

A GENERALIZATION OF THE BOUNDEDNESS OF CERTAIN INTEGRAL OPERATORS IN VARIABLE LEBESGUE SPACES

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Abstract. Let $n \in \mathbb{N}$. Let A_1, \dots, A_m be $n \times n$ invertible matrices. Let $0 \leq \alpha < n$ and $0 < \alpha_i < n$ such that $\alpha_1 + \dots + \alpha_m = n - \alpha$. We define

$$T_\alpha f(x) = \int \frac{1}{|x - A_1 y|^{\alpha_1} \dots |x - A_m y|^{\alpha_m}} f(y) dy.$$

In [8] we obtained the boundedness of this operator from $L^{p(\cdot)}(\mathbb{R}^n)$ into $L^{q(\cdot)}(\mathbb{R}^n)$ for $\frac{1}{q(\cdot)} = \frac{1}{p(\cdot)} - \frac{\alpha}{n}$, in the case that A_i is a power of certain fixed matrix A and for exponent functions p satisfying log-Hölder conditions and $p(Ay) = p(y)$, $y \in \mathbb{R}^n$. We will show now that the hypothesis on p , in certain cases, is necessary for the boundedness of T_α and we also prove the result for more general matrices A_i .

1. Introduction

Let $n \in \mathbb{N}$. Given a measurable function $p(\cdot) : \mathbb{R}^n \rightarrow [1, \infty)$, let $L^{p(\cdot)}(\mathbb{R}^n)$ be the Banach space of measurable functions f on \mathbb{R}^n such that for some $\lambda > 0$,

$$\int \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} dx < \infty,$$

with norm

$$\|f\|_{p(\cdot)} = \inf \left\{ \lambda > 0 : \int \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} dx \leq 1 \right\}.$$

These spaces are known as *variable exponent spaces* and are a generalization of the classical Lebesgue spaces $L^p(\mathbb{R}^n)$. They have been widely studied lately. See for example [1], [3] and [4]. The first step was to determine sufficient conditions on $p(\cdot)$ for the boundedness on $L^{p(\cdot)}$ of the Hardy Littlewood maximal operator

$$\mathcal{M}f(x) = \sup_B \frac{1}{|B|} \int_B |f(y)| dy,$$

where the supremum is taken over all balls B containing x . Let $p_- = \text{ess inf } p(x)$ and let $p_+ = \text{ess sup } p(x)$. In [3], D. Cruz Uribe, A. Fiorenza and C. J. Neugebauer proved the following result.

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THEOREM 1. Let $p(\cdot) : \mathbb{R}^n \rightarrow [1, \infty)$ be such that $1 < p_- \leq p_+ < \infty$. Suppose further that $p(\cdot)$ satisfies

$$|p(x) - p(y)| \leq \frac{c}{-\log|x-y|}, \quad |x-y| < \frac{1}{2}, \tag{1}$$

and

$$|p(x) - p(y)| \leq \frac{c}{\log(e+|x|)}, \quad |y| \geq |x|. \tag{2}$$

Then the Hardy Littlewood maximal operator is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.

We recall that a weight ω is a locally integrable and non negative function. The Muckenhoupt class \mathcal{A}_p , $1 < p < \infty$, is defined as the class of weights ω such that

$$\sup_Q \left[\left(\frac{1}{|Q|} \int_Q \omega \right) \left(\frac{1}{|Q|} \int_Q \omega^{-\frac{1}{p-1}} \right)^{p-1} \right] < \infty,$$

where Q is a cube in \mathbb{R}^n .

For $p = 1$, \mathcal{A}_1 is the class of weights ω satisfying that there exists $c > 0$ such that

$$\mathcal{M}\omega(x) \leq c\omega(x) \text{ a.e. } x \in \mathbb{R}^n.$$

We denote $[\omega]_{\mathcal{A}_1}$ the infimum of the constant c such that ω satisfies the above inequality.

In [5], B. Muckenhoupt y R.L. Wheeden define $\mathcal{A}(p, q)$, $1 < p < \infty$ and $1 < q < \infty$, as the class of weights ω such that

$$\sup_Q \left[\left(\frac{1}{|Q|} \int_Q \omega(x)^q dx \right)^{\frac{1}{q}} \left(\frac{1}{|Q|} \int_Q \omega(x)^{-p'} dx \right)^{\frac{1}{p'}} \right] < \infty.$$

When $p = 1$, $\omega \in \mathcal{A}(1, q)$ if only if

$$\sup_Q \left[\|\omega^{-1}\chi_Q\|_\infty \left(\frac{1}{|Q|} \int_Q \omega(x)^q dx \right)^{\frac{1}{q}} \right] < \infty.$$

Let $0 \leq \alpha < n$. For $1 \leq i \leq m$, let $0 < \alpha_i < n$, be such that

$$\alpha_1 + \dots + \alpha_m = n - \alpha.$$

Let T_α be the positive integral operator given by

$$T_\alpha f(x) = \int k(x, y) f(y) dy, \tag{3}$$

where

$$k(x, y) = \frac{1}{|x - A_1 y|^{\alpha_1}} \dots \frac{1}{|x - A_m y|^{\alpha_m}},$$

and where the matrices A_i are certain invertible matrices such that $A_i - A_j$ is invertible for $i \neq j$, $1 \leq i, j \leq m$.

In the paper [7] the authors studied this kind of integral operators and they obtained weighted (p, q) estimates, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, for weights $w \in A(p, q)$ such that $w(A_i x) \leq cw(x)$. In [8] we use extrapolation techniques to obtain $p(\cdot) - q(\cdot)$ and weak type estimates, in the case where $A_i = A^i$, for some invertible matrix A such that $A^N = I$, for some $N \in \mathbb{N}$. This technique allows us to replace the log-Hölder conditions about the exponent $p(\cdot)$ by a more general hypothesis concerning the boundedness of the maximal function \mathcal{M} . We obtain the following results.

THEOREM 2. *Let A be an invertible matrix such that $A^N = I$, for some $N \in \mathbb{N}$, let T_α be the integral operator given by (3), where $A_i = A^i$ and such that $A_i - A_j$ is invertible for $i \neq j$, $1 \leq i, j \leq m$. Let $p: \mathbb{R}^n \rightarrow [1, \infty)$ be such that $1 < p_- \leq p_+ < \frac{n}{\alpha}$ and such that $p(Ax) = p(x)$ a.e. $x \in \mathbb{R}^n$. Let $q(\cdot)$ be defined by $\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{\alpha}{n}$. If the maximal operator \mathcal{M} is bounded on $L^{\left(\frac{n-\alpha p_-}{np_-} q(\cdot)\right)'}$ then T is bounded from $L^{p(\cdot)}(\mathbb{R}^n)$ into $L^{q(\cdot)}(\mathbb{R}^n)$.*

THEOREM 3. *Let A be an invertible matrix such that $A^N = I$, for some $N \in \mathbb{N}$, let T_α be the integral operator given by (3), where $A_i = A^i$ and such that $A_i - A_j$ is invertible for $i \neq j$, $1 \leq i, j \leq m$. Let $p: \mathbb{R}^n \rightarrow [1, \infty)$ be such that $1 \leq p_- \leq p_+ < \frac{n}{\alpha}$ and such that $p(Ax) = p(x)$ a.e. $x \in \mathbb{R}^n$. Let $q(\cdot)$ be defined by $\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{\alpha}{n}$. If the maximal operator \mathcal{M} is bounded on $L^{\left(\frac{n-\alpha p_-}{np_-} q(\cdot)\right)'}$ then there exists $c > 0$ such that*

$$\|t\mathcal{X}_{\{x: T_\alpha f(x) > t\}}\|_{q(\cdot)} \leq c \|f\|_{p(\cdot)}.$$

We also showed that this technique applies in the case when each of the matrices A_i is either a power of an orthogonal matrix A or a power of A^{-1} .

In this paper we will prove that these theorems generalize to any invertible matrices A_1, \dots, A_m such that $A_i - A_j$ is invertible for $i \neq j$, $1 \leq i, j \leq m$. We will also show, in some cases, that the condition $p(A_i x) = p(x)$, $x \in \mathbb{R}^n$ is necessary to obtain $p(\cdot) - q(\cdot)$ boundedness.

2. Necessary conditions on p

Let A be a $n \times n$ invertible matrix and let $0 < \alpha < n$. We define

$$T_A f(x) = \int \frac{1}{|x - Ay|^{n-\alpha}} f(y) dy.$$

PROPOSITION 4. *Let A be a $n \times n$ invertible matrix. Let $p: \mathbb{R}^n \rightarrow [1, \infty)$ be a measurable function such that p is continuous at y_0 and at Ay_0 for some $y_0 \in \mathbb{R}^n$. If $p(Ay_0) > p(y_0)$ then there exists $f \in L^{p(\cdot)}(\mathbb{R}^n)$ such that $T_A f \notin L^{q(\cdot)}(\mathbb{R}^n)$ for $\frac{1}{q(\cdot)} = \frac{1}{p(\cdot)} - \frac{\alpha}{n}$.*

Proof. Since p is continuous at y_0 , there exists ball $B = B(y_0, r)$ such that $p(y) \sim p(y_0)$ for $y \in B$. We have that $p(y_0) < p(Ay_0)$. In this case we take

$$f(y) = \frac{\chi_B(y)}{|y - y_0|^\beta},$$

for certain $\beta < \frac{n}{p(y_0)}$ that will be chosen later. We will show that, for such β , $f \in L^{p(\cdot)}(\mathbb{R}^n)$ but $T_A f \notin L^{q(\cdot)}(\mathbb{R}^n)$. Indeed,

$$T_A f(x) = \int \frac{1}{|x - Ay|^{n-\alpha}} f(y) dy = \int_B \frac{1}{|x - Ay|^{n-\alpha} |y - y_0|^\beta} dy,$$

so

$$\begin{aligned} \int (T_A f(x))^{q(x)} dx &= \int \left(\int_B \frac{1}{|x - Ay|^{n-\alpha} |y - y_0|^\beta} dy \right)^{q(x)} dx \\ &\geq \int_{B(Ay_0, \varepsilon)} \left(\int_B \frac{1}{|x - Ay|^{n-\alpha} |y - y_0|^\beta} dy \right)^{q(x)} dx \\ &\geq \int_{B(Ay_0, \varepsilon)} \left(\int_{B \cap \{y: |Ay - Ay_0| < |Ay_0 - x|\}} \frac{1}{|x - Ay|^{n-\alpha} |y - y_0|^\beta} dy \right)^{q(x)} dx. \end{aligned}$$

Now, we denote by $M = \|A\| = \sup_{\|y\|=1} |Ay|$. For $\varepsilon < Mr$ and $x \in B(Ay_0, \varepsilon)$, $B(y_0, \frac{1}{M}|Ay_0 - x|) \subset B \cap \{y : |Ay - Ay_0| < |Ay_0 - x|\}$. Indeed, $|y - y_0| \leq \frac{1}{M}|Ay_0 - x| \leq \frac{1}{M}\varepsilon \leq r$ and $|Ay - Ay_0| \leq M|y - y_0| \leq |Ay_0 - x|$, so

$$\geq \int_{B(Ay_0, \varepsilon)} \left(\int_{B(y_0, \frac{1}{M}|Ay_0 - x|)} \frac{1}{|x - Ay|^{n-\alpha} |y - y_0|^\beta} dy \right)^{q(x)} dx,$$

also, for $y \in B(y_0, \frac{1}{M}|Ay_0 - x|)$

$$|x - Ay| \leq |x - Ay_0| + |Ay_0 - Ay| \leq |x - Ay_0| + M|y_0 - y| \leq 2|x - Ay_0|,$$

so

$$\begin{aligned} &\geq \int_{B(Ay_0, \varepsilon)} \left(\frac{1}{2^{n-\alpha} |x - Ay_0|^{n-\alpha}} \right)^{q(x)} \left(\int_{B(y_0, \frac{1}{M}|Ay_0 - x|)} \frac{1}{|y - y_0|^\beta} dy \right)^{q(x)} dx \\ &= \int_{B(Ay_0, \varepsilon)} \left(\frac{1}{2^{n-\alpha} |x - Ay_0|^{n-\alpha}} \right)^{q(x)} (c|Ay_0 - x|^{-\beta+n})^{q(x)} dx \\ &= \int_{B(Ay_0, \varepsilon)} \left(\frac{c}{2^{n-\alpha} |x - Ay_0|^{\beta-\alpha}} \right)^{q(x)} dx. \end{aligned}$$

Now, since $q(Ay_0) > q(y_0)$, $q(Ay_0) - \gamma > q(y_0)$ for $\gamma = \frac{q(Ay_0) - q(y_0)}{2}$. We observe that if $\frac{1}{q(y_0)} = \frac{1}{p(y_0)} - \frac{\alpha}{n}$, for $\beta_0 = \frac{n}{p(y_0)}$, $(\beta_0 - \alpha)q(y_0) = \left(\frac{n}{p(y_0)} - \alpha\right)q(y_0) = n$, so since $q(Ay_0) - \gamma > q(y_0)$, we obtain that $\left(\frac{n}{p(y_0)} - \alpha\right)(q(Ay_0) - \gamma) > n$ and still $(\beta - \alpha) \cdot (q(Ay_0) - \gamma) > n$ for $\beta = \frac{n}{p(y_0)} - \frac{1}{2} \left(\frac{n}{p(y_0)} - \left(\alpha + \frac{n}{q(Ay_0) - \gamma}\right)\right)$. So $\beta = \frac{n}{p(y_0)}(1 - \delta)$ for some $\delta > 0$. Since q is continuous, we chose ε so that, for $x \in B(Ay_0, \varepsilon)$, $q(x) > q(Ay_0) - \gamma$ and $\frac{c}{2^{n-\alpha}|x-Ay_0|^{\beta-\alpha}} > 1$ so this last integral is bounded from below by

$$c \int_{B(Ay_0, \varepsilon)} \left(\frac{1}{|x - Ay_0|^{\beta - \alpha}} \right)^{q(Ay_0) - \gamma} dx = \infty.$$

For this β we chose r to obtain that the ball $B = B(y_0, r) \subset \left\{y : p(y) < \frac{p(y_0)}{1 - \delta}\right\}$. In this way we obtain that $f \in L^{p(\cdot)}(\mathbb{R}^n)$ but $T_A f \notin L^{q(\cdot)}(\mathbb{R}^n)$. \square

COROLLARY 5. *If $A^N = I$ for some $N \in \mathbb{N}$, p is continuous and T_A is bounded from $L^{p(\cdot)}$ into $L^{q(\cdot)}$, then $p(Ay) = p(y)$ for all $y \in \mathbb{R}^n$.*

Proof. We suppose that $p(Ay_0) < p(y_0)$. Since p is continuous in y_0 , by the last proposition,

$$p(Ay_0) < p(y_0) = p(A^N y_0) \leq p(A^{N-1} y_0) \leq \dots \leq p(Ay_0) = p(y_0)$$

which is a contradiction. \square

3. The main results

Given $0 \leq \alpha < n$, we recall that we are studying fractional type integral operators of the form

$$T_\alpha f(x) = \int k(x, y) f(y) dy, \quad (4)$$

$f \in L_c^\infty(\mathbb{R}^n)$, with a kernel

$$k(x, y) = \frac{1}{|x - A_1 y|^{\alpha_1}} \cdots \frac{1}{|x - A_m y|^{\alpha_m}},$$

$$\alpha_1 + \dots + \alpha_m = n - \alpha, \quad 0 < \alpha_i < n.$$

THEOREM 6. *Let $m \in \mathbb{N}$, let A_1, \dots, A_m be invertible matrices such that $A_i - A_j$ is invertible for $i \neq j$, $1 \leq i, j \leq m$. Let T_α be the integral operator given by (4), let $p : \mathbb{R}^n \rightarrow [1, \infty)$ be such that $1 \leq p_- \leq p_+ < \frac{n}{\alpha}$ and such that $p(A_i x) = p(x)$ a.e. $x \in \mathbb{R}^n$, $1 \leq i \leq m$. Let $q(\cdot)$ be defined by $\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{\alpha}{n}$. If the maximal operator \mathcal{M} is bounded on $L\left(\frac{n - \alpha p_-}{np_-}, q(\cdot)\right)'$ then there exists $c > 0$ such that*

$$\left\| \mathcal{M}_{\{x: T_\alpha f(x) > t\}} \right\|_{q(\cdot)} \leq c \|f\|_{p(\cdot)},$$

$f \in L_c^\infty(\mathbb{R}^n)$.

REMARK 7. With the hypothesis of Theorem 6, if $f \in L^{p(\cdot)}(\mathbb{R}^n)$, the integral in (4) converges a.e. $x \in \mathbb{R}^n$, we still call it $T_\alpha f(x)$ and we have that there exists $c > 0$ such that

$$\|\lambda \chi_{\{x: T_\alpha f(x) > \lambda\}}\|_{q(\cdot)} \leq c \|f\|_{p(\cdot)}, \quad f \in L^{p(\cdot)}(\mathbb{R}^n).$$

Proof. We take $f \geq 0$ and a sequence $f_n \in L^\infty_c(\mathbb{R}^n)$ such that $f_n(x) \nearrow f(x)$ a.e. $x \in \mathbb{R}^n$. Then $T_\alpha f_n(x) \nearrow T_\alpha f(x)$ a.e. $x \in \mathbb{R}^n$ and then

$$\chi_{\{x: T_\alpha f_n(x) > \lambda\}}(x) \rightarrow \chi_{\{x: T_\alpha f(x) > \lambda\}}(x),$$

and so by Fatou’s Lemma, (see Th. 2.61, p.46 [2])

$$\begin{aligned} \|\lambda \chi_{\{x: T_\alpha f(x) > \lambda\}}\|_{q(\cdot)} &= \|\liminf \lambda \chi_{\{x: T_\alpha f_n(x) > \lambda\}}\|_{q(\cdot)} \\ &\leq \liminf \|\lambda \chi_{\{x: T_\alpha f_n(x) > \lambda\}}\|_{q(\cdot)} \leq \liminf \|f_n\|_{p(\cdot)} \leq \|f\|_{p(\cdot)}. \end{aligned}$$

For general f , as usual, we write $f = f^+ - f^-$. \square

THEOREM 8. Let $m \in \mathbb{N}$, let A_1, \dots, A_m be invertible matrices such that $A_i - A_j$ is invertible for $i \neq j$, $1 \leq i, j \leq m$. Let T_α be the integral operator given by (4), let $p : \mathbb{R}^n \rightarrow [1, \infty)$ be such that $1 < p_- \leq p_+ < \frac{n}{\alpha}$ and such that $p(A_i x) = p(x)$ a.e. $x \in \mathbb{R}^n$, $1 \leq i \leq m$. Let $q(\cdot)$ be defined by $\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{\alpha}{n}$. If the maximal operator \mathcal{M} is bounded on $L^{(\frac{n-\alpha p_-}{np_-} q(\cdot))'}$ then T_α is bounded from $L^{p(\cdot)}(\mathbb{R}^n)$ into $L^{q(\cdot)}(\mathbb{R}^n)$.

4. Proofs of the main results

LEMMA 9. If $f \in L^1_{loc}(\mathbb{R}^n)$ and A an invertible $n \times n$ matrix then

$$\mathcal{M}(f \circ A)(x) \leq c(\mathcal{M}(f) \circ A)(x).$$

Proof. Indeed,

$$\mathcal{M}(f \circ A) = \sup_B \frac{1}{|B|} \int_B |(f \circ A)(y)| dy,$$

where the supremum is taken over all balls B containing x . By a change of variable we see that,

$$\frac{1}{|B|} \int_B |(f \circ A)(y)| dy = |\det(A^{-1})| \frac{1}{|B|} \int_{A(B)} |f(z)| dz,$$

where $A(B) = \{Ay : y \in B\}$. Now, if $y \in B = B(x_0, r)$ then $|Ay - Ax_0| \leq M|y - x_0| \leq Mr$, where $M = \|A\|$. That is $Ay \in \tilde{B} = B(Ax_0, Mr)$. So

$$\leq \frac{M^n |\det(A^{-1})|}{|\tilde{B}|} \int_{\tilde{B}} f(z) dz \leq M^n |\det(A^{-1})| \mathcal{M}f(Ax).$$

Therefore we obtain that,

$$\mathcal{M}(f \circ A) \leq c(\mathcal{M}(f) \circ A),$$

with $c = M^n |\det(A^{-1})|$. \square

Proof of Theorem 6. We take $f \in L_c^\infty(\mathbb{R}^n)$. In [7] (See page 459) the authors prove that there exists $c > 0$ such that,

$$\sup_{\lambda > 0} \lambda (\omega^{q_0} \{x : |T_\alpha f(x)| > \lambda\})^{\frac{1}{q_0}} \leq \sup_{\lambda > 0} \lambda \left(\omega^{q_0} \{x : \sum_{i=1}^m \mathcal{M}_\alpha f(A_i^{-1}x) > c\lambda\} \right)^{\frac{1}{q_0}}$$

for all $\omega \in \mathcal{A}_\infty$ and $f \in L_c^\infty(\mathbb{R}^n)$.

Let $F_\lambda = \lambda^{q_0} \chi_{\{x: |T_\alpha f(x)| > \lambda\}}$. The last inequality implies that,

$$\int_{\mathbb{R}^n} F_\lambda(x) \omega(x)^{q_0} dx \leq \sup_{\lambda > 0} \int_{\mathbb{R}^n} \lambda^{q_0} \chi_{\{x: \sum_{i=1}^m \mathcal{M}_\alpha f(A_i^{-1}x) > c\lambda\}} \omega(x)^{q_0} dx \quad (5)$$

for some $c > 0$ and for all $\omega \in \mathcal{A}_\infty$. Now by Proposition 2.18 in [2], if $\tilde{q}(\cdot) = \frac{q(\cdot)}{q_0}$,

$$\begin{aligned} \|\lambda \chi_{\{x: |T_\alpha f(x)| > \lambda\}}\|_{q(\cdot)}^{q_0} &= \|\lambda^{q_0} \chi_{\{x: |T_\alpha f(x)| > \lambda\}}\|_{\tilde{q}(\cdot)} \\ &= \|F_\lambda\|_{\tilde{q}(\cdot)} \leq c \sup_{\|h\|_{\tilde{q}(\cdot)'} = 1} \int_{\mathbb{R}^n} F_\lambda(x) h(x) dx. \end{aligned}$$

We define an iteration algorithm on $L^{\tilde{q}(\cdot)'}$ by

$$\mathcal{B}h(x) = \sum_{k=0}^{\infty} \frac{\mathcal{M}^k h(x)}{2^k \|\mathcal{M}\|_{\tilde{q}(\cdot)'}^k}, \quad (6)$$

where, for $k \geq 1$, \mathcal{M}^k denotes k iteration of the maximal operator \mathcal{M} and $\mathcal{M}^0(h) = |h|$. We will check that

- a) $|h(x)| \leq \mathcal{B}h(x)$ $x \in \mathbb{R}^n$,
- b) For all $j : 1, \dots, m$, $\|\mathcal{B}h \circ A_j\|_{\tilde{q}(\cdot)' } \leq c \|h\|_{\tilde{q}(\cdot)'}$,
- c) For all $j : 1, \dots, m$, $\mathcal{B}h^{\frac{1}{q_0}} \circ A_j \in \mathcal{A}(p_-, q_0)$

Indeed, a) is evident from the definition. To verify b),

$$\|\mathcal{B}h \circ A_j\|_{\tilde{q}(\cdot)' } \leq \sum_{k=0}^{\infty} \frac{\|\mathcal{M}^k h \circ A_j\|_{\tilde{q}(\cdot)' }}{2^k \|\mathcal{M}\|_{\tilde{q}(\cdot)' }^k}$$

and

$$\|\mathcal{M}^k h \circ A_j\|_{\tilde{q}(\cdot)'} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \left(\frac{\mathcal{M}^k h(A_j x)}{\lambda} \right)^{\tilde{q}(x)'} dx \leq 1 \right\}.$$

But, by a change of variable and using the hypothesis on the exponent,

$$\int_{\mathbb{R}^n} \left(\frac{\mathcal{M}^k h(A_j x)}{\lambda} \right)^{\tilde{q}(x)'} dx = |\det(A_j^{-1})| \int_{\mathbb{R}^n} \left(\frac{\mathcal{M}^k h(y)}{\lambda} \right)^{\tilde{q}(A_j^{-1}y)'} dy,$$

put $D = \max \{ |\det(A_j^{-1})|, j = 1 \dots m \}$

$$\leq D \int_{\mathbb{R}^n} \left(\frac{\mathcal{M}^k h(y)}{\lambda} \right)^{\tilde{q}(y)'} dy. \quad (7)$$

If $D \leq 1$,

$$\|\mathcal{M}^k h \circ A_j\|_{\tilde{q}(\cdot)'} \leq \|\mathcal{M}^k h\|_{\tilde{q}(\cdot)'}$$

So,

$$\|\mathcal{R}h \circ A_j\|_{\tilde{q}(\cdot)'} \leq \sum_{k=0}^{\infty} \frac{\|\mathcal{M}^k h(x)\|_{\tilde{q}(\cdot)'}}{2^k \|\mathcal{M}\|_{\tilde{q}(\cdot)'}^k} \leq \|h\|_{\tilde{q}(\cdot)'} \sum_{k=0}^{\infty} \frac{1}{2^k} = 2 \|h\|_{\tilde{q}(\cdot)'}$$

If $D > 1$ then from (7) it follows that

$$D \int_{\mathbb{R}^n} \left(\frac{\mathcal{M}^k h(y)}{\lambda} \right)^{\tilde{q}(y)'} dy = \int_{\mathbb{R}^n} \left(\frac{M^k h(y)}{\lambda C^{\frac{1}{\tilde{q}(y)'}}} \right)^{\tilde{q}(y)'} dy$$

and $D = \frac{1}{C}$ where $C = \min \{ |\det(A_j)|, j = 1 \dots m \}$. So,

$$\leq \int_{\mathbb{R}^n} \left(\frac{M^k h(y)}{\lambda C^{\frac{1}{\tilde{q}(y)'}}} \right)^{\tilde{q}(y)'} dy.$$

That is,

$$\int_{\mathbb{R}^n} \left(\frac{\mathcal{M}^k h(A_j x)}{\lambda} \right)^{\tilde{q}(x)'} dx \leq \int_{\mathbb{R}^n} \left(\frac{M^k h(x)}{\lambda C^{\frac{1}{\tilde{q}(x)'}}} \right)^{\tilde{q}(x)'} dx.$$

From this last inequality it follows that

$$\|\mathcal{M}^k h \circ A_j\|_{\tilde{q}(\cdot)'} \leq D^{\frac{1}{\tilde{q}(\cdot)'}} \|\mathcal{M}^k h\|_{\tilde{q}(\cdot)'}$$

and so b) is verified with $c = 2D^{\frac{1}{\tilde{q}(\cdot)'}}$. To see c), by Lemma 9,

$$\mathcal{M}(\mathcal{R}h^{\frac{1}{q_0}} \circ A_j)(x) \leq c \mathcal{M}(\mathcal{R}h^{\frac{1}{q_0}})(A_j x)$$

$\mathcal{R}h \in \mathcal{A}_1$ (see [2]) implies that $\mathcal{R}h^{\frac{1}{q_0}} \in \mathcal{A}_1$ and so,

$$\leq c \mathcal{R}h^{\frac{1}{q_0}}(A_j x) = c(\mathcal{R}h^{\frac{1}{q_0}} \circ A_j)(x).$$

Then c) follows since a weight $\omega \in \mathcal{A}_1$ implies that $\omega \in \mathcal{A}(p_-, q_0)$.

And so,

$$\begin{aligned} c \sup_{\|h\|_{\mathcal{A}'(\cdot)}=1} \int_{\mathbb{R}^n} F_\lambda(x) h(x) dx &\leq c \sup_{\|h\|_{\mathcal{A}'(\cdot)}=1} \int_{\mathbb{R}^n} F_\lambda(x) \mathcal{R}h(x) dx \\ &= c \sup_{\|h\|_{\mathcal{A}'(\cdot)}=1} \int_{\mathbb{R}^n} F_\lambda(x) (\mathcal{R}h^{\frac{1}{q_0}}(x))^{q_0} dx, \end{aligned}$$

and by (5), since $\mathcal{R}h^{\frac{1}{q_0}} \in \mathcal{A}(p_-, q_0)$ and $Rh \in \mathcal{A}_1 \subset \mathcal{A}_\infty$,

$$\leq c \sup_{\|h\|_{\mathcal{A}'(\cdot)}=1} \sup_{\lambda > 0} \int_{\mathbb{R}^n} \lambda^{q_0} \chi_{\{x: \sum_{i=1}^m \mathcal{M}_{\alpha f}(A_i^{-1}x) > c\lambda\}} (\mathcal{R}h^{\frac{1}{q_0}}(x))^{q_0} dx.$$

Since,

$$\left\{ x : \sum_{i=1}^m \mathcal{M}_{\alpha f}(A_i^{-1}x) > c\lambda \right\} \subseteq \bigcup_{i=1}^m \left\{ x : \mathcal{M}_{\alpha f}(A_i^{-1}x) > \frac{c\lambda}{m} \right\}$$

then,

$$\chi_{\{x: \sum_{i=1}^m \mathcal{M}_{\alpha f}(A_i^{-1}x) > c\lambda\}} \leq \sum_{i=1}^m \chi_{\{x: \mathcal{M}_{\alpha f}(A_i^{-1}x) > \frac{c\lambda}{m}\}},$$

so

$$\begin{aligned} &\leq c \sup_{\|h\|_{\mathcal{A}'(\cdot)}=1} \sup_{\lambda > 0} \sum_{i=1}^m \int_{\mathbb{R}^n} \lambda^{q_0} \chi_{\{x: \mathcal{M}_{\alpha f}(A_i^{-1}x) > \frac{c\lambda}{m}\}} (\mathcal{R}h^{\frac{1}{q_0}}(x))^{q_0} dx \\ &= c \sup_{\|h\|_{\mathcal{A}'(\cdot)}=1} \sup_{\lambda > 0} \sum_{i=1}^m \int_{\{x: \mathcal{M}_{\alpha f}(A_i^{-1}x) > \frac{c\lambda}{m}\}} \lambda^{q_0} (\mathcal{R}h^{\frac{1}{q_0}}(x))^{q_0} dx \\ &= c \sup_{\|h\|_{\mathcal{A}'(\cdot)}=1} \sup_{\lambda > 0} \sum_{i=1}^m \lambda^{q_0} |\det(A_i)| \int_{A_i^{-1}\{x: \mathcal{M}_{\alpha f}(A_i^{-1}x) > \frac{c\lambda}{m}\}} (\mathcal{R}h^{\frac{1}{q_0}}(A_i y))^{q_0} dy, \\ &\leq c \sup_{\|h\|_{\mathcal{A}'(\cdot)}=1} \sup_{\lambda > 0} \sum_{i=1}^m \lambda^{q_0} \int_{\{y: \mathcal{M}_{\alpha f}(y) > \frac{c\lambda}{m}\}} (\mathcal{R}h^{\frac{1}{q_0}}(A_i y))^{q_0} dy \\ &\leq c \sup_{\|h\|_{\mathcal{A}'(\cdot)}=1} \sup_{\lambda > 0} \sum_{i=1}^m \left(\int_{\mathbb{R}^n} |f(y)|^{p_-} (\mathcal{R}h^{\frac{p_-}{q_0}}(A_i y)) dy \right)^{\frac{q_0}{p_-}} \\ &= c \sup_{\|h\|_{\mathcal{A}'(\cdot)}=1} \sum_{i=1}^m \left(\int_{\mathbb{R}^n} |f(y)|^{p_-} (\mathcal{R}h^{\frac{p_-}{q_0}}(A_i y)) dy \right)^{\frac{q_0}{p_-}}. \end{aligned}$$

We denote by $\tilde{p}(\cdot) = \frac{p(\cdot)}{p_-}$. Hölder’s inequality, 2) and Proposition 2.18 in [2] and again the hypothesis about A_i and p give

$$\begin{aligned} \|\lambda \chi_{\{x:|T_\alpha f(x)|>\lambda\}}\|_{q(\cdot)}^{q_0} &\leq C \|f\|^{p_-} \sup_{\|h\|_{\tilde{q}(\cdot)'}=1} \sum_{j=1}^m \left\| \left(\mathcal{R}h^{\frac{p_-}{q_0}} \right) \circ A_j \right\|_{\tilde{p}(\cdot)}^{\frac{q_0}{p_-}} \\ &\leq \sup_{\|h\|_{\tilde{q}(\cdot)'}=1} C m \|f\|_{p(\cdot)}^{q_0} \|h\|_{\tilde{q}(\cdot)'} \leq C \|f\|_{p(\cdot)}^{q_0}. \end{aligned}$$

Now f is bounded and with compact support, so $T_\alpha f \in L^s(\mathbb{R}^n)$ for $\frac{n}{n-\alpha} < s < \infty$, (see Lemma 2.2 in [7]) thus $\|\lambda \chi_{\{x:|T_\alpha f(x)|>\lambda\}}\|_{q(\cdot)} < \infty$. \square

Proof of Theorem 7. In the paper [7] the authors obtain an estimate of the form

$$\int (T_\alpha f)^p(x)w(x)dx \leq c \sum_{j=1}^m \int (\mathcal{M}_\alpha f)^p(x)w(A_jx)dx, \tag{8}$$

for any $w \in \mathcal{A}_\infty$ and $0 < p < \infty$ (See the last lines of page 454 in [7]). We denote $\tilde{q}(\cdot) = \frac{q(\cdot)}{q_0}$, we define an iteration algorithm on $L^{\tilde{q}(\cdot)'}$ as in the last proof (see (6)). We have

- a) For all $x \in \mathbb{R}^n, |h(x)| \leq \mathcal{R}h(x)$,
- b) For all $j : 1, \dots, m, \|\mathcal{R}h \circ A_j\|_{\tilde{q}(\cdot)'} \leq c \|h\|_{\tilde{q}(\cdot)'}$,
- c) For all $j : 1, \dots, m, \mathcal{R}h^{\frac{1}{q_0}} \circ A_j \in \mathcal{A}(p_-, q_0)$.

We now take a bounded function f with compact support. So as in Theorem 5.24 in [2],

$$\begin{aligned} \|T_\alpha f\|_{q(\cdot)}^{q_0} &= \|(T_\alpha f)^{q_0}\|_{\tilde{q}(\cdot)} = C \sup_{\|h\|_{\tilde{q}(\cdot)'}=1} \int (T_\alpha f)^{q_0}(x)h(x)dx \\ &\leq C \sup_{\|h\|_{\tilde{q}(\cdot)'}=1} \int (T_\alpha f)^{q_0}(x)\mathcal{R}h(x)dx \\ &\leq C \sup_{\|h\|_{\tilde{q}(\cdot)'}=1} \sum_{j=1}^m \int (\mathcal{M}_\alpha f)^{q_0}(x)\mathcal{R}h(A_jx)dx \\ &\leq C \sup_{\|h\|_{\tilde{q}(\cdot)'}=1} \sum_{j=1}^m \left(\int |f(x)|^{p_-} \mathcal{R}h^{\frac{p_-}{q_0}}(A_jx)dx \right)^{\frac{q_0}{p_-}}, \end{aligned}$$

where the last inequality follows since $\mathcal{R}h^{\frac{1}{q_0}} \circ A_i$ are weights in $\mathcal{A}(p_-, q_0)$ (by c)). Now, following as in the last proof,

$$\leq C \|f\|_{p(\cdot)}^{q_0}.$$

Also, as in the last proof, we show that $\|T_{\alpha}f\|_{q(\cdot)} < \infty$. The theorem follows since bounded functions with compact support are dense in $L^{p(\cdot)}(\mathbb{R}^n)$ (See Corollary 2.73 in [2]). \square

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