(S^{1/2}x), S^{-1/2}S_{\mathbb{J}}S^{-1/2}(S^{1/2}x) \right\rangle \\
&\quad + \left\langle S^{-1/2}S_{\mathbb{J}^c}S^{-1/2}(S^{1/2}x), S^{-1/2}S_{\mathbb{J}^c}S^{-1/2}(S^{1/2}x) \right\rangle \\
&\leq \lambda \left\langle S^{-1/2}S_{\mathbb{J}}S^{-1/2}(S^{1/2}x), (S^{1/2}x) \right\rangle + \lambda \left\langle S^{-1/2}S_{\mathbb{J}^c}S^{-1/2}(S^{1/2}x), (S^{1/2}x) \right\rangle \\
&= \lambda \langle S_{\mathbb{J}}x, x \rangle + \lambda \langle S_{\mathbb{J}^c}f, f \rangle = \lambda \langle (S_{\mathbb{J}} + S_{\mathbb{J}^c})x, x \rangle \\
&= \lambda \langle Sx, x \rangle = \lambda \sum_{i \in \mathbb{I}} w_i^2 \langle \pi_{W_i}x, \pi_{W_i}x \rangle.
\end{aligned}$$

The proof is completed. \square

In case of Parseval fusion frames, we immediately get the following result.

COROLLARY 5. *Let $\{(W_i, w_i)\}_{i \in \mathbb{I}}$ be a Parseval fusion frame for \mathcal{H} . Then for any $\lambda \in [1, 2]$, for all $\mathbb{J} \subset \mathbb{I}$ and any $x \in \mathcal{H}$, we have*

$$\begin{aligned}
0 &\leq \sum_{i \in \mathbb{J}} w_i^2 \langle \pi_{W_i}x, \pi_{W_i}x \rangle - \sum_{i \in \mathbb{I}} w_i^2 \langle \pi_{W_i}S_{\mathbb{J}}x, \pi_{W_i}S_{\mathbb{J}}x \rangle \\
&\leq (\lambda - 1) \sum_{i \in \mathbb{J}^c} w_i^2 \langle \pi_{W_i}x, \pi_{W_i}x \rangle + \left(1 - \frac{\lambda}{2}\right)^2 \sum_{i \in \mathbb{I}} w_i^2 \langle \pi_{W_i}x, \pi_{W_i}x \rangle,
\end{aligned}$$

and

$$\begin{aligned}
&\left(2\lambda - \frac{\lambda^2}{2} - 1\right) \sum_{i \in \mathbb{J}} w_i^2 \langle \pi_{W_i}x, \pi_{W_i}x \rangle + \left(1 - \frac{\lambda^2}{2}\right) \sum_{i \in \mathbb{J}^c} w_i^2 \langle \pi_{W_i}x, \pi_{W_i}x \rangle \\
&\leq \sum_{i \in \mathbb{I}} w_i^2 \langle \pi_{W_i}S_{\mathbb{J}}x, \pi_{W_i}S_{\mathbb{J}}x \rangle + \sum_{i \in \mathbb{I}} w_i^2 \langle \pi_{W_i}S_{\mathbb{J}^c}x, \pi_{W_i}S_{\mathbb{J}^c}x \rangle \leq \lambda \langle x, x \rangle.
\end{aligned}$$

In Corollary 5, if we take $\langle x, x \rangle = |x|^2$ for all $x \in \mathcal{H}$, then we have the following result.

COROLLARY 6. Let $\{(W_i, w_i)\}_{i \in I}$ be a C -tight fusion frame for \mathcal{H} . If we set $\langle x, x \rangle = |x|^2$ for all $x \in \mathcal{H}$, then for any $\lambda \in [1, 2]$, for all $\mathbb{J} \subset \mathbb{I}$ and any $x \in \mathcal{H}$, we have

$$\begin{aligned} 0 &\leq \sum_{i \in \mathbb{J}} w_i^2 |\pi_{W_i x}|^2 - \frac{1}{C} \left| \sum_{i \in \mathbb{J}} w_i^2 \langle \pi_{W_i x}, \pi_{W_i x} \rangle \right|^2 \\ &\leq (\lambda - 1) \sum_{i \in \mathbb{J}^c} w_i^2 |\pi_{W_i x}|^2 + C \left(1 - \frac{\lambda}{2}\right)^2 |x|^2, \end{aligned}$$

and

$$\begin{aligned} &(2\lambda - \frac{\lambda^2}{2} - 1) \sum_{i \in \mathbb{J}} w_i^2 |\pi_{W_i x}|^2 + (1 - \frac{\lambda^2}{2}) \sum_{i \in \mathbb{J}^c} w_i^2 |\pi_{W_i x}|^2 \\ &\leq \frac{1}{C} \left| \sum_{i \in \mathbb{J}} w_i^2 \langle \pi_{W_i x}, \pi_{W_i x} \rangle \right|^2 + \frac{1}{C} \left| \sum_{i \in \mathbb{J}^c} w_i^2 \langle \pi_{W_i x}, \pi_{W_i x} \rangle \right|^2 \leq \lambda |x|^2. \end{aligned}$$

REMARK 3. If we take $\lambda = 1$ in Theorem 5 and Theorem 6, we can obtain the inequalities of the version of fusion frame in Hilbert C^* -modules for Theorem 2.4 in [21].

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