

## SOME PROPERTIES OF GRAND SOBOLEV–MORREY SPACES WITH DOMINANT MIXED DERIVATIVES

ALIK M. NAJAFOV AND SAIN T. ALEKBERLI

(Communicated by A. Meskhi)

*Abstract.* In this paper we construct a grand Sobolev–Morrey spaces with dominant mixed derivatives, and by means of integral representation we study differential and differential-difference properties of functions from this spaces.

### 1. Introduction and preliminary notes

The Sobolev  $S_p^l W$ , ( $l \in \mathbb{N}^n$ ) and Nikolskii  $S_p^l H$  ( $l \in (0, \infty)^n$ ) spaces with dominant mixed derivatives were introduced and studied by S. M. Nikolskii [15] and later by A. D. Djabraïlov [3]. The spaces  $S_p^l W$  were extended to the case when  $l = (l_1, l_2, \dots, l_n)$ , where  $l_j \geq 0$  cannot be an integer, also in the paper by T. I. Amanov [1]. The spaces of Morrey type with dominant mixed derivatives were introduced and studied in [11], [12], [14].

Note that at studying of differential equations with the higher order mixed partial derivatives, it became necessary to study such spaces with dominant mixed derivatives.

EXAMPLE 1. Let's consider an equation of the form

$$u_{x^2 y^2}^{(4)} + u_{x^2 y}^{(3)} + u_{xy^2}^{(3)} + u_{xy}^{(2)} + u'_x + u'_y + u = f.$$

In our case the solution of this equation is sought in the space  $S^{(2,2)} W$ . One can look for the solution of equation in the space  $W^{(4,4)}$ , but then this solution will require additional derivatives, in other words, in our case the solution belongs to a wider class.

In the paper we introduce a grand Sobolev–Morrey space with dominant mixed derivatives

$$S_{p) , a, \varkappa}^l W(G), \tag{1.1}$$

where  $G \subset \mathbb{R}^n$  is a bounded domain  $l \in \mathbb{N}^n$ ,  $p \in (1, \infty)$ ,  $a \in [0, 1]^n$ ,  $\varkappa \in (0, \infty)^n$  and by using integral representation for generalized mixed derivatives functions defined on  $n$ -dimensional domains satisfying the flexible horn conditions in [11], received inequality

*Mathematics subject classification* (2010): 46E35, 46E30, 26D15.

*Keywords and phrases:* Grand Sobolev–Morrey spaces with dominant mixed derivatives, integral representation, embedding theorem, Hölder condition.

type Sobolev’s in this spaces and also was proved generalized mixed derivatives the functions belongs to a Hölder class.

The grand Lebesgue spaces  $L_p(G)$  for a measurable set  $G \subset \mathbb{R}^n$  of finite Lebesgue measure were introduced in the paper of T. Iwaniec and C. Sbordone in [4]. Later a vast amount of research about grand Lebesgue and grand Lebesgue–Morrey spaces (with different norms) has been done by many mathematicians (see, e.g., [5]–[10],[13],[16]–[19]).

We note that in this paper it is investigated the grand Sobolev–Morrey spaces  $S^l_{p),a,\varkappa} W(G)$  with dominant mixed derivatives giving possibility to increase “Hölder exponents” (see Theorem 2) more than in the case of the Sobolev–Morrey type spaces  $S^l_{p,a,\varkappa,\tau} W(G)$  with dominant mixed derivatives studied in [11].

Let  $e_n = \{1, 2, \dots, n\}$ ,  $e \subseteq e_n$ ;  $l = (l_1, \dots, l_n)$ ,  $l_j > 0$  are integers ( $j \in e_n$ ), and let  $l^e = (l^e_1, \dots, l^e_n)$ ,  $l^e_j = l_j$  for  $j \in e$ ;  $l^e_j = 0$  for  $j \in e_n \setminus e = e'$ .

Let for any  $x \in \mathbb{R}^n$ ,

$$G_{I^{\varkappa}}(x) = G \cap I_{I^{\varkappa}}(x) = G \cap \left\{ y : |y_j - x_j| < \frac{1}{2} t_j^{\varkappa_j}, j \in e_n \right\},$$

and

$$\int_{a^e}^{b^e} f(x) dx^e = \left( \prod_{j \in e} \int_{a_j}^{b_j} dx_j \right) f(x),$$

i.e. integration is carried out only with respect to the variables  $x_j$  whose indices belong to  $e$ .

DEFINITION 1. Denote by  $S^l_{p),a,\varkappa} W(G)$  the space of locally summable functions  $f$  on  $G$  having the generalized derivatives  $D^{l^e} f$  ( $e \subseteq e_n$ ) with the finite norm

$$\|f\|_{S^l_{p),a,\varkappa} W(G)} = \sum_{e \subseteq e_n} \left\| D^{l^e} f \right\|_{p),a,\varkappa;G}, \tag{1.2}$$

where

$$\begin{aligned} & \|f\|_{p),a,\varkappa;G} = \|f\|_{L_p),a,\varkappa(G)} = \\ & = \sup_{\substack{x \in G \\ 0 < t_j \leq d_j, j \in e_n \\ 0 < \varepsilon < p-1}} \left( \frac{1}{\prod_{j \in e_n} t_j^{\varkappa_j a_j}} \frac{\varepsilon}{|G_{I^{\varkappa}}(x)|} \int_{G_{I^{\varkappa}}(x)} |f(y)|^{p-\varepsilon} dy \right)^{\frac{1}{p-\varepsilon}}, \end{aligned} \tag{1.3}$$

$d_j (j \in e_n)$  is diagonals of  $I_{I^{\varkappa}}(x)$ .

Observe some properties of the spaces  $L_p),a,\varkappa(G)$  and  $S^l_{p),a,\varkappa} W(G)$ .

1. The following embeddings hold:

$$L_p),a,\varkappa(G) \hookrightarrow L_p(G), \quad S^l_{p),a,\varkappa} W(G) \hookrightarrow S^l_p) W(G),$$

and,

$$\|f\|_{p,G} \leq c \|f\|_{p,a,\varkappa;G}; \quad \|f\|_{S'_p W(G)} \leq c \|f\|_{S'_{p,a,\varkappa} W(G)}, \tag{1.4}$$

where

$$\|f\|_{S'_p W(G)} = \sum_{e \subseteq e_n} \left\| D^{I^e} f \right\|_{p,G},$$

$$\|f\|_{p,G} = \|f\|_{L_p(G)} = \sup_{0 < \varepsilon < p-1} \left( \frac{\varepsilon}{|G|} \int_G |f(x)|^{p-\varepsilon} dx \right)^{\frac{1}{p-\varepsilon}}.$$

Indeed,

$$\|f\|_{p,a,\varkappa;G} = \sup_{\substack{x \in G \\ 0 < t_j \leq d_j, j \in e_n \\ 0 < \varepsilon < p-1}} \left( \frac{1}{\prod_{j \in e_n} t_j^{\varkappa_j a_j}} \frac{\varepsilon}{|G_{t^{\varkappa}}(x)|} \int_{G_{t^{\varkappa}}(x)} |f(y)|^{p-\varepsilon} dy \right)^{\frac{1}{p-\varepsilon}} \geq$$

$$\geq \prod_{j \in e_n} d_j^{-\frac{\varkappa_j a_j}{p-\varepsilon}} \sup_{0 < \varepsilon < p-1} \left( \frac{\varepsilon}{|G|} \int_G |f(x)|^{p-\varepsilon} dx \right)^{\frac{1}{p-\varepsilon}} = c \|f\|_{p,G}.$$

- 2.  $L_{p,a,\varkappa}(G)$  and  $S'_{p,a,\varkappa} W(G)$  are complete.
- 3. For every real  $c > 0$

$$\|f\|_{p,a,c,\varkappa;G} = \|f\|_{p,a,\varkappa;G}; \quad \|f\|_{S'_{p,a,c,\varkappa} W(G)} = \|f\|_{S'_{p,a,\varkappa} W(G)};$$

- 4.  $\|f\|_{p,0,\varkappa;G} = \|f\|_{p,G}, \quad \|f\|_{S'_{p,0,\varkappa} W(G)} = \|f\|_{S'_p W(G)}.$

DEFINITION 2. We say that the open set  $G \subset R^n$  satisfies the condition flexible-horn, if for any  $x \in G$  and  $T \in (0, \infty)^n$  there exists a vector-function

$$\rho(t, x) = (\rho_1(t_1, x), \dots, \rho_n(t_n, x)), 0 \leq t_j \leq T_j, j \in e_n,$$

with the following properties:

- 1) for all  $j \in e_n$ , the functions  $\rho_j(t_j, x)$  are absolutely continuous with respect to  $t_j$  on  $[0, T_j]$ , and  $|\rho_j(t_j, x)| \leq 1$  for almost all  $t_j \in [0, T_j]$ , where  $\rho_j(t_j, x) = \frac{\partial}{\partial t_j} \rho_j(t_j, x)$ ;
- 2)  $\rho_j(0, x) = 0$  for all  $j \in e_n$ ,

$$x + V(x, \omega) = x + \bigcup_{\substack{0 \leq t_j \leq T_j \\ j \in e_n}} [\rho(t, x) + t\omega I] \subset G,$$

where  $\omega = (\omega_1, \dots, \omega_n)$   $\omega_j \in [0, 1]$  for  $j \in e_n$ ,  $I = [-1, 1]^n$ ,  $t\omega I = \{(t_1 \omega_1 y_1, \dots, t_n \omega_n y_n) : y \in I\}$ . If  $t_1 = t^{\lambda_1}, \dots, t_n = t^{\lambda_n}$ ,  $\lambda = (\lambda_1, \dots, \lambda_n)$ ,  $\rho(t, x) = \rho(t^\lambda, x)$ ,  $\omega^\lambda = (\omega^{\lambda_1}, \dots, \omega^{\lambda_n})$ ,  $\omega \in (0, 1]$ , then  $V(x, \lambda, \omega) = \bigcup_{0 \leq t \leq T} [\rho(t^\lambda, x) + t^\lambda \omega^\lambda I]$

is a flexible  $\lambda$ -horn introduced by O. V. Besov [2].

Let  $M(\cdot, y, z) \in C_0^\infty(\mathbb{R}^n)$  be such that

$$S(M) = \text{supp } M \subset I_1 = \left\{ y : |y_j| < \frac{1}{2}; j \in e_n \right\}$$

and let  $0 < T_j \leq \min\{1, d_j\}$ ,  $j \in e_n$  and assume

$$V = \bigcup_{0 < t_j \leq T_j} \left\{ y : \left( \frac{y}{t^e + T^{e'}} \right) \in S(M) \right\},$$

where we have  $t^e + T^{e'} = t_j$ ,  $j \in e$ ;  $t^e + T^{e'} = T'_j$ ,  $j \in e'$  and clearly,  $V \subset I_T = \left\{ x : |x_j| < \frac{1}{2}T_j : j \in e_n \right\}$ . Let  $U$  be an open set contained in the domain  $G$ ; henceforth we always assume that  $U + V \subset G$ , and put

$$G_{T^*}(U) = (U + I_{T^*}(x)) \cap G.$$

Obviously, if  $0 < \varkappa_j \leq 1$  ( $j \in e_n$ ), then  $I_T \subset I_{T^*}$  and thereby  $U + V \subset G_{T^*}(U) = Z$ .

LEMMA 1. Let  $1 < p < q \leq r \leq \infty$ ,  $0 \leq \varkappa_j \leq \frac{1}{1+a_j}$ ,  $0 < t_j, \eta_j \leq T_j \leq \min\{1, d_j\}$ ,  $0 < \gamma_j < \gamma_{j,0}$ ,  $\nu = (\nu_1, \nu_2, \dots, \nu_n)$ ,  $\nu_j \geq 0$  are integers ( $j \in e_n$ ),  $\Psi \in L_{p,a,\varkappa}(G)$  and

$$\mu_j = l_j - \nu_j - (1 - \varkappa_j - \varkappa_j a_j) \left( \frac{1}{p-\varepsilon} - \frac{1}{q-\varepsilon} \right),$$

$$E_\eta^e(x) = \int_{0^e}^{\eta^e} \prod_{j \in e} t^{l_j - \nu_j - 2} \times \\ \times \int_{\mathbb{R}^n} \Psi(x+y) M \left( \frac{y}{t^e + T^{e'}}, \frac{\rho(t^e + T^{e'}, x)}{t^e + T^{e'}}, \rho'(t^e + T^{e'}, x) \right) dy dt^e, \tag{1.5}$$

$$E_{\eta,T}^e(x) = \int_{\eta^e}^{T^e} \prod_{j \in e} t^{l_j - \nu_j - 2} \times \\ \times \int_{\mathbb{R}^n} \Psi(x+y) M \left( \frac{y}{t^e + T^{e'}}, \frac{\rho(t^e + T^{e'}, x)}{t^e + T^{e'}}, \rho'(t^e + T^{e'}, x) \right) dy dt^e. \tag{1.6}$$

Then

$$\sup_{\tilde{x} \in U} \|E_\eta^e\|_{q-\varepsilon, U_{\rho^*}(\tilde{x})} \leq C_1 \|\Psi\|_{p,a,\varkappa;Z} \varepsilon^{-\frac{1}{p-\varepsilon}} \times \\ \times \prod_{j \in e_n} \gamma_j^{\frac{\varkappa_j a_j + \varkappa_j}{q-\varepsilon}} \prod_{j \in e'} T_j^{1 - (\varkappa_j - \varkappa_j a_j) \left( \frac{1}{p-\varepsilon} - \frac{1}{q-\varepsilon} \right)} \prod_{j \in e} \eta_j^{\mu_j} \quad (\mu_j > 0), \tag{1.7}$$

$$\begin{aligned} & \sup_{\bar{x} \in U} \|E_{\eta, T}^e\|_{q-\varepsilon, U_{\gamma^\varepsilon}(\bar{x})} \leq \\ & \leq C_2 \|\Psi\|_{p, a, \varepsilon, Z} \varepsilon^{-\frac{1}{p-\varepsilon}} \prod_{j \in e_n} \gamma_j^{\frac{\varepsilon_j a_j + \varepsilon_j}{q-\varepsilon}} \prod_{j \in e'} T_j^{1 - (\varepsilon_j - \varepsilon_j a_j) \left(\frac{1}{p-\varepsilon} - \frac{1}{q-\varepsilon}\right)} \times \\ & \times \begin{cases} \prod_{j \in e} T_j^{\mu_j}, & \text{for } \mu_j > 0, \\ \prod_{j \in e} \ln \frac{T_j}{\eta_j}, & \text{for } \mu_j = 0, \\ \prod_{j \in e} \eta_j^{\mu_j}, & \text{for } \mu_j < 0, \end{cases} \end{aligned} \tag{1.8}$$

where  $U_{\gamma^\varepsilon}(\bar{x}) = \left\{x : |x_j - \bar{x}_j| < \frac{1}{2} \gamma_j^{\varepsilon_j}, j \in e_n\right\}$  and  $C_1$  and  $C_2$  are constants independent of  $\Psi, \gamma, \eta, T$  and  $\varepsilon$ .

*Proof.* Applying the generalized Minkowskii inequality we deduce

$$\|E_{\eta}^e\|_{q-\varepsilon, U_{\gamma^\varepsilon}(\bar{x})} \leq \int_0^{\eta^e} \prod_{j \in e} t^{l_j - v_j - 2} \|\varphi(\cdot, t)\|_{q-\varepsilon, U_{\gamma^\varepsilon}(\bar{x})} dt^e, \tag{1.9}$$

for every  $\bar{x} \in U$ , where

$$\varphi(x, t) = \int_{\mathbb{R}^n} \Psi(x+y) M \left( \frac{y}{t^e + T^{e'}}, \frac{\rho(t^e + T^{e'}, x)}{t^e + T^{e'}}, \rho'(t^e + T^{e'}, x) \right) dy. \tag{1.10}$$

Estimate of the norm  $\|\varphi(\cdot, t)\|_{q-\varepsilon, U_{\gamma^\varepsilon}(\bar{x})}$ . From Holder’s inequality ( $q \leq r$ ) we obtain

$$\|\varphi(\cdot, t)\|_{q-\varepsilon, U_{\gamma^\varepsilon}(\bar{x})} \leq \|\varphi(\cdot, t)\|_{r-\varepsilon, U_{\gamma^\varepsilon}(\bar{x})} \prod_{j \in e_n} \gamma_j^{\varepsilon_j \left(\frac{1}{q-\varepsilon} - \frac{1}{r-\varepsilon}\right)}. \tag{1.11}$$

Let  $\chi$  be the characteristic function of  $S(M)$ . Using the fact that  $1 < p \leq r \leq \infty$ ,  $s \leq r$   $\left(\frac{1}{s} = 1 - \frac{1}{p-\varepsilon} + \frac{1}{r-\varepsilon}\right)$  and

$$|\Psi M| = (|\Psi|^{p-\varepsilon} |M|^s)^{\frac{1}{r-\varepsilon}} (|\Psi|^{p-\varepsilon} \chi)^{\frac{1}{p-\varepsilon} - \frac{1}{r-\varepsilon}} (|M|^s)^{\frac{1}{s} - \frac{1}{r-\varepsilon}}$$

and applying again Holder’s inequality  $\left(\frac{1}{r-\varepsilon} + \left(\frac{1}{p-\varepsilon} - \frac{1}{r-\varepsilon}\right) + \left(\frac{1}{s} - \frac{1}{r-\varepsilon}\right) = 1\right)$ , we have

$$\begin{aligned} \|\varphi(\cdot, t)\|_{r-\varepsilon, U_{\gamma^\varepsilon}(\bar{x})} & \leq \sup_{x \in U_{\gamma^\varepsilon}(\bar{x})} \left( \int_{\mathbb{R}^n} |\Psi(x+y)|^{p-\varepsilon} \chi \left( \frac{y}{t^e + T^{e'}} \right) dy \right)^{\frac{1}{p-\varepsilon} - \frac{1}{r-\varepsilon}} \times \\ & \times \sup_{y \in V} \left( \int_{U_{\gamma^\varepsilon}(\bar{x})} |\Psi(x+y)|^{p-\varepsilon} dx \right)^{\frac{1}{r-\varepsilon}} \left( \int_{\mathbb{R}^n} \left| M_1 \left( \frac{y}{t^e + T^{e'}} \right) \right|^s dy \right)^{\frac{1}{s}}, \end{aligned} \tag{1.12}$$

here it is assumed that  $|M(x, y, z)| \leq |M_1(x)|$ .

Obviously, if  $\varkappa_j \leq 1, 0 < t_j \leq T_j, j \in e_n$ , then  $Z_{t^e+T^{e'}}(x) \subset Z_{t^{\varkappa^e}+T^{\varkappa^{e'}}}(x)$ . For every  $x \in U$  we have

$$\begin{aligned} \int_{\mathbb{R}^n} |\Psi(x+y)|^{p-\varepsilon} \chi\left(\frac{y}{t^e+T^{e'}}\right) dy &\leq \int_{Z_{t^e+T^{e'}}(x)} |\Psi(y)|^{p-\varepsilon} dy \leq \\ &\leq \int_{Z_{t^{\varkappa^e}+T^{\varkappa^{e'}}}(x)} |\Psi(y)|^{p-\varepsilon} dy \leq \|\Psi\|_{p-\varepsilon, Z_{\varkappa^e}(x)}^{p-\varepsilon} \leq \\ &\leq \|\Psi\|_{p, Z_{\varkappa^e}(x)}^{p-\varepsilon} \frac{|Z_{\varkappa^e}(x)|}{\varepsilon} \leq \|\Psi\|_{p, a, \varkappa; Z}^{p-\varepsilon} \varepsilon^{-1} \prod_{j \in e'} T_j^{\varkappa_j(1+a_j)} \prod_{j \in e} t_j^{\varkappa_j(1+a_j)}. \end{aligned} \tag{1.13}$$

For  $y \in V$

$$\int_{U_{\varkappa^e}(\bar{x})} |\Psi(x+y)|^{p-\varepsilon} dx \leq \int_{Z_{\varkappa^e}(\bar{x}+y)} |\Psi(x)|^{p-\varepsilon} dx \leq \|\Psi\|_{p, a, \varkappa; Z}^{p-\varepsilon} \prod_{j \in e_n} \gamma_j^{\varkappa_j(1+a_j)}, \tag{1.14}$$

$$\int_{\mathbb{R}^n} \left| M_1\left(\frac{y}{t^e+T^{e'}}\right) \right|^s dy = \|M_1\|_s^s \prod_{j \in e'} T_j \prod_{j \in e} t_j. \tag{1.15}$$

It follows from (1.11)-(1.15) for  $r = q$  that

$$\begin{aligned} \|\varphi(\cdot, t)\|_{q-\varepsilon, U_{\varkappa^e}(\bar{x})} &\leq \\ &\leq C \|M_1\|_s \|\Psi\|_{p, a, \varkappa; Z} \varepsilon^{-\frac{1}{p-\varepsilon}} \prod_{j \in e_n} \gamma_j^{\frac{\varkappa_j a_j + \varkappa_j}{q-\varepsilon}} \prod_{j \in e'} T_j^{1-(1-\varkappa_j-\varkappa_j a_j)\left(\frac{1}{p-\varepsilon} - \frac{1}{q-\varepsilon}\right)} \times \\ &\quad \times \prod_{j \in e} t_j^{1-(1-\varkappa_j-\varkappa_j a_j)\left(\frac{1}{p-\varepsilon} - \frac{1}{q-\varepsilon}\right)}. \end{aligned} \tag{1.16}$$

Unseating this inequality in (1.9), for all  $\bar{x} \in U$ , we see that

$$\begin{aligned} \|E_\eta^e\|_{q-\varepsilon, U_{\varkappa^e}(\bar{x})} &\leq C_1 \|\Psi\|_{p, a, \varkappa; Z} \varepsilon^{-\frac{1}{p-\varepsilon}} \times \\ &\times \prod_{j \in e_n} \gamma_j^{\frac{\varkappa_j a_j + \varkappa_j}{q-\varepsilon}} \prod_{j \in e'} T_j^{1-(1-\varkappa_j-\varkappa_j a_j)\left(\frac{1}{p-\varepsilon} - \frac{1}{q-\varepsilon}\right)} \prod_{j \in e_n} \eta_j^{\mu_j} \quad (\mu_j > 0). \end{aligned}$$

Similarly, we can prove (1).

### 2. Main results

Proved two theorems on the properties of the functions from the space  $S^l_{p,a,\varkappa}W(G)$ .

**THEOREM 1.** *Let open bounded set  $G \subset \mathbb{R}^n$  satisfy the flexible-horn condition,  $1 < p < q \leq \infty$ ,  $\varkappa_j \leq \frac{1}{1+a_j}$ ;  $\mathbf{v} = (v_1, v_2, \dots, v_n)$ ,  $v_j \geq 0$  are integers ( $j \in e_n$ ) and  $\mu_j > 0$  ( $j \in e_n$ ), and  $f \in S^l_{p,a,\varkappa}W(G)$ .*

*Then  $D^{\mathbf{v}} : S^l_{p,a,\varkappa}W(G) \hookrightarrow L_{q-\varepsilon}(G)$  holds for any  $0 < \varepsilon < p - 1$ . Moreover, the following inequality is valid*

$$\|D^{\mathbf{v}}f\|_{q-\varepsilon,G} \leq C(\varepsilon) \sum_{e \subseteq e_n} \prod_{j \in e_n} T_j^{s_{e,j}} \|D^{t^e}f\|_{p,a,\varkappa;G}, \tag{2.1}$$

where  $s_{e,j} = \begin{cases} \mu_j, & j \in e, \\ -v_j - (1 - \varkappa_j - \varkappa_j a_j) \left( \frac{1}{p-\varepsilon} - \frac{1}{q-\varepsilon} \right), & j \in e'. \end{cases}$

In particular, if

$$\mu_{j,0} = l_j - v_j - (1 - \varkappa_j - \varkappa_j a_j) \frac{1}{p-\varepsilon} > 0 \quad (j \in e_n),$$

then  $D^{\mathbf{v}}f$  is continuous on  $G$  and

$$\sup_{x \in G} |D^{\mathbf{v}}f(x)| \leq C(\varepsilon) \sum_{e \subseteq e_n} \prod_{j \in e_n} T_j^{s_{e,j,0}} \|D^{t^e}f\|_{p,a,\varkappa;G}, \tag{2.2}$$

where  $0 < T_j \leq d_j$ , ( $j \in e_n$ )  $C(\varepsilon)$  is a constant independent of  $f$  and  $T$ .

*Proof.* Under the conditions of the theorem, the generalized derivatives  $D^{\mathbf{v}}f$  exists [11],[12]. Indeed, if  $\mu_j > 0, (j \in e_n)$  and  $l_j - v_j > 0 (j \in e_n)$ . Since  $p < q, \varkappa_j \leq \frac{1}{1+a_j}, j \in e_n$  and  $S^l_{p,a,\varkappa}W(G) \hookrightarrow S^l_p W(G) \hookrightarrow S^{l-p-\varepsilon}W(G)$  ( $p - \varepsilon > 1$ ). Then  $D^{\mathbf{v}}f$  exists on  $G$  and belongs to  $L_{p-\varepsilon}(G)$  and for almost each point  $x \in G$  the integral identity received in [11]:

$$D^{\mathbf{v}}f(x) = \sum_{e \subseteq e_n} (-1)^{|v|} \prod_{j \in e'} T_j^{-1-v_j} \int \int_{0^e \mathbb{R}^n} \prod_{j \in e} t_j^{l_j-v_j-2} \times K_e^{(v)} \left( \frac{y}{t^e + T^e}, \frac{\rho(t^e + T^e, x)}{t^e + T^e}, \rho'(t^e + T^e, x) \right) D^{t^e}f(x+y) dy dt^e, \tag{2.3}$$

$0 < T_j \leq d_j$ , and  $K_e(\cdot, y, z) \in C^0_0(\mathbb{R}^n)$ . Recall that the flexible horn  $x + V \subset G$  is the support of the representation (2). Hence, using Minkowski’s inequality, we arrive at

$$\|D^{\mathbf{v}}f\|_{q-\varepsilon,G} \leq \sum_{e \subseteq e_n} \prod_{j \in e'} T_j^{-1-v_j} \|E^e_T\|_{q-\varepsilon,G}. \tag{2.4}$$

By means of inequality (1) for  $U = G, D^l f = \Psi, K_e^{(v)} = M, \eta_j = T_j (j \in e_n)$  we get inequality (2.1).

Now, let conditions  $\mu_{j,0} > 0 (j \in e_n)$  be satisfied, then based around identity (2) from inequality (2.4) we get

$$\left\| D^v f - f_T^{(v)} \right\|_{\infty, G} \leq C \sum_{\emptyset \neq e \subseteq e_n} \prod_{j \in e'} T_j^{s_{e,j,0}} \left\| D^l f \right\|_{p), a, \varkappa; G}.$$

As  $T_j \rightarrow 0 (j \in e_n)$ , the left side of this inequality tends to zero, since  $f_T^{(v)}$  is continuous on  $G$ , in our case the convergence in  $L_\infty(G)$  coincides with uniform convergence. Then the limit function  $D^v f$  is continuous on  $G$ . The theorem is proved.

Let  $\gamma$  be an  $n$  dimensional vector.

**THEOREM 2.** *Let all the condition of Theorem 1. If  $\mu_j > 0 (j \in e_n)$  then  $D^v f$  satisfies the Hölder condition with exponent  $\beta_j$  on  $G$  in the metric of  $L_{q-\varepsilon}$ ; more exactly*

$$\left\| \Delta(\gamma, G) D^v f \right\|_{q-\varepsilon, G} \leq C(\varepsilon) \|f\|_{S_{p), a, \varkappa}^l W(G)} \prod_{j \in e_n} |\gamma_j|^{\beta_j}, \tag{2.5}$$

where  $\beta_j$  is an arbitrary number satisfying the inequalities:

$$\begin{aligned} 0 \leq \beta_j \leq 1, & \text{ if } \mu_j > 1, j \in e; \\ 0 \leq \beta_j < 1, & \text{ if } \mu_j = 1, j \in e; 0 \leq \beta_j \leq 1, j \in e'; \\ 0 \leq \beta_j \leq \mu_j, & \text{ if } \mu_j < 1, j \in e, \end{aligned} \tag{2.6}$$

If  $\mu_{j,0} > 0 (j \in e_n)$  then

$$\sup_{x \in G} |\Delta(\gamma, G) D^v f(x)| \leq C(\varepsilon) \|f\|_{S_{p), a, \varkappa}^l W(G)} \prod_{j \in e_n} |\gamma_j|^{\beta_{j,0}}, \tag{2.7}$$

where  $\beta_{j,0}$  satisfy the same conditions as  $\beta_j$  with  $\mu_{j,0}$  instead of  $\mu_j$ .

*Proof.* By Lemma 8.6 of [2], there is a domain  $G_\sigma \subset G (\sigma = (\sigma_1, \dots, \sigma_n), \sigma_j = \xi_j r(x), \xi_j > 0, r(x) = \text{dist}(x, \partial G), x \in G)$ . Suppose that  $|\gamma_j| < \sigma_j (j \in e_n)$ . Then, for every  $x \in G_\sigma$ , then segment joining the points  $x$  and  $x + \gamma$  is contained in  $G$ . Identity (2) are valid for all points of the segment with the same kernel. Making simple transformations, we obtain

$$\begin{aligned} & |\Delta(\gamma, G) D^v f| \leq \\ & \leq C_1 \sum_{e \subseteq e_n} (-1)^{|v|} \prod_{j \in e'} T_j^{-1-v_j} \int_{0^e}^{\gamma^e} \frac{dt^e}{\prod_{j \in e} t_j^{2+v_j-l_j}} \times \\ & \times \int_{\mathbb{R}^n} \left| K_e^{(v)} \left( \frac{y}{t^e + T^e}, \frac{\rho(t^e + T^e, x)}{t^e + T^e}, \rho'(t^e + T^e, x) \right) \right| |\Delta(\gamma, G) D^l f(x+y)| dy + \end{aligned}$$

$$\begin{aligned}
 & +C_1 \sum_{e \subseteq e_n} (-1)^{|v|} \prod_{j \in e'} T_j^{-2-v_j} \prod_{j \in e_n} |\gamma_j| \int \frac{dt^e}{|\gamma^e| \prod_{j \in e} t_j^{2+v_j-t_j}} \times \\
 & \times \int_{\mathbb{R}^n} \left| K_e^{(v+1)} \left( \frac{y}{t^e + T^{e'}}, \frac{\rho(t^e + T^{e'}, x)}{t^e + T^{e'}}, \rho'(t^e + T^{e'}, x) \right) \right| \times \\
 & \times \int_0^1 \cdots \int_0^1 |D^{l^e} f(x+y+\gamma_1 u_1 + \dots + \gamma_n u_n)| du dy = \\
 & = C_1 \sum_{e \subseteq e_n} F_1^e(x, \gamma) + C_2 \sum_{e \subseteq e_n} F_2^e(x, \gamma), \tag{2.8}
 \end{aligned}$$

where  $0 < T_j \leq d_j, j \in e_n, |\gamma^e| = (|\gamma_1^e|, \dots, |\gamma_n^e|), |\gamma_j^e| = |\gamma_j|$  for  $j \in e; |\gamma_j^e| = 0$  for  $j \in e'$ . We also assume that  $|\gamma_j| < T_j, (j \in e_n)$  and consequently  $|\gamma_j| \leq \min(\sigma_j, T_j), j \in e_n$ . If  $x \in G \setminus G_\sigma$  then by definition  $\Delta(\gamma, G)D^v f(x) = 0$ .

By (2)

$$\begin{aligned}
 & \|\Delta(\gamma, G)D^v f\|_{q-\varepsilon, G} = \|\Delta(\gamma, G)D^v f\|_{q-\varepsilon, G_\sigma} \leq \\
 & \leq C_1 \sum_{e \subseteq e_n} \|F_1^e(\cdot, \gamma)\|_{q-\varepsilon, G_\sigma} + C_2 \sum_{e \subseteq e_n} \|F_2^e(\cdot, \gamma)\|_{q-\varepsilon, G_\sigma}. \tag{2.9}
 \end{aligned}$$

By means of inequality (1) for  $D^{l^e} f = \Psi, \eta_j = |\gamma_j| (j \in e)$  we get

$$\|F_1^e(\cdot, \gamma)\|_{q-\varepsilon, G_\sigma} \leq C_1 \left\| D^{l^e} f \right\|_{(p), a, \varepsilon, \zeta; G} \prod_{j \in e} |\gamma_j|^{\mu_j}, \tag{2.10}$$

and by means of inequality (1) for  $D^{l^e} f = \Psi, \eta_j = |\gamma_j| (j \in e)$  we get

$$\|F_2^e(\cdot, \zeta)\|_{q-\varepsilon, G_\sigma} \leq C_2 \left\| D^{l^e} f \right\|_{(p), a, \varepsilon, \zeta; G} \prod_{j \in e_n} |\gamma_j|^{\sigma_j}. \tag{2.11}$$

From inequalities (2)-(2.11) we get the required inequality.

Now suppose that  $|\gamma_j| \geq \min(\sigma_j, T_j), j \in e_n$ . Then

$$\|\Delta(\gamma, G)D^v f\|_{q-\varepsilon, G} \leq 2 \|D^v f\|_{q-\varepsilon, G} \leq C(\sigma, T) \|D^v f\|_{q-\varepsilon, G} \prod_{j \in e_n} |\gamma_j|^{\sigma_j}.$$

Estimating  $\|D^v f\|_{q-\varepsilon, G}$  by means of (2.4), we obtain the sought inequality in this case as well. The theorem is proved.

*Acknowledgement.* The authors would like to thank the referees for their valuable comments which helped to improve the manuscript.

## REFERENCES

- [1] T. I. AMANOV, *Representation and embedding theorems for function spaces SB*, Trudy Mat. Inst. Steklov., 77, Nauka, 1965, pp. 5–34.
- [2] O. V. BESOV, V. P. IL'YIN AND S. M. NIKOLSKII, *Integral representations of functions and embeddings theorems*, M. Nauka, 1996, 480 pp.
- [3] A. D. DJABRAILOV, *Families of spaces of function whose mixed derivatives satisfy a multiple-integral Hölder condition*, Trudy Mat.Inst. Steklov, 1972, v.117, pp. 139–158 (in Russian).
- [4] T. IWANIEC AND C. SBORDONE, *On the integrability of the Jacobian under minimal hypotheses*, Arch. Ration. Mech. Anal., 119, 1992, pp. 129–143.
- [5] A. FIORENZA AND C. E. KARADZHOV, *Grand and small Lebesgue spaces and their analogs*, J. Anal. Appl., vol. (23)(4) (2004), pp. 657–681.
- [6] V. KOKILASHVILI, *The Riemann boundary value problem for analytic functions in the frame of grand  $L^p$  spaces*, Bull. Georgian Nat. Acad. Sci., vol. 4, No 1, 2010, pp. 5–7.
- [7] V. KOKILASHVILI, A. MESKHI AND H. RAFEIRO, *Estimates for nondivergence elliptic equations with VMO coefficients in generalized grand Morrey spaces*, Comp. Var. Ellip. Equations, 8 (59), 2014, pp. 1169–1184.
- [8] V. KOKILASHVILI AND A. MESKHI, *Trace inequalities for fractional integrals in grand Lebesgue spaces*, Studia Math., 210(2), 2012, pp. 159–176.
- [9] A. MESKHI, *Maximal functions, potentials and singular integrals in grand Morrey spaces*, Comp. Var. Ellip. Equations, 2011, DOI: 10.1080/17476933: 2010, 534793.
- [10] Y. MIZUTA, T. OHNO, *Trudingers exponential integrability for Riesz potentials of function in generalized grand Morrey spaces*, J. Math. Anal. Appl., 420(1)(2014), pp. 268–278.
- [11] A. M. NADZHAFOV, *Embedding theorems in the Sobolev-Morrey type spaces  $S_{p,a,\alpha,\tau}^1 W(G)$  with dominant mixed derivatives*, Sib. Math. J., (47)(3), 2006, pp. 505–516.
- [12] A. M. NAJAFOV, *Some properties of function from the intersection of Besov-Morrey type spaces with dominant mixed derivatives*, Proc. of A. Ramzmadze Math. Inst., 2005, v.139, pp. 71–82.
- [13] A. M. NAJAFOV AND N. R. RUSTAMOVA, *Some differential properties of anisotropic grand Sobolev-Morrey spaces*, Trans. of A. Razmadze Mathematical Institute Vol. 172, Issue 1, 2018, pp. 82–89.
- [14] A. M. NAJAFOV, *The differential properties of functions from Sobolev-Morrey type spaces of fractional order*, Journal of Mathematical Research, v.7, no 3, 2015, pp. 1–10.
- [15] S. M. NIKOLSKII, *Functions with dominant mixed derivative satisfying a multiple Hölder condition*, Sib.M.Jour., 1963, v.4(6), pp. 1342–1364.
- [16] H. RAFEIRO, *A note on boundedness of operators in grand Morrey spaces*, Arxiv: 1109, 2550 v1 [math. FA], 2011.
- [17] S. G. SAMKO AND S. M. UMARKHADZHIEV, *On Iwaniec - Sbordone spaces on sets which may have infinite measure*, Azerb. Jour. Math. 2011, vol. 1, No 1, pp. 67–84, v.1, no 2, pp. 143–144.
- [18] C. SBORDONE, *Grand Sobolev spaces and their applications to variational problems*, Le Matematiche, vol. LI, 1996, Fasc. II, pp. 335–347.
- [19] S. M. UMARKHADZHIEV, *The boundedness of the Riesz potential operator from generalized grand Lebesgue spaces to generalized grand Morrey spaces*, Operator Theory: Advances and Applications, Basel, 2014, vol.242, pp. 363–373.

(Received June 4, 2018)

Alik M. Najafov  
Azerbaijan University of Architecture and Construction  
Institute of Mathematics and Mechanics  
National Academy of Sciences of Azerbaijan  
Az-1141, Baku, Azerbaijan  
e-mail: aliknajafov@gmail.com

Sain T. Alekberli  
Baku Engineering University  
Baku, Azerbaijan  
e-mail: sain.elekberli@bk.ru