Journal of
Mathematical
Inequalities
Volume 14, Number 2 (2020), 473–481 doi:10.7153/jmi-2020-14-30

A NOTE ON GAUSSIAN INTEGRAL MEANS OF ENTIRE FUNCTIONS

HAIYING LI AND YANGYANG WANG

(*Communicated by S. Li*)

Abstract. In this paper, we investigate the convexity of function $r \mapsto \ln M_{2,1}(f(z),r)$ in $\ln r$ on $(0,\infty)$, where $f(z) = z^3 + C$.

1. Introduction

Let dA be the Euclidean area measure on the finite complex plane $\mathbb{C}$. Suppose α is real and $0 < p < \infty$. For any entire function $f : \mathbb{C} \mapsto \mathbb{C}$, we recall that the Gaussian integral means of f is defined by

$$M_{p,\alpha}(f,r) = \frac{\int_{|z|<r} |f(z)|^p e^{-\alpha|z|^2} dA(z)}{\int_{|z|<r} e^{-\alpha|z|^2} dA(z)}, \quad \forall r \in (0,\infty).$$

The Gaussian integral means is derived from the related theory of Fock spaces, see [1, 2, 3, 4].

In [1], suppose k is a nonnegative integer and $0 < p < \infty$. If $0 < \alpha < \infty$, then the function $r \mapsto \ln M_{p,\alpha}(z^k,r)$ is concave in $\ln r$. If $-\infty < \alpha \leqslant 0$, then there exists some c (depending on k and α) on $(0,\infty)$ such that the function $r \mapsto \ln M_{p,\alpha}(z^k,r)$ is convex in $\ln r$ on $(0,c]$ and concave in $\ln r$ on $[c,\infty)$.

Recall the Remark 9 in [1] we have that the integral means of all monomials are logarithmically concave when $\alpha > 0$. However, this is not true for all entire functions, even for linear mappings. And we have an instance in [1], and just choose $p = 2$, $\alpha = 1$ and $f(z) = a + z$ which proved $M_{2,1}(a+z,r)$ is logarithmically concave on $(\sqrt{\lambda},\infty)$, for any $a \in \mathbb{C}$.

In this paper, we consider the convexity of function $r \mapsto \ln M_{2,1}(f,r)$ in $\ln r$, when $f(z) = z^3 + C$. The main result of this paper is the following theorem.

THEOREM. *Suppose $p = 2$, $\alpha = 1$, $f(z) = z^3 + C$ and $|C|^2 = c$. There exists a point $P_1 \in (0,\infty)$ (depending on c) on $(0,\infty)$ such that the function $r \mapsto \ln M_{2,1}(f(z),r)$ is convex in $\ln r$ on $(0,P_1)$. And the function $r \mapsto \ln M_{2,1}(f(z),r)$ is concave in $\ln r$ on $(2\sqrt{2},\infty)$.*

Mathematics subject classification (2010): 30H10, 30H20.
Keywords and phrases: Gaussian integral means, entire function, logarithmic convexity.

© ELEMENT, Zagreb
Paper JMI-14-30

2. Preliminaries

The proof of the Theorem is elementary but very laborious, it requires several preliminary results which we present in this section. Throughout the paper we use the symbol $\equiv$ when a new notation is being introduced. We will also use the notation $A \sim B$ to mean that A and B have the same sign as a convention in [5].

In order to give the proof of our main theorem,we need the following three lemmas. The first one comes from [6]. For convenience, the proof is given.

LEMMA 1. *If we write every analytic function $f : \mathbb{C} \longmapsto \mathbb{C}$ in the form of a power series*

$$f(z) = \sum_{k=0}^{\infty} a_k z^k,$$

then we can immediately obtain that

$$\frac{1}{2\pi}\int_0^{2\pi} \left|f(re^{i\theta})\right|^2 d\theta = \sum_{k=0}^{\infty} |a_k|^2 r^{2k}.$$

Proof.

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi} \left|f(re^{i\theta})\right|^2 d\theta &= \frac{1}{2\pi}\int_0^{2\pi} f(re^{i\theta})\overline{f(re^{i\theta})}d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \sum_{k=0}^{\infty} a_k (re^{i\theta})^k \sum_{j=0}^{\infty} \overline{a_j}(\overline{re^{i\theta}})^j d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \sum_{k=0}^{\infty} a_k r^k e^{ik\theta} \sum_{j=0}^{\infty} \overline{a_j} r^j e^{-ij\theta} d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} a_k \overline{a_j} r^{k+j} e^{(k-j)i\theta} d\theta \\
&= \frac{1}{2\pi}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} a_k \overline{a_j} r^{k+j} \int_0^{2\pi} e^{(k-j)i\theta} d\theta = \sum_{k=0}^{\infty} |a_k|^2 r^{2k}.
\end{aligned}$$

This completes the proof of the lemma. □

The second lemma comes directly from [5] with $(0,1)$ being replaced by $(0,\infty)$. For completeness, we give the proof of this lemma.

LEMMA 2. (*i*) *Suppose f is twice differentiable on $(0,\infty)$. Then $f(x)$ is convex in $\ln x$ if and only if $f'(x) + xf''(x) \geqslant 0$ on $(0,\infty)$.*

(*ii*) *Suppose f is twice differentiable on $(0,\infty)$. Then $f(x)$ is convex in $\ln x$ if and only if $f(x^2)$ is convex in $\ln x$ and $f(x)$ is concave in $\ln x$ if and only if $f(x^2)$ is concave in $\ln x$.*

(iii) *Suppose f is positive twice differentiable on $(0,\infty)$. Let*

$$D(f(x)) \equiv \frac{f'(x)}{f(x)} + x\frac{f''(x)}{f(x)} - x\left(\frac{f'(x)}{f(x)}\right)^2.$$

Then the function $\ln f(x)$ is convex in $\ln x$ if and only if $D(f(x)) \geqslant 0$ on $(0,\infty)$ and the function $\ln f(x)$ is concave in $\ln x$ if and only if $D(f(x)) \leqslant 0$ on $(0,\infty)$.

Proof. (i) Let $t = \ln x$, It follows easily from the Chain Rule that

$$\frac{d^2 f}{dt^2} = x[f'(x) + xf''(x)].$$

Thus f is convex in $\ln x$ if and only if $f'(x) + xf''(x) \geqslant 0$ on $(0,\infty)$.

(ii) For the function $g(x) = f(x^2)$, we easily compute that

$$g'(x) + xg''(x) = 4x[f'(x^2) + x^2 f''(x^2)].$$

The desired result then follows from (i).

(iii) Apply (i) to the function $g(x) = \ln f(x)$. The desired result follows immediately.

The third lemma comes from [7] and [8].

LEMMA 3. *If $f = \frac{f_1}{f_2}$ is a quotient of two positive and twice differentiable functions f_1 and f_2 on $(0,\infty)$, then*

$$D(f(x)) = D(f_1(x)) - D(f_2(x)), \quad \forall x \in (0,\infty).$$

3. Proof of Theorem

This section is devoted to the proof of the theorem.

Proof. Let $p = 2$, $\alpha = 1$ and $f(z) = z^3 + C$. According to

$$M_{p,\alpha}(f,r) = \frac{\int_{|z|<r} |f(z)|^p e^{-\alpha|z|^2} dA(z)}{\int_{|z|<r} e^{-\alpha|z|^2} dA(z)}, \forall r \in (0,\infty),$$

by polar coordinates, an obvious change of variables and Lemma 1, we have

$$\begin{aligned} M_{2,1}(f,r) &= \frac{\int_{|z|<r} |z^3 + C|^2 e^{-|z|^2} dA(z)}{\int_{|z|<r} e^{-|z|^2} dA(z)} \\ &= \frac{\int_0^r (\rho^6 + |C|^2) e^{-\rho^2} \rho d\rho}{\int_0^r e^{-\rho^2} \rho d\rho} \\ &= \frac{\int_0^r (\rho^6 + |C|^2) e^{-\rho^2} d\rho^2}{\int_0^r e^{-\rho^2} d\rho^2} \end{aligned}$$

$$= \frac{\int_0^x (t^3 + |C|^2) e^{-t} dt}{\int_0^x e^{-t} dt}.$$

Let $|C|^2 = c > 0$, we rewrite

$$\frac{\int_0^x (t^3 + c) e^{-t} dt}{\int_0^x e^{-t} dt} \equiv F(x).$$

Since

$$\int_0^x e^{-t} dt = 1 - e^{-x},$$

$$\int_0^x t^3 e^{-t} dt = -e^{-x}(x^3 + 3x^2 + 6x + 6) + 6,$$

so

$$\begin{aligned}\int_0^x (t^3 + c) e^{-t} dt &= \int_0^x t^3 e^{-t} dt + \int_0^x c e^{-t} dt \\ &= -e^{-x}(x^3 + 3x^2 + 6x + 6) + 6 + c(1 - e^{-x}) \\ &= -e^{-x}(x^3 + 3x^2 + 6x + 6 + c) + (6 + c).\end{aligned}$$

Therefore

$$F(x) = \frac{-e^{-x}(x^3 + 3x^2 + 6x + 6 + c) + (6 + c)}{1 - e^{-x}} \equiv \frac{g(x)}{h(x)}.$$

We immediately obtain

$$g'(x) = e^{-x}(x^3 + c),$$

$$g''(x) = e^{-x}(-x^3 + 3x^2 - c),$$

$$\begin{aligned}g'(x)g(x) = &-e^{-2x}(x^6 + 3x^5 + 6x^4 + (6 + 2c)x^3 + 3cx^2 + 6cx + 6c + c^2) \\ &+ e^{-x}((6 + c)x^3 + 6c + c^2),\end{aligned}$$

$$\begin{aligned}xg''(x)g(x) = &-xe^{-2x}(-x^6 + 3x^4 + (12 - 2c)x^3 + 18x^2 - 6cx - (6c + c^2)) \\ &+ xe^{-x}(-(6 + c)x^3 + (18 + 3c)x^2 - (6c + c^2)),\end{aligned}$$

$$x(g'(x))^2 = xe^{-2x}(x^6 + 2cx^3 + c^2),$$

$$\begin{aligned}g^2(x) = &\, e^{-2x}(x^6 + 21x^4 + 6x^5 + (48 + 2c)x^3 + (72 + 6c)x^2 + (72 + 12c)x \\ &+ (36 + 12c + c^2)) - e^{-x}((12 + 2c)x^3 + (36 + 6c)x^2 + (72 + 12c)x \\ &+ (72 + 24c + 2c^2)) + (36 + 12c + c^2).\end{aligned}$$

Employing the D-notation in Lemma 2 (iii), we have

$$\begin{aligned} D(g(x)) &= \frac{g'(x)}{g(x)} + x\frac{g''(x)}{g(x)} - x\left(\frac{g'(x)}{g(x)}\right)^2 \\ &= \frac{g'(x)g(x) + xg''(x)g(x) - x(g'(x))^2}{g^2(x)} \\ &= \frac{m(x)}{g^2(x)}, \end{aligned}$$

where

$$\begin{aligned} m(x) = {} & e^{-2x}(-x^6 - 6x^5 - 18x^4 - (24+2c)x^3 + 3cx^2 - (6c+c^2)) \\ & + e^{-x}(-(6+c)x^4 + (24+4c)x^3 - (6c+c^2)x + (6c+c^2)). \end{aligned}$$

From the following calculations

$$\begin{aligned} & h(x) = 1 - e^{-x}, h'(x) = e^{-x}, h''(x) = -e^{-x}, h(x)h'(x) = e^{-x} - e^{-2x}, \\ & h''(x)h(x) = -e^{-x} + e^{-2x}, (h'(x))^2 = e^{-2x}, h^2(x) = 1 - 2e^{-x} + e^{-2x}, \end{aligned}$$

we can get

$$\begin{aligned} D(h(x)) &= \frac{h'(x)}{h(x)} + x\frac{h''(x)}{h(x)} - x\left(\frac{h'(x)}{h(x)}\right)^2 \\ &= \frac{h'(x)h(x) + xh''(x)h(x) - x(h'(x))^2}{h^2(x)} \\ &= \frac{e^{-x}(1 - x - e^{-x})}{h^2(x)}. \end{aligned}$$

Hence, by Lemma 3,

$$\begin{aligned} D(F(x)) &= D(g(x)) - D(h(x)) \\ &= \frac{m(x)(1 - 2e^{-x} + e^{-2x}) - e^{-x}(1 - x - e^{-x})g^2(x)}{g^2(x)h^2(x)} \\ &\sim m(x)(1 - 2e^{-x} + e^{-2x}) - e^{-x}(1 - x - e^{-x})g^2(x) \\ &\equiv n(x). \end{aligned}$$

By calculation, we have

$$\begin{aligned} n(x) = {} & e^{-4x}(3x^4 + 24x^3 + (72+9c)x^2 + (72+12c)x + (36+6c)) \\ & + e^{-3x}(x^7 + 7x^6 + 27x^5 + (57+c)x^4 + (84+10c)x^3 \\ & - (36+6c)x^2 - (108+18c)x - (108+18c)) + e^{-2x}(-x^6 \\ & - 6x^5 - 18x^4 - (96+14c)x^3 - (36+3c)x^2 + (108+18c)) \\ & + e^{-x}(-(6+c)x^4 + (24+4c)x^3 + (36+6c)x - (36+6c)). \end{aligned}$$

We can get

$$n(x) \sim n(x) \cdot e^{4x} \equiv G(x).$$

Therefore,

$$\begin{aligned}
G'(x) = {} & 12x^3 + 72x^2 + (144+18c)x + (72+12c) + e^x(x^7 + 14x^6 \\
& +69x^5 + (192+c)x^4 + (312+14c)x^3 + (216+24c)x^2 \\
& -(180+30c)x - (216+36c)) + e^{2x}(-2x^6 - 18x^5 \\
& -66x^4 - (264+28c)x^3 - (360+48c)x^2 - (72+6c)x \\
& +(216+36c)) + e^{3x}(-(18+3c)x^4 + (48+8c)x^3 \\
& +(72+12c)x^2 + (108+18c)x - (72+12c)),
\end{aligned}$$

$$\begin{aligned}
G''(x) = {} & 36x^2 + 144x + (144+18c) + e^x(x^7 + 21x^6 + 153x^5 + (537+c)x^4 \\
& +(1080+18c)x^3 + (1152+66c)x^2 + (252+18c)x - (396+66c)) \\
& +e^{2x}(-4x^6 - 48x^5 - 222x^4 - (792+56c)x^3 - (1512+180c)x^2 \\
& -(864+108c)x + (360+66c)) + e^{3x}(-(54+9c)x^4 \\
& +(72+12c)x^3 + (360+60c)x^2 + (468+78c)x \\
& -(108+18c)),
\end{aligned}$$

$$\begin{aligned}
G'''(x) = {} & 72x + 144 + e^x(x^7 + 28x^6 + 279x^5 + (1302+c)x^4 \\
& +(3228+22c)x^3 + (4392+120c)x^2 + (2556+150c)x - (144+48c)) \\
& +e^{2x}(-8x^6 - 120x^5 - 684x^4 - (2472+112c)x^3 - (5400+528c)x^2 \\
& -(4752+576c)x - (144-24c)) + e^{3x}(-(162+27c)x^4 + (1296+216c)x^2 \\
& +(2124+354c)x + (144+24c)),
\end{aligned}$$

$$\begin{aligned}
G^{(4)}(x) = {} & 72 + e^x(x^7 + 35x^6 + 447x^5 + (2697+c)x^4 + (8436+26c)x^3 \\
& +(14076+186c)x^2 + (11340+390c)x + (2412+102c)) \\
& +e^{2x}(-16x^6 - 288x^5 - 1968x^4 - (7680+224c)x^3 \\
& -(18216+1392c)x^2 - (20304+2208c)x - (5040+528c)) \\
& +e^{3x}(-(486+81c)x^4 - (648+108c)x^3 + (3888+648c)x^2 \\
& +(8964+1494c)x + (2556+426c)).
\end{aligned}$$

$$\begin{aligned}
G^{(5)}(x) = {} & e^x(x^7 + 42x^6 + 657x^5 + (4932+c)x^4 + (19224+30c)x^3 \\
& +(39384+264c)x^2 + (39492+762c)x + (13752+492c)) \\
& +e^{2x}(-32x^6 - 672x^5 - 5376x^4 - (23232+448c)x^3 \\
& -(59472+3456c)x^2 - (77040+7200c)x - (30384+3264c)) \\
& +e^{3x}(-(1458+243c)x^4 - (3888+648c)x^3 + (9720+1620c)x^2
\end{aligned}$$

$$+(34668+5778c)x+(16632+2772c)).$$

Since

$$G^{(5)}(x) \sim G^{(5)}(x)\cdot e^{-x} \equiv H(x),$$

then

$$\begin{aligned} H'(x) = {} & 7x^6+252x^5+3285x^4+(19728+4c)x^3+(57672+90c)x^2 \\ & +(78768+528c)x+(39492+762c)+e^x(-32x^6-864x^5-8736x^4 \\ & -(44736+448c)x^3-(129168+4800c)x^2-(195984+14112c)x \\ & -(107424+10464c))+e^{2x}(-(2916+486c)x^4-(13608+2268c)x^3 \\ & +(7776-1296c)x^2+(88776+14796c)x+(67932+11322c)). \end{aligned}$$

After calculation,

$$G(0)=G'(0)=G''(0)=G'''(0)=G^{(4)}(0)=G^{(5)}(0)=H(0)=0,$$

$$H'(0)=39492+726c-(107424+10464c)+67932+11322c=1620c.$$

Since $H'(0)>0$, there exists a point $x_1\in(0,\infty)$ such that $H'(x)>0$ for $x\in(0,x_1)$, therefore $H(x)$ is increasing on $x\in(0,x_1)$. Since $H(0)=0$, there exists a point $x_2\in(x_1,\infty)$ such that $H(x)>0$ for $x\in(0,x_2)$, that is, $G^{(5)}(x)>0$ for $x\in(0,x_2)$, therefore, $G^{(4)}(x)$ is increasing on $x\in(0,x_2)$. Since $G^{(4)}(0)=G'''(0)=G''(0)=G'(0)=G(0)=0$, it have same properties. So, there exists a point $x^*\in(x_2,\infty)$ such that $G(x)>0$ for $x\in(0,x^*)$, that is, $n(x)>0$ for $x\in(0,x^*)$. Thus we can obtain $D(F(x))>0$ for $r\in(0,x^*)$, which means that there exists a point $P_1\in(0,\infty)$ (depending on c), function $r\mapsto \ln M_{2,1}((z^3+C),r)$ is convex in $\ln r$ for $r\in(0,P_1)$.

Next, we consider the case of bigger r. Since $x=r^2$, that is the case of bigger x. We change $n(x)$ into $n_1(x)$

$$n_1(x)=e^{-4x}a_1+e^{-3x}a_2+e^{-2x}a_3+e^{-x}a_4,$$

where
$a_1=3x^4+24x^3+(72+9c)x^2+(72+12c)x+(36+6c)$,

$$\begin{aligned} a_2 = {} & x^7+7x^6+27x^5+(57+c)x^4+(84+10c)x^3-(36+6c)x^2-(108+18c)x \\ & -(108+18c), \end{aligned}$$

$a_3=-x^6-6x^5-18x^4-(96+14c)x^3-(36+3c)x^2+(108+18c)$,
$a_4=-(6+c)x^4+(24+4c)x^3+(36+6c)x-(36+6c)$.

Suppose $x\geqslant 1$, since e^x is monotone increasing on $x\in[1,\infty)$, then $e^{-4x}\leqslant e^{-3x}\leqslant e^{-2x}\leqslant e^{-x}$. Since $c\geqslant 0$, $x\leqslant e^x=1+x+\cdots$, we have $xe^{-3x}\leqslant e^{-2x}$, $xe^{-2x}\leqslant e^{-x}$.

$$\begin{aligned} n(x) &= e^{-4x}a_1+e^{-3x}a_2+e^{-2x}a_3+e^{-x}a_4 \\ &\leqslant e^{-3x}(x^7+7x^6+27x^5+(60+c)x^4+(108+10c)x^3+(36+3c)x^2 \end{aligned}$$

$$
\begin{aligned}
&-(36+6c)x-(72+12c))+e^{-2x}a_3+e^{-x}a_4\\
\leqslant\ & xe^{-3x}(x^6+7x^5+27x^4+(60+c)x^3+(108+10c)x^2+(36+3c)x)\\
&+e^{-2x}a_3+e^{-x}a_4\\
\leqslant\ & e^{-2x}(x^5+9x^4-(36+13c)x^3+(72+7c)x^2+(36+3c)x\\
&+(108+18c))+e^{-x}a_4\\
\leqslant\ & e^{-2x}(x^5+9x^4+(72+7c)x^2+(36+3c)x+(108+18c))+e^{-x}a_4\\
\leqslant\ & xe^{-2x}(x^4+9x^3+(72+7c)x+(144+21c))+e^{-x}a_4\\
\leqslant\ & e^{-x}(-(5+c)x^4+(33+4c)x^3+(108+13c)x+(108+15c))\\
\sim\ & -(5+c)x^4+(33+4c)x^3+(108+13c)x+(108+15c)\equiv N(x).
\end{aligned}
$$

Also we have

$$
\begin{aligned}
N'(x) &= -(20+4c)x^3+(99+12c)x^2+(108+13c),\\
N''(x) &= -(60+12c)x^2+(198+24c)x,\\
N'''(x) &= -(120+24c)x+(198+24c),\\
N^{(4)}(x) &= -(120+24c).
\end{aligned}
$$

Since $N^{(4)}(x)<0$, $N'''(x)$ is monotone decreasing on $x\in(1,\infty)$. We notice that $N'''(2)=-240-48c+198+24c=-42-24c<0$, so we can get $N'''(x)<0$ on $x\in(2,\infty)$, so $N''(x)$ is monotone decreasing on $x\in(2,\infty)$. When $x=4$, $N''(4)=-168-96c<0$, so $N''(x)<0$ on $x\in(4,\infty)$, that is, $N'(x)$ is monotone decreasing on $x\in(4,\infty)$. When $x=6$, $N'(6)=-648-419c<0$, so $N'(x)<0$ on $x\in(6,\infty)$, that is, $N(x)$ is monotone decreasing on $x\in(6,\infty)$. When $x=8$, $N(8)=-2612-1929c<0$, so $N(x)<0$ on $x\in(8,\infty)$, that is, the function $r\mapsto \ln M_{2,1}((z^3+C),r)$ is concave in $\ln r$ for $r\in(2\sqrt{2},\infty)$.

Acknowledgement. This work was supported by the National Natural Science Foundation of China (No.11771126) and the Key Scientific Research Foundation of the Higher Education Institutions of Henan Province (No. 18A110022).

REFERENCES

[1] C. WANG, J. XIAO, *Gaussian integral means of entire functions*, Complex. Anal. Oper. Theory, **8**(2014), 1487–1505.

[2] H. LI, T. LIU, *Convexities of Gaussian integral means and weighted integral means for analytic functions*, Czechoslovak Math. J., **69**(2019), 525–543.

[3] K. ZHU, *Analysis on Fock spaces*, Springer, New York, 2012.

[4] O. CONSTANTIN, *A Volterra-type intergration operator on Fock spaces*, Proc. Amer. Math. Soc., **140**(2012), 4247–4257.

[5] C. WANG, K. ZHU, *Logarithmic convexity of area integral means for analytic functions*, Math. Scand., **114**(2014), 149–160.

[6] P. DUREN, *Theory of H^p Space*, Academic Press, 1970.

[7] C. WANG, J. XIAO, *Addendum to "Gaussian integral means of entire functions"*, Complex. Anal. Oper. Theory, **10**(2016), 495–503.
[8] C. WANG, J. XIAO, K. ZHU, *Logarithmic convexity of area integral means for analytic functions II*, J. Aust. Math. Soc., **98**(2015), 117–128.

(Received July 26, 2018)

Haiying Li
Henan Engineering Laboratory for Big Data Statistical Analysis and Optimal Control
School of Mathematics and Information Science, Henan Normal University
Xinxiang 453007
e-mail: haiyingli2012@yahoo.com

Yangyang Wang
Henan Engineering Laboratory for Big Data Statistical Analysis and Optimal Control
School of Mathematics and Information Science, Henan Normal University
Xinxiang 453007
e-mail: YYWang601@163.com