

SOME OSTROWSKI TYPE INEQUALITIES FOR p -CONVEX FUNCTIONS VIA GENERALIZED FRACTIONAL INTEGRALS

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Abstract. In this paper, some new Ostrowski type inequalities for generalized fractional integrals are obtained. An identity via generalized fractional integrals and differentiable mappings, together with a new concept are used.

1. Introduction

During the last decades fractional calculus, differential equations and inequalities have been studied extensively. As a matter of fact, fractional derivatives and integrals provide a more excellent tool for the description of memory and hereditary properties of various materials and processes than integer derivatives. Engineers and scientists have developed new precisely models which involved fractional equations and inequalities. These models have been applied successfully, e.g., in physics, biomathematics, blood flow phenomena, ecology, environmental issues, viscoelasticity, aerodynamics, electro-dynamics of complex medium, electrical circuits, electron-analytical chemistry, control theory, etc. For a systematic development of the topic, we refer the books [1]-[7]. As an important issue for the theory of fractional differential equations, the existence of solutions to kinds of initial and boundary value problems has attracted many scholars attention, and lots of excellent results have been obtained by means of fixed point theorems, upper and lower solutions technique, differential and integral inequalities and so forth. Fractional differential/integral inequalities are an important tool to investigate properties of solutions of various fractional problems, such as existence, uniqueness, boundedness, stability, asymptotic behavior, and oscillation etc. A variety of results on initial and boundary value problems of fractional differential equations and inclusions can easily be found in the literature on the topic. For some recent results, we can refer to [8]-[16] and references cited therein.

In recent years, Ostrowski type inequalities and Hermite-Hadamard type inequalities were studied extensively by many researchers and numerous generalizations, extensions and variants of them appeared in a number of papers [21]-[26] and references therein.

In [24] Set established a new fractional integral identity via differentiable mappings and Riemann-Liouville fractional integrals.

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LEMMA 1.1. Let $f : [a, b] \rightarrow \mathbb{R}$ be a differentiable mapping on (a, b) with $a < b$. If $f' \in L[a, b]$, then the following equality for fractional integrals holds

$$\begin{aligned} & \frac{(x-a)^\alpha + (b-x)^\alpha}{b-a} f(x) - \frac{\Gamma(\alpha+1)}{b-a} [{}_R L J_{b-}^\alpha f(a) + {}_R L J_{a+}^\alpha f(b)] \\ &= \frac{(x-a)^{\alpha+1}}{b-a} \int_0^1 t^\alpha f'(tx + (1-t)a) dt - \frac{(b-x)^{\alpha+1}}{b-a} \int_0^1 t^\alpha f'(tx + (1-t)b) dt, \end{aligned}$$

where ${}_R L J_{a+}^\alpha f$ and ${}_R L J_{b-}^\alpha f$ denote the left-sided and right-sided Riemann-Liouville fractional integrals of order $\alpha \in \mathbb{R}^+$ defined by

$$({}_R L J_{a+}^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \quad 0 \leq a < x \leq b,$$

and

$$({}_R L J_{b-}^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt, \quad 0 \leq a \leq x < b,$$

respectively.

By using the above established integral identity in Lemma 1.1 via s -convex mappings in the second sense, Set [24] established many Ostrowski type inequalities for Riemann-Liouville fractional integrals, which generalized the classical Ostrowski inequality (see [27] or [28]).

Recently, Wang *et al.* in [26], established a new identity for Hadamard fractional integrals, which is similar to the identity of Lemma 1.1 for Riemann-Liouville fractional integrals.

LEMMA 1.2. Let $f : [a, b] \rightarrow \mathbb{R}$ be a differentiable mapping on (a, b) with $0 < a < b$. If $f' \in L[a, b]$, then the following equality for fractional integrals holds

$$\begin{aligned} & \frac{(\log x - \log a)^\alpha + (\log b - \log x)^\alpha}{\log b - \log a} f(x) - \frac{\Gamma(\alpha+1)}{\log b - \log a} [{}_H J_{b-}^\alpha f(a) + {}_H J_{a+}^\alpha f(b)] \\ &= \frac{(\log x - \log a)^{\alpha+1}}{\log b - \log a} \int_0^1 t^\alpha e^{t \log x + (1-t) \log a} f'(e^{t \log x + (1-t) \log a}) dt \\ & \quad - \frac{(\log b - \log x)^{\alpha+1}}{\log b - \log a} \int_0^1 t^\alpha e^{t \log x + (1-t) \log b} f'(e^{t \log x + (1-t) \log b}) dt, \end{aligned}$$

where ${}_H J_{a+}^\alpha f$ and ${}_H J_{b-}^\alpha f$ denote the left-sided and right-sided Hadamard fractional integrals of order $\alpha \in \mathbb{R}^+$ defined by

$$({}_H J_{a+}^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left(\log \frac{x}{t}\right)^{\alpha-1} f(t) \frac{dt}{t}, \quad 0 < a < x \leq b,$$

and

$$({}_H J_{b-}^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \left(\log \frac{t}{x}\right)^{\alpha-1} f(t) \frac{dt}{t}, \quad 0 < a \leq x < b,$$

respectively.

Some new Ostrowski type inequalities for Hadamard fractional integrals were established in [26], by using the new concept “s-e-condition” defined as follows:

DEFINITION 1.1. A function $f : I \subset (0, \infty) \rightarrow \mathbb{R}$ is said to satisfy s-e-condition if

$$f(e^{\lambda x + (1-\lambda)y}) \leq \lambda^s f(e^x) + (1-\lambda)^s f(e^y),$$

for all $x, y \in I$, $\lambda \in [0, 1]$ and for some fixed $s \in (0, 1]$.

If $f : I \subset (0, \infty) \rightarrow \mathbb{R}$ is a nondecreasing and convex function, then f satisfies the above s-e-condition. We say that $f : [a, b] \rightarrow \mathbb{R}$ is convex if it satisfies the following inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

where $t \in [0, 1]$ and $x, y \in [a, b]$.

In [18] Iscan gave the definition of p -convex function as follows:

DEFINITION 1.2. Let $I \subset (0, \infty)$ be a real interval and $p \in \mathbb{R} \setminus \{0\}$. A function $f : I \rightarrow \mathbb{R}$ is said to be p -convex, if

$$f\left([tx^p + (1-t)y^p]^{1/p}\right) \leq tf(x) + (1-t)f(y), \quad (1.1)$$

for all $x, y \in I$ and $t \in [0, 1]$. If the inequality (1.1) is reversed, then f is said to be p -concave.

According to Definition 1.2, it can be easily seen that for $p = 1$ and $p = -1$, p -convexity reduces to ordinary convexity and harmonically convexity of functions defined on $I \subset (0, \infty)$, respectively. For some results related to p -convex functions and its generalizations, we refer the reader to [19, 20] and references cited therein.

In this paper, by using Definition 1.2, we will establish a new identity for generalized fractional integrals (known as *Katugampola integral*), introduced in [17], which is similar to the identities in Lemmas 1.1 and 1.2 for Riemann-Liouville and Hadamard fractional integrals.

DEFINITION 1.3. [17] The left-sided and right-sided generalized Riemann-Liouville fractional integrals of order $\alpha \in \mathbb{R}^+$ of a function f are defined by

$$({}^\rho I_{a+}^\alpha f)(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^x \frac{\tau^{\rho-1} f(\tau)}{(x^\rho - \tau^\rho)^{1-\alpha}} d\tau, \quad x > a, \quad \alpha > 0, \quad \rho > 0,$$

and

$$({}^\rho I_{b-}^\alpha f)(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_x^b \frac{\tau^{\rho-1} f(\tau)}{(\tau^\rho - x^\rho)^{1-\alpha}} d\tau, \quad x < b, \quad \alpha > 0, \quad \rho > 0.$$

REMARK 1.1. From the above definition it follows that when $p = 1$ we arrive at the standard Riemann-Liouville fractional integral, which is used to define both the Riemann-Liouville and Caputo fractional derivatives, while when $p \rightarrow 0$ we have

$$\lim_{\rho \rightarrow 0} {}^{\rho}I^q f(t) = \frac{1}{\Gamma(q)} \int_0^t \left(\log \frac{t}{s}\right)^{q-1} \frac{f(s)}{s} ds,$$

which is the famous Hadamard fractional integral. See [17].

2. Identity via generalized fractional integrals and differentiable mappings

LEMMA 2.1. Let $f : [a, b] \rightarrow \mathbb{R}$ be a differentiable mapping on (a, b) with $a < b$ and $p \in \mathbb{R} \setminus \{0\}$. If $f' \in L[a, b]$, then the following equality holds:

$$\begin{aligned} & \frac{(x^p - a^p)^{\alpha+1}}{p^{1+\alpha}(b-a)} \int_0^1 t^\alpha (ta^p + (1-t)x^p)^{\frac{1-p}{p}} f' \left(\sqrt[p]{ta^p + (1-t)x^p} \right) dt \\ & - \frac{(b^p - x^p)^{\alpha+1}}{p^{1+\alpha}(b-a)} \int_0^1 t^\alpha (tb^p + (1-t)x^p)^{\frac{1-p}{p}} f' \left(\sqrt[p]{tb^p + (1-t)x^p} \right) dt \\ & = - \frac{(x^p - a^p)^\alpha f(a) + (b^p - x^p)^\alpha f(b)}{p^\alpha(b-a)} + \frac{\Gamma(\alpha + 1)}{b-a} \left[({}^pI_{a+}^\alpha f)(x) + ({}^pI_{b-}^\alpha f)(x) \right]. \end{aligned}$$

Proof. By integrating by parts, we have

$$\begin{aligned} & \int_0^1 t^\alpha (ta^p + (1-t)x^p)^{\frac{1-p}{p}} f' \left(\sqrt[p]{ta^p + (1-t)x^p} \right) dt \\ & = \left[\frac{pt^\alpha f \left(\sqrt[p]{ta^p + (1-t)x^p} \right)}{a^p - x^p} \right]_0^1 - \int_0^1 \frac{p\alpha t^{\alpha-1} f \left(\sqrt[p]{ta^p + (1-t)x^p} \right)}{a^p - x^p} dt \\ & = \frac{pf(a)}{a^p - x^p} - \int_0^1 \frac{p\alpha t^{\alpha-1} f \left(\sqrt[p]{ta^p + (1-t)x^p} \right)}{a^p - x^p} dt \\ & = \frac{pf(a)}{a^p - x^p} - \frac{p^2 \alpha}{x^p - a^p} \int_x^a \left(\frac{x^p - u^p}{x^p - a^p} \right)^{\alpha-1} f(u) \frac{u^{p-1}}{x^p - a^p} du \quad \left[u = \sqrt[p]{ta^p + (1-t)x^p} \right] \\ & = \frac{pf(a)}{a^p - x^p} + \frac{p^2 \alpha}{(x^p - a^p)^{\alpha+1}} \int_a^x (x^p - u^p)^{\alpha-1} f(u) u^{p-1} du \\ & = \frac{pf(a)}{a^p - x^p} + \frac{p^2 \alpha \Gamma(\alpha)}{p^{1-\alpha} (x^p - a^p)^{\alpha+1} \Gamma(\alpha)} \int_a^x u^{p-1} (x^p - u^p)^{\alpha-1} f(u) du \\ & = \frac{pf(a)}{a^p - x^p} + \frac{p^{1+\alpha} \Gamma(\alpha + 1)}{(x^p - a^p)^{\alpha+1} \Gamma(\alpha)} \int_a^x u^{p-1} (x^p - u^p)^{\alpha-1} f(u) du \\ & = \frac{pf(a)}{a^p - x^p} + \frac{p^{1+\alpha} \Gamma(\alpha + 1)}{(x^p - a^p)^{\alpha+1}} ({}^pI_{a+}^\alpha f)(x). \tag{2.1} \end{aligned}$$

By a similar way we find

$$\begin{aligned}
 & \int_0^1 t^\alpha (tb^p + (1-t)x^p)^{\frac{1-p}{p}} f' \left(\sqrt[p]{tb^p + (1-t)x^p} \right) dt \\
 &= \frac{pf(b)}{b^p - x^p} - \frac{p^2\alpha}{b^p - x^p} \int_x^b \left(\frac{u^p - x^p}{b^p - x^p} \right)^{\alpha-1} f(u) \frac{u^{p-1}}{b^p - x^p} du \quad \left[u = \sqrt[p]{tb^p + (1-t)x^p} \right] \\
 &= \frac{pf(b)}{b^p - x^p} - \frac{p^2\alpha}{(b^p - x^p)^{\alpha+1}} \int_x^b (u^p - x^p)^{\alpha-1} f(u) u^{p-1} du \\
 &= \frac{pf(b)}{b^p - x^p} - \frac{p^2\alpha\Gamma(\alpha)}{p^{1-\alpha}(b^p - x^p)^{\alpha+1}} \frac{p^{1-\alpha}}{\Gamma(\alpha)} \int_x^b u^{p-1} (u^p - x^p)^{\alpha-1} f(u) du \\
 &= \frac{pf(b)}{b^p - x^p} - \frac{p^{1+\alpha}\Gamma(\alpha+1)}{(b^p - x^p)^{\alpha+1}} \frac{p^{1-\alpha}}{\Gamma(\alpha)} \int_x^b u^{p-1} (u^p - x^p)^{\alpha-1} f(u) du \\
 &= \frac{pf(b)}{b^p - x^p} - \frac{p^{1+\alpha}\Gamma(\alpha+1)}{(b^p - x^p)^{\alpha+1}} ({}^pI_{b^-}^\alpha f)(x). \tag{2.2}
 \end{aligned}$$

Multiplying both sides of (2.1) and (2.2) by $\frac{(x^p - a^p)^{\alpha+1}}{p^{1+\alpha}(b-a)}$ and $\frac{(b^p - x^p)^{\alpha+1}}{p^{1+\alpha}(b-a)}$ respectively, we have

$$\begin{aligned}
 & \frac{(x^p - a^p)^{\alpha+1}}{p^{1+\alpha}(b-a)} \int_0^1 t^\alpha (ta^p + (1-t)x^p)^{\frac{1-p}{p}} f' \left(\sqrt[p]{ta^p + (1-t)x^p} \right) dt \\
 &= -\frac{(x^p - a^p)^\alpha f(a)}{p^\alpha(b-a)} + \frac{\Gamma(\alpha+1)}{(b-a)} ({}^pI_{a^+}^\alpha f)(x), \tag{2.3}
 \end{aligned}$$

and

$$\begin{aligned}
 & \frac{(b^p - x^p)^{\alpha+1}}{p^{1+\alpha}(b-a)} \int_0^1 t^\alpha (tb^p + (1-t)x^p)^{\frac{1-p}{p}} f' \left(\sqrt[p]{tb^p + (1-t)x^p} \right) dt \\
 &= \frac{(b^p - x^p)^\alpha f(b)}{p^\alpha(b-a)} - \frac{\Gamma(\alpha+1)}{(b-a)} ({}^pI_{b^-}^\alpha f)(x). \tag{2.4}
 \end{aligned}$$

From (2.3) and (2.4) we have the desired result. The proof is completed. □

3. Ostrowski type inequalities via generalized fractional integrals

THEOREM 3.1. *Let $f : [a, b] \subset (0, \infty) \rightarrow \mathbb{R}$ be a differentiable mapping on (a, b) with $a < b$ such that $f' \in L[a, b]$. In addition, assume that $|f'|$ is p -convex function, $|f'(x)| \leq M, \forall x \in [a, b]$ and $\alpha > 0$.*

(i) *If $p \in (1, \infty)$, then the following inequality holds*

$$\begin{aligned}
 & \left| \frac{(x^p - a^p)^\alpha f(a) + (b^p - x^p)^\alpha f(b)}{p^\alpha(b-a)} - \frac{\Gamma(\alpha+1)}{b-a} [({}^pI_{a^+}^\alpha f)(x) + ({}^pI_{b^-}^\alpha f)(x)] \right| \\
 & \leq \frac{a^{1-p}M}{p^{1+\alpha}(\alpha+1)} \left[\frac{(x^p - a^p)^{\alpha+1} + (b^p - x^p)^{\alpha+1}}{(b-a)} \right], \quad x \in (a, b). \tag{3.1}
 \end{aligned}$$

(ii) If $p \in (-\infty, 0) \cup (0, 1)$, then we have

$$\begin{aligned} & \left| \frac{(x^p - a^p)^\alpha f(a) + (b^p - x^p)^\alpha f(b)}{p^\alpha (b - a)} - \frac{\Gamma(\alpha + 1)}{b - a} [({}^pI_{a+}^\alpha f)(x) + ({}^pI_{b-}^\alpha f)(x)] \right| \\ & \leq \frac{b^{1-p}M}{p^{1+\alpha}(\alpha + 1)} \left[\frac{(x^p - a^p)^{\alpha+1} + (b^p - x^p)^{\alpha+1}}{(b - a)} \right], \quad x \in (a, b). \end{aligned} \tag{3.2}$$

Proof. To prove (i), from Lemma 2.1, we have

$$\begin{aligned} & \left| -\frac{(x^p - a^p)^\alpha f(a) + (b^p - x^p)^\alpha f(b)}{p^\alpha (b - a)} + \frac{\Gamma(\alpha + 1)}{b - a} [({}^pI_{a+}^\alpha f)(x) + ({}^pI_{b-}^\alpha f)(x)] \right| \\ & \leq \frac{(x^p - a^p)^{\alpha+1}}{p^{1+\alpha}(b - a)} \int_0^1 t^\alpha (ta^p + (1 - t)x^p)^{\frac{1-p}{p}} \left| f' \left(\sqrt[p]{ta^p + (1 - t)x^p} \right) \right| dt \\ & \quad + \frac{(b^p - x^p)^{\alpha+1}}{p^{1+\alpha}(b - a)} \int_0^1 t^\alpha (tb^p + (1 - t)x^p)^{\frac{1-p}{p}} \left| f' \left(\sqrt[p]{tb^p + (1 - t)x^p} \right) \right| dt \\ & \leq \frac{(x^p - a^p)^{\alpha+1}}{p^{1+\alpha}(b - a)} \int_0^1 t^\alpha (ta^p + (1 - t)x^p)^{\frac{1-p}{p}} [t|f'(a)| + (1 - t)|f'(x)|] dt \\ & \quad + \frac{(b^p - x^p)^{\alpha+1}}{p^{1+\alpha}(b - a)} \int_0^1 t^\alpha (tb^p + (1 - t)x^p)^{\frac{1-p}{p}} [t|f'(b)| + (1 - t)|f'(x)|] dt. \end{aligned}$$

Since if $p \in (1, \infty)$, we deduce that

$$(tb^p + (1 - t)x^p)^{\frac{1-p}{p}} \leq (ta^p + (1 - t)x^p)^{\frac{1-p}{p}} \leq a^{1-p}, \tag{3.3}$$

which implies that the inequality in (3.1) holds via

$$\int_0^1 t^\alpha dt = \frac{1}{\alpha + 1}.$$

To prove (ii), let $p \in (-\infty, 0) \cup (0, 1)$, then we obtain the requested inequality in (3.2) by applying the fact that

$$(ta^p + (1 - t)x^p)^{\frac{1-p}{p}} \leq (tb^p + (1 - t)x^p)^{\frac{1-p}{p}} \leq b^{1-p}. \tag{3.4}$$

The proof is completed. □

THEOREM 3.2. Let $f : [a, b] \subset (0, \infty) \rightarrow \mathbb{R}$ be a differentiable mapping on (a, b) with $a < b$ such that $f' \in L[a, b]$. Suppose that $|f'|^q$ is p -convex function, $|f'(x)| \leq M, \forall x \in [a, b]$ and given constants $\alpha > 0$ and $r > 1$.

(i) If $p \in (1, \infty)$, then the following inequality is true

$$\begin{aligned} & \left| \frac{(x^p - a^p)^\alpha f(a) + (b^p - x^p)^\alpha f(b)}{p^\alpha (b - a)} - \frac{\Gamma(\alpha + 1)}{b - a} [({}^pI_{a+}^\alpha f)(x) + ({}^pI_{b-}^\alpha f)(x)] \right| \\ & \leq \frac{a^{1-p}M}{p^{1+\alpha}(1 + r\alpha)^{1/r}} \left[\frac{(x^p - a^p)^{\alpha+1} + (b^p - x^p)^{\alpha+1}}{b - a} \right], \quad x \in (a, b). \end{aligned} \tag{3.5}$$

(ii) If $p \in (-\infty, 0) \cup (0, 1)$, then the following inequality is satisfied

$$\begin{aligned} & \left| \frac{(x^p - a^p)^\alpha f(a) + (b^p - x^p)^\alpha f(b)}{p^\alpha(b - a)} - \frac{\Gamma(\alpha + 1)}{b - a} [({}^pI_{a+}^\alpha f)(x) + ({}^pI_{b-}^\alpha f)(x)] \right| \\ & \leq \frac{b^{1-p}M}{p^{1+\alpha}(1+r\alpha)^{1/r}} \left[\frac{(x^p - a^p)^{\alpha+1} + (b^p - x^p)^{\alpha+1}}{b - a} \right], \quad x \in (a, b). \end{aligned} \tag{3.6}$$

Proof. Since $r > 1$, then there exists a constant q such that $1/r + 1/q = 1$. We prove (i), for If $p \in (1, \infty)$. Applying Lemma 2.1, relation (3.3), and using the well known Hölder inequality, we have

$$\begin{aligned} & \left| -\frac{(x^p - a^p)^\alpha f(a) + (b^p - x^p)^\alpha f(b)}{p^\alpha(b - a)} + \frac{\Gamma(\alpha + 1)}{b - a} [({}^pI_{a+}^\alpha f)(x) + ({}^pI_{b-}^\alpha f)(x)] \right| \\ & \leq \frac{(x^p - a^p)^{\alpha+1}}{p^{1+\alpha}(b - a)} \int_0^1 t^\alpha (ta^p + (1-t)x^p)^{\frac{1-p}{p}} \left| f' \left(\sqrt[p]{ta^p + (1-t)x^p} \right) \right| dt \\ & \quad + \frac{(b^p - x^p)^{\alpha+1}}{p^{1+\alpha}(b - a)} \int_0^1 t^\alpha (tb^p + (1-t)x^p)^{\frac{1-p}{p}} \left| f' \left(\sqrt[p]{tb^p + (1-t)x^p} \right) \right| dt \\ & \leq \frac{a^{1-p}(x^p - a^p)^{\alpha+1}}{p^{1+\alpha}(b - a)} \int_0^1 t^\alpha \left| f' \left(\sqrt[p]{ta^p + (1-t)x^p} \right) \right| dt \\ & \quad + \frac{a^{1-p}(b^p - x^p)^{\alpha+1}}{p^{1+\alpha}(b - a)} \int_0^1 t^\alpha \left| f' \left(\sqrt[p]{tb^p + (1-t)x^p} \right) \right| dt \\ & \leq \frac{a^{1-p}(x^p - a^p)^{\alpha+1}}{p^{1+\alpha}(b - a)} \left(\int_0^1 t^{r\alpha} dt \right)^{1/r} \left(\int_0^1 \left| f' \left(\sqrt[p]{ta^p + (1-t)x^p} \right) \right|^q dt \right)^{1/q} \\ & \quad + \frac{a^{1-p}(b^p - x^p)^{\alpha+1}}{p^{1+\alpha}(b - a)} \left(\int_0^1 t^{r\alpha} dt \right)^{1/r} \left(\int_0^1 \left| f' \left(\sqrt[p]{tb^p + (1-t)x^p} \right) \right|^q dt \right)^{1/q}. \end{aligned}$$

Since $|f'|^q$ is p -convex and $|f'(x)| \leq M, \forall x \in [a, b]$, we get

$$\begin{aligned} \int_0^1 \left| f' \left(\sqrt[p]{ta^p + (1-t)x^p} \right) \right|^q dt & \leq \int_0^1 |tf'(a) + (1-t)f'(x)|^q dt \\ & \leq M^q \int_0^1 |t + (1-t)|^q dt \\ & \leq M^q, \end{aligned}$$

and

$$\int_0^1 \left| f' \left(\sqrt[p]{tb^p + (1-t)x^p} \right) \right|^q dt \leq M^q.$$

Then we obtain the inequality in (3.5) using $\int_0^1 t^{r\alpha} dt = \frac{1}{r\alpha + 1}$.

The Case (ii) can be proved by using the above process with relation (3.4). The proof is completed. □

REMARK 3.1. *Theorem 3.2 can be proved also by using the power mean inequality.*

To prove this, by Lemma 2.1, relation (3.3) and applying the well known power mean inequality, we have

$$\begin{aligned} & \left| -\frac{(x^p - a^p)^\alpha f(a) + (b^p - x^p)^\alpha f(b)}{p^\alpha(b - a)} + \frac{\Gamma(\alpha + 1)}{b - a} [({}^pI_{a^+}^\alpha f)(x) + ({}^pI_{b^-}^\alpha f)(x)] \right| \\ & \leq \frac{(x^p - a^p)^{\alpha+1}}{p^{1+\alpha}(b - a)} \int_0^1 t^\alpha (ta^p + (1-t)x^p)^{\frac{1-p}{p}} \left| f' \left(\sqrt[p]{ta^p + (1-t)x^p} \right) \right| dt \\ & \quad + \frac{(b^p - x^p)^{\alpha+1}}{p^{1+\alpha}(b - a)} \int_0^1 t^\alpha (tb^p + (1-t)x^p)^{\frac{1-p}{p}} \left| f' \left(\sqrt[p]{tb^p + (1-t)x^p} \right) \right| dt \\ & \leq \frac{a^{1-p}(x^p - a^p)^{\alpha+1}}{p^{1+\alpha}(b - a)} \int_0^1 t^\alpha \left| f' \left(\sqrt[p]{ta^p + (1-t)x^p} \right) \right| dt \\ & \quad + \frac{a^{1-p}(b^p - x^p)^{\alpha+1}}{p^{1+\alpha}(b - a)} \int_0^1 t^\alpha \left| f' \left(\sqrt[p]{tb^p + (1-t)x^p} \right) \right| dt \\ & \leq \frac{a^{1-p}(x^p - a^p)^{\alpha+1}}{p^{1+\alpha}(b - a)} \left(\int_0^1 t^{\alpha q} \left| f' \left(\sqrt[p]{ta^p + (1-t)x^p} \right) \right|^q dt \right)^{1/q} \\ & \quad + \frac{a^{1-p}(b^p - x^p)^{\alpha+1}}{p^{1+\alpha}(b - a)} \left(\int_0^1 t^{\alpha q} \left| f' \left(\sqrt[p]{tb^p + (1-t)x^p} \right) \right|^q dt \right)^{1/q}. \end{aligned}$$

Since $|f'|^q$ is p -convex and $|f'(x)| \leq M, \forall x \in [a, b]$, we get

$$\begin{aligned} \int_0^1 t^{\alpha q} \left| f' \left(\sqrt[p]{ta^p + (1-t)x^p} \right) \right|^q dt & \leq \int_0^1 t^{\alpha q+1} |f'(a)|^q dt + \int_0^1 t^{\alpha q}(1-t) |f'(x)|^q dt \\ & = \frac{|f'(a)|^q}{\alpha q + 2} + \frac{|f'(x)|^q}{(\alpha q + 1)(\alpha q + 2)} \\ & \leq \frac{M^q}{\alpha q + 1}, \end{aligned}$$

and

$$\int_0^1 t^{\alpha q} \left| f' \left(\sqrt[p]{tb^p + (1-t)x^p} \right) \right|^q dt \leq \frac{M^q}{\alpha q + 1}.$$

It follows that inequality in (3.5) is fulfilled. Similarly, the bound (3.6) holds proving by power mean inequality with relation in (3.4).

THEOREM 3.3. *Let $f : [a, b] \subset (0, \infty) \rightarrow \mathbb{R}$ be a differentiable mapping on (a, b) with $a < b$ such that $f' \in L[a, b]$. Assume that $|f'|$ is p -convex function, $|f'(x)| \leq M, \forall x \in [a, b]$ and constants $\alpha > 0, r, q > 1$ such that $1/r + 1/q = 1$.*

(i) *If $p \in (1, \infty)$, then we obtain inequality as*

$$\left| \frac{(x^p - a^p)^\alpha f(a) + (b^p - x^p)^\alpha f(b)}{p^\alpha(b - a)} - \frac{\Gamma(\alpha + 1)}{b - a} [({}^pI_{a^+}^\alpha f)(x) + ({}^pI_{b^-}^\alpha f)(x)] \right|$$

$$\leq \frac{(x^p - a^p)^{\alpha+1} + (b^p - x^p)^{\alpha+1}}{p^{1+\alpha}(b-a)} \left[\frac{(a^{1-p})^r}{r(\alpha r + 1)} + \frac{M^q}{q} \right], \quad x \in (a, b). \tag{3.7}$$

(ii) If $p \in (-\infty, 0) \cup (0, 1)$, then we get the estimate

$$\begin{aligned} & \left| \frac{(x^p - a^p)^\alpha f(a) + (b^p - x^p)^\alpha f(b)}{p^\alpha(b-a)} - \frac{\Gamma(\alpha + 1)}{b-a} [({}^pI_{a+}^\alpha f)(x) + ({}^pI_{b-}^\alpha f)(x)] \right| \\ & \leq \frac{(x^p - a^p)^{\alpha+1} + (b^p - x^p)^{\alpha+1}}{p^{1+\alpha}(b-a)} \left[\frac{(b^{1-p})^r}{r(\alpha r + 1)} + \frac{M^q}{q} \right], \quad x \in (a, b). \end{aligned} \tag{3.8}$$

Proof. According to the well-know Young’s inequality,

$$XY \leq \frac{X^r}{r} + \frac{Y^q}{q}, \quad \forall X, Y \geq 0, \quad r, q > 1, \quad \frac{1}{r} + \frac{1}{q} = 1,$$

with Lemma 2.1 and relation (3.3), we have

$$\begin{aligned} & \left| -\frac{(x^p - a^p)^\alpha f(a) + (b^p - x^p)^\alpha f(b)}{p^\alpha(b-a)} + \frac{\Gamma(\alpha + 1)}{b-a} [({}^pI_{a+}^\alpha f)(x) + ({}^pI_{b-}^\alpha f)(x)] \right| \\ & \leq \frac{(x^p - a^p)^{\alpha+1}}{p^{1+\alpha}(b-a)} \int_0^1 \left(\frac{1}{r} \left| t^\alpha (ta^p + (1-t)x^p)^{\frac{1-p}{p}} \right|^r + \frac{1}{q} \left| f' \left(\sqrt[p]{ta^p + (1-t)x^p} \right) \right|^q \right) dt \\ & \quad + \frac{(b^p - x^p)^{\alpha+1}}{p^{1+\alpha}(b-a)} \int_0^1 \left(\frac{1}{r} \left| t^\alpha (tb^p + (1-t)x^p)^{\frac{1-p}{p}} \right|^r + \frac{1}{q} \left| f' \left(\sqrt[p]{tb^p + (1-t)x^p} \right) \right|^q \right) dt \\ & \leq \frac{(x^p - a^p)^{\alpha+1}}{p^{1+\alpha}(b-a)} \int_0^1 \left(\frac{t^{\alpha r}}{r} \left| (ta^p + (1-t)x^p)^{\frac{1-p}{p}} \right|^r + \frac{1}{q} |t f'(a)| + (1-t)|f'(x)| \right)^q dt \\ & \quad + \frac{(b^p - x^p)^{\alpha+1}}{p^{1+\alpha}(b-a)} \int_0^1 \left(\frac{t^{\alpha r}}{r} \left| (tb^p + (1-t)x^p)^{\frac{1-p}{p}} \right|^r + \frac{1}{q} |t f'(b)| + (1-t)|f'(x)| \right)^q dt \\ & \leq \frac{(x^p - a^p)^{\alpha+1} + (b^p - x^p)^{\alpha+1}}{p^{1+\alpha}(b-a)} \left[\frac{(a^{1-p})^r}{r(\alpha r + 1)} + \frac{M^q}{q} \right], \end{aligned}$$

which means that the first part of this theorem holds.

To prove the second part, we use the above method with relation (3.4). This completes the proof. □

THEOREM 3.4. Let $f : [a, b] \subset (0, \infty) \rightarrow \mathbb{R}$ be a differentiable mapping on (a, b) with $a < b$ such that $f' \in L[a, b]$. In addition, we suppose that $|f'|$ is p -convex function, $|f'(x)| \leq M, \forall x \in [a, b]$ and constants $\alpha > 0, r, q > 0$ such that $r + q = 1$.

(i) If $p \in (1, \infty)$, then we deduce that the following inequality hold

$$\begin{aligned} & \left| \frac{(x^p - a^p)^\alpha f(a) + (b^p - x^p)^\alpha f(b)}{p^\alpha(b-a)} - \frac{\Gamma(\alpha + 1)}{b-a} [({}^pI_{a+}^\alpha f)(x) + ({}^pI_{b-}^\alpha f)(x)] \right| \\ & \leq \frac{(x^p - a^p)^{\alpha+1} + (b^p - x^p)^{\alpha+1}}{p^{1+\alpha}(b-a)} \left[\frac{ra^{1-p}}{\alpha + 1} + Mq \right], \quad x \in (a, b). \end{aligned} \tag{3.9}$$

(ii) If $p \in (-\infty, 0) \cup (0, 1)$, then we have the relation

$$\begin{aligned} & \left| \frac{(x^p - a^p)^\alpha f(a) + (b^p - x^p)^\alpha f(b)}{p^\alpha(b - a)} - \frac{\Gamma(\alpha + 1)}{b - a} [({}^pI_{a+}^\alpha f)(x) + ({}^pI_{b-}^\alpha f)(x)] \right| \\ & \leq \frac{(x^p - a^p)^{\alpha+1} + (b^p - x^p)^{\alpha+1}}{p^{1+\alpha}(b - a)} \left[\frac{rb^{1-p}}{\alpha + 1} + Mq \right], \quad x \in (a, b). \end{aligned} \tag{3.10}$$

Proof. By using the well-know Weighted AM-GM inequality,

$$X^r Y^q \leq rX + qY, \quad \forall X, Y \geq 0, \quad r, q > 0, \quad r + q = 1,$$

with Lemma 2.1 and relation (3.3), we have

$$\begin{aligned} & \left| -\frac{(x^p - a^p)^\alpha f(a) + (b^p - x^p)^\alpha f(b)}{p^\alpha(b - a)} + \frac{\Gamma(\alpha + 1)}{b - a} [({}^pI_{a+}^\alpha f)(x) + ({}^pI_{b-}^\alpha f)(x)] \right| \\ & \leq \frac{(x^p - a^p)^{\alpha+1}}{p^{1+\alpha}(b - a)} \int_0^1 \left[t^\alpha (ta^p + (1 - t)x^p)^{\frac{1-p}{p}} \right]^r \left[f' \left(\sqrt[p]{ta^p + (1 - t)x^p} \right) \right]^q dt \\ & \quad + \frac{(b^p - x^p)^{\alpha+1}}{p^{1+\alpha}(b - a)} \int_0^1 \left[t^\alpha (tb^p + (1 - t)x^p)^{\frac{1-p}{p}} \right]^r \left[f' \left(\sqrt[p]{tb^p + (1 - t)x^p} \right) \right]^q dt \\ & \leq \frac{(x^p - a^p)^{\alpha+1}}{p^{1+\alpha}(b - a)} \left[\int_0^1 rt^\alpha (ta^p + (1 - t)x^p)^{\frac{1-p}{p}} dt + \int_0^1 qf' \left(\sqrt[p]{ta^p + (1 - t)x^p} \right) dt \right] \\ & \quad + \frac{(b^p - x^p)^{\alpha+1}}{p^{1+\alpha}(b - a)} \left[\int_0^1 rt^\alpha (tb^p + (1 - t)x^p)^{\frac{1-p}{p}} dt + \int_0^1 qf' \left(\sqrt[p]{tb^p + (1 - t)x^p} \right) dt \right] \\ & \leq \frac{(x^p - a^p)^{\alpha+1}}{p^{1+\alpha}(b - a)} \left[\int_0^1 rt^\alpha \left| (ta^p + (1 - t)x^p)^{\frac{1-p}{p}} \right| dt + \int_0^1 q \left| f' \left(\sqrt[p]{ta^p + (1 - t)x^p} \right) \right| dt \right] \\ & \quad + \frac{(b^p - x^p)^{\alpha+1}}{p^{1+\alpha}(b - a)} \left[\int_0^1 rt^\alpha \left| (tb^p + (1 - t)x^p)^{\frac{1-p}{p}} \right| dt + \int_0^1 q \left| f' \left(\sqrt[p]{tb^p + (1 - t)x^p} \right) \right| dt \right] \\ & \leq \frac{(x^p - a^p)^{\alpha+1}}{p^{1+\alpha}(b - a)} \left[\int_0^1 a^{1-p} rt^\alpha dt + \int_0^1 q |t|f'(a)| + (1 - t)|f'(x)| dt \right] \\ & \quad + \frac{(b^p - x^p)^{\alpha+1}}{p^{1+\alpha}(b - a)} \left[\int_0^1 a^{1-p} rt^\alpha dt + \int_0^1 q |t|f'(b)| + (1 - t)|f'(x)| dt \right] \\ & \leq \frac{(x^p - a^p)^{\alpha+1} + (b^p - x^p)^{\alpha+1}}{p^{1+\alpha}(b - a)} \left[\frac{ra^{1-p}}{\alpha + 1} + Mq \right]. \end{aligned}$$

Thus the first part of this theorem is derived.

For the second part, we use the same method to prove (i) with relation (3.4). \square

REMARK 3.2. For $p = 1$ the results of cases (i) and (ii) in the above theorems are identical.

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