

INHOMOGENEOUS MULTI-PARAMETER BESOV AND TRIEBEL-LIZORKIN SPACES ASSOCIATED WITH DIFFERENT HOMOGENEITIES AND BOUNDEDNESS OF COMPOSITION OPERATORS

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Abstract. In this paper, the author establishes inhomogeneous multi-parameter Besov and Triebel-Lizorkin spaces associated with different homogeneities. Moreover, the boundedness of the composition of two inhomogeneous Calderón-Zygmund singular integrals of order (ε, σ) with different homogeneities is obtained.

1. Introduction and statement of main results

The new multi-parameter function spaces associated with the underlying mixed homogeneity arising from the two singular integral operators attracted many authors' attentions, which is motivated by Phong and Stein's work in [9]. Recently, the weighted Besov spaces and Triebel-Lizorkin spaces associated with different homogeneities were introduced in [16]. Weighted Carleson measure spaces associated with different homogeneities were considered in [17]. Y-C. Han and Y-S. Han [3] introduced a new class of Lipschitz spaces associated with different homogeneities and characterized these spaces via the Littlewood-Paley theory.

On the other hand, inhomogeneous function spaces have their own interests. For instance, the principle that Hardy space H^p is like Lebesgue space L^p when $0 < p < 1$ breaks down at a number of key points namely: (1) Hardy space H^p does not contain Schwartz space \mathcal{S} ; (2) Hardy space H^p is not semi-local; (3) pseudo-differential operators are not bounded on Hardy spaces. To circumvent those drawbacks, Goldberg [2] introduced inhomogeneous Hardy spaces h^p , $0 < p < \infty$. In many applications, moreover, it does not work to use the homogeneous spaces \mathcal{E}^s rather than the inhomogeneous Hölder spaces \mathcal{E}^s . For example, the continuity property of pseudo-differential operators $T \in \mathcal{O}pS_{1,0}^m$ (whose symbols fulfilling $|D_\xi^\alpha D_x^\beta \sigma(x, \xi)| \leq C(\alpha, \beta)(1 + |\xi|_e)^{m-|\alpha|}$) in the inhomogeneous Hölder spaces \mathcal{E}^s is considered in [11]. Also, $T \in \mathcal{O}pS_{1,1}^0$ (whose symbols satisfy that $|D_\xi^\alpha D_x^\beta \sigma(x, \xi)| \leq C(\alpha, \beta)(1 + |\xi|_e)^{|\beta|-|\alpha|}$)

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is continuous on all inhomogeneous Besov spaces $\mathcal{B}_p^{s,q}$, where $s > 0$ and $1 < p, q < \infty$ (see [8]). Recently, Stein and Yung [12] show that $T \in \mathcal{O}pS^{\varepsilon, -2\varepsilon}$ or $\mathcal{O}pS^{-\varepsilon, \varepsilon}$ (whose symbol $\sigma(x, \xi) = \sigma_0(x, M_x \xi)$ for some symbol $\sigma_0(x, \xi)$ that satisfies $|D_{\xi'}^\alpha D_{\xi_n}^\beta D_x^\gamma \sigma_0(x, \xi)| \leq C(\alpha, \beta, I)(1 + |\xi|_e)^{m-|\beta|}(1 + |\xi|_h)^{n-|\alpha|}$, where $m = \varepsilon$, $n = -2\varepsilon$ or $m = -\varepsilon$, $n = \varepsilon$) preserve the isotropic and non-isotropic inhomogeneous Lipschitz spaces. There has been a lot of work on the inhomogeneous function spaces studied by many authors, see [1, 5, 6, 10, 13, 14, 15].

In this paper, we will introduce a new class of inhomogeneous multi-parameter Besov and Triebel-Lizorkin spaces associated with different homogeneities and prove that the composition of two inhomogeneous Calderón-Zygmund singular integral operator associated with different homogeneities is bounded on these new spaces.

Before we introduce the new inhomogeneous multi-parameter Besov and Triebel-Lizorkin spaces, we need some notions. Throughout this paper, we use C to denote positive constants, whose value may vary from line to line. Constants with subscripts, such as C_1 , do not change in different occurrences. We denote by $f \lesssim g$ that there exists a constant $C > 0$ independent of the main parameters such that $f \leq Cg$. We also denote by $f \sim g$ that there exists a constant $C > 0$ independent of the main parameters such that $C^{-1}g \leq f \leq Cg$. Now we can introduce the definition of inhomogeneous Lipschitz space associated with different homogeneities.

For $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$, we denote $|x|_e = (|x'|^2 + |x_n|^2)^{1/2}$ and $|x|_h = (|x'|^2 + |x_n|)^{1/2}$. Note that both $|x|_e$ and $|x|_h$ satisfy the triangular inequality. We also use notations $j \wedge k = \min\{j, k\}$ and $j \vee k = \max\{j, k\}$. Let $\psi^{(1)} \in \mathcal{S}(\mathbb{R}^n)$ satisfy

$$\text{supp}\widehat{\psi^{(1)}}(\xi) \subset \{\xi : 1/2 < |\xi|_e \leq 2\},$$

and $\varphi^{(1)}$ whose Fourier transform does not vanish at the origin with

$$\text{supp}\widehat{\varphi^{(1)}} \subset \{\xi : |\xi|_e \leq 2\}.$$

satisfy

$$|\widehat{\varphi^{(1)}}(\xi)|^2 + \sum_{j=1}^{\infty} |\widehat{\psi^{(1)}}(2^{-j}\xi)|^2 = 1, \quad \text{for all } \xi \in \mathbb{R}^n.$$

And let $\psi^{(2)} \in \mathcal{S}(\mathbb{R}^n)$ with

$$\text{supp}\widehat{\psi^{(2)}}(\xi) \subset \{\xi : 1/2 < |\xi|_h \leq 2\},$$

and $\varphi^{(2)}$ whose Fourier transform does not vanish at the origin with

$$\text{supp}\widehat{\varphi^{(2)}} \subset \{\xi : |\xi|_h \leq 2\}$$

fulfill

$$|\widehat{\varphi^{(2)}}(\xi', \xi_n)|^2 + \sum_{k=1}^{\infty} |\widehat{\psi^{(2)}}(2^{-k}\xi', 2^{-2k}\xi_n)|^2 = 1, \quad \text{for all } (\xi', \xi_n) \in \mathbb{R}^{n-1} \times \mathbb{R}.$$

Let $\psi_j^{(1)}(x) = 2^{jn}\psi^{(1)}(2^jx', 2^jx_n)$, $\psi_k^{(2)}(x) = 2^{k(n+1)}\psi^{(2)}(2^kx', 2^{2k}x_n)$ and $\psi_{j,k}(x) = \psi_j^{(1)} * \psi_k^{(2)}(x)$. We denote $\varphi^{(i)} = \psi_0^{(i)}$, where $i = 1, 2$. Notice that $\psi_j^{(1)} * \psi_k^{(2)}(x) = 0$ unless $k \lesssim j \lesssim 2k$ for $x \in \mathbb{R}^n$.

For $f \in L^2$, we have the inhomogeneous continuous Calderón’s identity

$$f = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \psi_{j,k} * \psi_{j,k} * f$$

via taking the Fourier transform, where the series converges in $L^2(\mathbb{R}^n)$ norm.

First we obtain the following discrete Calderón’s identity.

THEOREM 1.1. *Suppose that $\psi^{(1)}$, $\psi^{(2)}$ and $\psi_{j,k}$ are defined above. Then*

$$\begin{aligned} f(x', x_n) &= \sum_{(l', l_n) \in \mathbb{Z}^{n-1} \times \mathbb{Z}} (\psi_{0,0} * f)(l', l_n) \psi_{j,k}(x' - l', x_n - l_n) \\ &\quad + \sum_{j=1}^{\infty} \sum_{(l', l_n) \in \mathbb{Z}^{n-1} \times \mathbb{Z}} (\psi_{j,0} * f)(l', l_n) \psi_{j,0}(x' - l', x_n - l_n) \\ &\quad + \sum_{k=1}^{\infty} \sum_{(l', l_n) \in \mathbb{Z}^{n-1} \times \mathbb{Z}} (\psi_{0,k} * f)(l', l_n) \psi_{0,k}(x' - l', x_n - l_n) \\ &\quad + \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \sum_{(l', l_n) \in \mathbb{Z}^{n-1} \times \mathbb{Z}} 2^{-(n-1)(j \wedge k)} 2^{-(j \wedge 2k)} (\psi_{j,k} * f)(2^{-(j \wedge k)} l', 2^{-(j \wedge 2k)} l_n) \\ &\quad \times \psi_{j,k}(x' - 2^{-(j \wedge k)} l', x_n - 2^{-(j \wedge 2k)} l_n) \\ &= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{(l', l_n) \in \mathbb{Z}^{n-1} \times \mathbb{Z}} 2^{-(n-1)(j \wedge k)} 2^{-(j \wedge 2k)} (\psi_{j,k} * f)(2^{-(j \wedge k)} l', 2^{-(j \wedge 2k)} l_n) \\ &\quad \times \psi_{j,k}(x' - 2^{-(j \wedge k)} l', x_n - 2^{-(j \wedge 2k)} l_n), \end{aligned}$$

where the series converges in L^2 if $f \in L^2$, and the series also converges in \mathcal{S} if $f \in \mathcal{S}$. Furthermore, the convergence of the right-hand, as well as the equality, is in \mathcal{S}' .

Now we can introduce the definition of inhomogeneous multi-parameter Besov and Triebel-Lizorkin spaces associated with different homogeneities.

DEFINITION 1.1. Let $s = (s_1, s_2) \in \mathbb{R}^2$ and $p, q \in (0, \infty)$. The inhomogeneous Triebel-Lizorkin space associated with different homogeneities $\mathcal{F}_{p,q}^s$ is defined to be a collection of all $f \in \mathcal{S}'$ such that

$$\begin{aligned} &\|f\|_{\mathcal{F}_{p,q}^s} \\ &= \left\| \sum_{(l', l_n) \in \mathbb{Z}^{n-1} \times \mathbb{Z}} |(\psi_{0,0} * f)(l', l_n)| \chi_I \chi_J \right\|_{L^p(\mathbb{R}^n)} \end{aligned}$$

$$\begin{aligned}
 &+ \left\| \left(\sum_{j=1}^{\infty} \sum_{(l', l_n) \in \mathbb{Z}^{n-1} \times \mathbb{Z}} 2^{js_1 q} |(\psi_{j,0} * f)(l', l_n)|^q \chi_I \chi_J \right) \right\|_{L^p(\mathbb{R}^n)}^{1/q} \\
 &+ \left\| \left(\sum_{k=1}^{\infty} \sum_{(l', l_n) \in \mathbb{Z}^{n-1} \times \mathbb{Z}} 2^{ks_2 q} |(\psi_{0,k} * f)(l', l_n)|^q \chi_I \chi_J \right) \right\|_{L^p(\mathbb{R}^n)}^{1/q} \\
 &+ \left\| \left(\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \sum_{(l', l_n) \in \mathbb{Z}^{n-1} \times \mathbb{Z}} 2^{js_1 q} 2^{ks_2 q} |(\psi_{j,k} * f)(2^{-(j \wedge k)} l', 2^{-(j \wedge 2k)} l_n)|^q \chi_I \chi_J \right) \right\|_{L^p(\mathbb{R}^n)}^{1/q} \\
 &< \infty
 \end{aligned}$$

and the inhomogeneous multi-parameter Besov space associated with different homogeneities $\mathcal{B}_{p,q}^s$ is defined to be a collection of all $f \in \mathcal{S}'$ such that

$$\begin{aligned}
 &\|f\|_{\mathcal{B}_{p,q}^s}^\Psi \\
 &= \left\| \sum_{(l', l_n) \in \mathbb{Z}^{n-1} \times \mathbb{Z}} |(\psi_{0,0} * f)(l', l_n)| \chi_I \chi_J \right\|_{L^p(\mathbb{R}^n)} \\
 &+ \left(\sum_{j=1}^{\infty} 2^{js_1 q} \left\| \sum_{(l', l_n) \in \mathbb{Z}^{n-1} \times \mathbb{Z}} (\psi_{j,0} * f)(l', l_n) \chi_I \chi_J \right\|_{L^p(\mathbb{R}^n)}^q \right)^{1/q} \\
 &+ \left(\sum_{k=1}^{\infty} 2^{ks_2 q} \left\| \sum_{(l', l_n) \in \mathbb{Z}^{n-1} \times \mathbb{Z}} (\psi_{0,k} * f)(l', l_n) \chi_I \chi_J \right\|_{L^p(\mathbb{R}^n)}^q \right)^{1/q} \\
 &+ \left(\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} 2^{js_1 q} 2^{ks_2 q} \left\| \sum_{(l', l_n) \in \mathbb{Z}^{n-1} \times \mathbb{Z}} |(\psi_{j,k} * f)(2^{-(j \wedge k)} l', 2^{-(j \wedge 2k)} l_n)|^q \chi_I \chi_J \right\|_{L^p(\mathbb{R}^n)} \right)^{1/q} \\
 &< \infty.
 \end{aligned}$$

REMARK 1.1. Let all notation be as in Definition 1.1, observe that

$$\begin{aligned}
 &\|f\|_{\mathcal{F}_{p,q}^s}^\Psi \\
 &\sim \left\| \left(\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{(l', l_n) \in \mathbb{Z}^{n-1} \times \mathbb{Z}} 2^{js_1 q} 2^{ks_2 q} |(\psi_{j,k} * f)(2^{-(j \wedge k)} l', 2^{-(j \wedge 2k)} l_n)|^q \chi_I \chi_J \right) \right\|_{L^p(\mathbb{R}^n)}^{1/q}
 \end{aligned}$$

as well as

$$\|f\|_{\mathcal{B}_{p,q}^s}^\Psi$$

$$\sim \left(\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} 2^{js_1q} 2^{ks_2q} \left\| \sum_{(l', l_n) \in \mathbb{Z}^{n-1} \times \mathbb{Z}} |(\psi_{j,k} * f)(2^{-(j \wedge k)} l', 2^{-(j \wedge 2k)} l_n)|^q \chi_l \chi_j \right\|_{L^p(\mathbb{R}^n)} \right)^{1/q}.$$

Next we will show that the definitions of the Besov spaces $\mathcal{B}_{p,q}^s$ and the Triebel-Lizorkin spaces $\mathcal{F}_{p,q}^s$ are independent of the choice of $(\psi^{(1)}, \psi^{(2)}, \varphi^{(1)}, \varphi^{(2)})$. Therefore, the Besov spaces $\mathcal{B}_{p,q}^s$ and the Triebel-Lizorkin spaces $\mathcal{F}_{p,q}^s$ are well defined.

THEOREM 1.2. *Let $s = (s_1, s_2) \in \mathbb{R}^2$ and $p, q \in (0, \infty)$. If $\varphi_{j,k}$ satisfies the same conditions as $\psi_{j,k}$, then for $f \in \mathcal{S}'$,*

$$\|f\|_{\mathcal{B}_{p,q}^s}^\psi \sim \|f\|_{\mathcal{B}_{p,q}^s}^\varphi, \quad \|f\|_{\mathcal{F}_{p,q}^s}^\psi \sim \|f\|_{\mathcal{F}_{p,q}^s}^\varphi.$$

REMARK 1.2. As a consequence of Theorem 1.2, $L^2 \cap \mathcal{B}_{p,q}^s$ $L^2 \cap \mathcal{F}_{p,q}^s$ is dense in $\mathcal{B}_{p,q}^s$ and $\mathcal{F}_{p,q}^s$, respectively. Indeed, we only need to observe that \mathcal{S} is dense in $\mathcal{B}_{p,q}^s$ and $\mathcal{F}_{p,q}^s$. The proof of this claim is similar to the proof of Corollary 3.3 in [4].

Then we will state that the composition of two inhomogeneous Calderón-Zygmund singular integral operators of order (ε, σ) with different homogeneities is a bounded operator on the new inhomogeneous multi-parameter Besov spaces.

We consider two kinds of homogeneities

$$\delta : (x', x_n) \rightarrow (\delta x', \delta x_n), \quad \delta > 0$$

and

$$\delta : (x', x_n) \rightarrow (\delta x', \delta^2 x_n), \quad \delta > 0.$$

The first are the classical isotropic dilations occurring in the inhomogeneous Calderón-Zygmund singular integrals, while the second are non-isotropic and related to the heat equations. The inhomogeneous singular integrals were originally introduced by Meyer and Coifman [8].

DEFINITION 1.2. Let $K_1 \in L^1_{loc}(\mathbb{R}^n \setminus \{0\})$ and let $\varepsilon, \delta > 0$. K_1 is called an inhomogeneous Calderón-Zygmund kernel of type (ε, δ) associated with isotropic homogeneity if there exist $C_1 > 0$ such that:

- (i) $|K_1(x)| \leq C_1 \frac{1}{|x|_e^n}$ for $x \in \mathbb{R}^n$;
- (ii) $|K_1(x)| \leq C_1 \frac{1}{|x|_e^{n+\delta}}$ for $|x|_e \geq 1$;
- (iii) $|K_1(x) - K_1(x')| \leq C_1 \frac{|x-x'|_e^\varepsilon}{|x|_e^{n+\varepsilon}}$ for $|x-x'|_e \leq 1/2|x|_e$;
- (iv) $\int_{r_1 < |x|_e < r_2} K_1(x) dx = 0$ for all $0 < r_1 < r_2 < \infty$.

We say that an operator T_1 is an inhomogeneous Calderón-Zygmund singular integral operators associated with isotropic homogeneity if $T_1 f(x) = p.v.(K_1 * f)(x)$, where K_1 fulfills condition (i) – (iv).

DEFINITION 1.3. Let $K_2 \in L^1_{loc}(\mathbb{R}^n \setminus \{0\})$ and let $\varepsilon, \delta > 0$. K_2 is said to be an inhomogeneous Calderón-Zygmund kernel of type (ε, δ) associated with the non-isotropic homogeneity if there exist $C_2 > 0$ such that:

- (v) $|K_2(x)| \leq C_2 \frac{1}{|x|_h^{n+1}}$ for $x \in \mathbb{R}^n$;
- (vi) $|K_2(x)| \leq C_2 \frac{1}{|x|_h^{n+\delta+1}}$ for $|x|_h \geq 1$;
- (vii) $|K_2(x) - K_2(x')| \leq C_2 \frac{|x-x'|_h^\varepsilon}{|x|_h^{n+\varepsilon+1}}$ for $|x-x'|_h \leq 1/2|x-y|_h$;
- (viii) $\int_{r_1 < |x|_h < r_2} K_2(x) dx = 0$ for all $0 < r_1 < r_2 < \infty$.

We say that an operator T_2 is an inhomogeneous Calderón-Zygmund singular integral operators associated with the non-isotropic homogeneity if $T_2 f(x) = p.v.(K_2 * f)(x)$, where K_2 fulfills condition (v) – (viii).

We see that T_1 and T_2 are, respectively, bounded on L^2 . We omit the details of the proofs and refer reader to [7, Section 2.1] for further details. Now we state the main results.

THEOREM 1.3. Let $s = (s_1, s_2) \in \mathbb{R}^2$ and $0 < p, q < \infty$. Suppose that T_1 and T_2 are, respectively, the inhomogeneous Calderón-Zygmund singular integral operators associated with the isotropic and non-isotropic homogeneity and the kernel satisfying (i) – (iv) of Definition 1.2 and (v) – (viii) of Definition 1.3, respectively. Then both T_1 and T_2 can be extended to bounded linear operators on $\mathcal{F}_{p,q}^s$ for $(\frac{n-1}{\sigma'_1+n-1} \vee \frac{1}{\sigma'_2+1} \vee \frac{n-1}{\sigma'_3+n-1} \vee \frac{1}{\sigma'_4+1}) \leq (p \wedge q)$ and $\mathcal{B}_{p,q}^s$ for $(\frac{n-1}{\sigma'_1+n-1} \vee \frac{1}{\sigma'_2+1} \vee \frac{n-1}{\sigma'_3+n-1} \vee \frac{1}{\sigma'_4+1}) \leq p$, where $\sigma'_1 + \sigma'_2 = \sigma'_3 + 2\sigma'_4 = \sigma'$, $\sigma' = (\delta \wedge \varepsilon)$. Furthermore, $T = T_1 \circ T_2$ can be also extended to a bounded linear operator on $\mathcal{F}_{p,q}^s$ and $\mathcal{B}_{p,q}^s$.

2. Proof of Theorem 1.1

As mentioned in Section 1, for $f \in L^2(\mathbb{R}^n)$, we have the inhomogeneous continuous Calderón’s identity

$$f = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \psi_{j,k} * \psi_{j,k} * f, \tag{1}$$

where the series convergences in $L^2, \mathcal{S}, \mathcal{S}'$.

Now we get a discrete version of Calderón identity. First we need to decompose $\psi_{j,k} * \psi_{j,k} * f$ in (1). Set $g = \psi_{j,k} * f$ and $h = \psi_{j,k}$. The Fourier transforms of g and h are given by $\widehat{g}(\xi', \xi_n) = \widehat{\psi}^{(1)}(2^{-j}\xi', 2^{-j}\xi_n) \widehat{\psi}^{(2)}(2^{-k}\xi', 2^{-2k}\xi_n) \widehat{f}(\xi', \xi_n)$ and $\widehat{h}(\xi', \xi_n) = \widehat{\psi}^{(1)}(2^{-j}\xi', 2^{-j}\xi_n) \widehat{\psi}^{(2)}(2^{-k}\xi', 2^{-2k}\xi_n)$. Note that the Fourier transforms of g and h are both compactly supported. More precisely,

$$\text{supp } \widehat{g}, \text{supp } \widehat{h} \subseteq \{(\xi', \xi_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : |\xi'| \leq 2^{j \wedge k} \pi, |\xi_n| \leq 2^{j \wedge 2k} \pi\}.$$

Thus, we first expand \widehat{g} in a Fourier series on the rectangle $R_{j,k} = \{\xi' \in \mathbb{R}^{n-1}, \xi_n \in \mathbb{R} : |\xi'| \leq 2^{j \wedge k} \pi, |\xi_n| \leq 2^{j \wedge 2k} \pi\}$:

$$\begin{aligned} \widehat{g}(\xi', \xi_n) &= \sum_{(\ell', \ell_n) \in \mathbb{Z}^{n-1} \times \mathbb{Z}} 2^{-(n-1)(j \wedge k)} 2^{-(j \wedge 2k)} (2\pi)^{-n} \\ &\quad \times \int_{R_{j,k}} \widehat{g}(\eta', \eta_n) e^{i(2^{-(j \wedge k)} \ell' \cdot \eta' + 2^{-(j \wedge 2k)} \ell_n \eta_n)} d\eta' d\eta_n \\ &\quad \times e^{-i(2^{-(j \wedge k)} \ell' \cdot \xi' + 2^{-(j \wedge 2k)} \ell_n \xi_n)} \end{aligned}$$

and then replace $R_{j,k}$ by \mathbb{R}^n since \widehat{g} is supported in $R_{j,k}$. Finally, we obtain

$$\begin{aligned} \widehat{g}(\xi', \xi_n) &= \sum_{(\ell', \ell_n) \in \mathbb{Z}^{n-1} \times \mathbb{Z}} 2^{-(n-1)(j \wedge k)} 2^{-(j \wedge 2k)} \\ &\quad \times g(2^{-(j \wedge k)} \ell', 2^{-(j \wedge 2k)} \ell_n) e^{-i(2^{-(j \wedge k)} \ell' \cdot \xi' + 2^{-(j \wedge 2k)} \ell_n \xi_n)}. \end{aligned}$$

Multiplying $\widehat{h}(\xi', \xi_n)$ from both sides yields

$$\begin{aligned} \widehat{g}(\xi', \xi_n) \widehat{h}(\xi', \xi_n) &= \sum_{(\ell', \ell_n) \in \mathbb{Z}^{n-1} \times \mathbb{Z}} 2^{-(n-1)(j \wedge k)} 2^{-(j \wedge 2k)} g(2^{-(j \wedge k)} \ell', 2^{-(j \wedge 2k)} \ell_n) \\ &\quad \times \widehat{h}(\xi', \xi_n) e^{-i(2^{-(j \wedge k)} \ell' \cdot \xi' + 2^{-(j \wedge 2k)} \ell_n \xi_n)}. \end{aligned}$$

Note that $\widehat{h}(\xi', \xi_n) e^{-i(2^{-k} \ell' \cdot \xi' + 2^{-j} \ell_n \xi_n)} = [h(\cdot - 2^{-k} \ell', \cdot - 2^{-j} \ell_n)]^\wedge(\xi', \xi_n)$. Therefore, applying the identity $g * h = (\widehat{g} \widehat{h})^\vee$ implies that

$$\begin{aligned} (g * h)(x', x_n) &= \sum_{(\ell', \ell_n) \in \mathbb{Z}^{n-1} \times \mathbb{Z}} 2^{-(n-1)(j \wedge k)} 2^{-(j \wedge 2k)} g(2^{-(j \wedge k)} \ell', 2^{-(j \wedge 2k)} \ell_n) \\ &\quad \times h(x' - 2^{-(j \wedge k)} \ell', x_n - 2^{-(j \wedge 2k)} \ell_n). \end{aligned} \tag{2}$$

Substituting g by $\psi_{j,k} * f$ and h by $\psi_{j,k}$ into Calderón’s identity in (1) gives the inhomogeneous discrete Calderón’s identity and the convergence of the series in the $L^2(\mathbb{R}^n)$.

It remains to prove that the series in the inhomogeneous discrete Calderón’s identity converges in $\mathcal{S}'(\mathbb{R}^n)$. To do this, it suffices to show that

$$\sum_{j \geq N_1} \sum_{(\ell', \ell_n) \in \mathbb{Z}^{n-1} \times \mathbb{Z}} (\psi_{j,0} * f)(\ell', \ell_n) \psi_{j,0}(x' - \ell', x_n - \ell_n), \tag{3}$$

$$\sum_{k \geq N_2} \sum_{(\ell', \ell_n) \in \mathbb{Z}^{n-1} \times \mathbb{Z}} (\psi_{0,k} * f)(\ell', \ell_n) \psi_{0,k}(x' - \ell', x_n - \ell_n) \tag{4}$$

and

$$\sum_{j \geq N_3, k \geq N_4} \sum_{(\ell', \ell_n) \in \mathbb{Z}^{n-1} \times \mathbb{Z}} 2^{-(n-1)(j \wedge k)} 2^{-(j \wedge 2k)} (\psi_{j,k} * f)(2^{-(j \wedge k)} \ell', 2^{-(j \wedge 2k)} \ell_n) \times \psi_{j,k}(x' - 2^{-(j \wedge k)} \ell', x_n - 2^{-(j \wedge 2k)} \ell_n) \tag{5}$$

tend to zero in $\mathcal{S}(\mathbb{R}^n)$ as N_1, N_2, N_3 and N_4 tend to infinity.

When $j \in \mathbb{Z}_+, k = 0$, the above limit (3) will follow from the following estimates: for any fixed j and any given integer $M > 0, |\alpha| \geq 0$, there exists a constant $C = C(M, \alpha) > 0$ which is independent of j such that

$$\left| \sum_{(\ell', \ell_n) \in \mathbb{Z}^{n-1} \times \mathbb{Z}} (\psi_{j,0} * f)(\ell', \ell_n) D^\alpha \psi_{j,0}(x' - \ell', x_n - \ell_n) \right| \leq C 2^{-j} (1 + |x'| + |x_n|)^{-M}.$$

Notice that we can get that

$$|(\psi_j^{(1)} * (\psi_0^{(2)} * f))(u', u_n)| \leq C 2^{-jL} \frac{1}{(1 + |u'| + |u_n|)^M}.$$

On the other hand, from the size conditions of the functions $\psi^{(1)}$ and $\psi^{(0)}$, we have that for any fixed large M ,

$$\begin{aligned} & |D^\alpha \psi_{j,0}(u', u_n)| \\ &= |D^\alpha (\psi_j^{(1)} * \psi_0^{(2)})(u', u_n)| \\ &\leq C 2^{j|\alpha|} \int_{\mathbb{R}^{n-1} \times \mathbb{R}} \frac{2^{jn}}{(1 + 2^j |u' - v'| + 2^j |u_n - v_n|)^M} \frac{1}{(1 + |v'| + |v_n|)^M} dv' dv_n \\ &\leq C 2^{j|\alpha|} \frac{1}{(1 + |u'| + |u_n|)^M}. \end{aligned}$$

Therefore,

$$\begin{aligned} & \left| \sum_{(\ell', \ell_n) \in \mathbb{Z}^{n-1} \times \mathbb{Z}} (\psi_{j,0} * f)(\ell', \ell_n) D^\alpha \psi_{j,0}(x' - \ell', x_n - \ell_n) \right| \\ &\leq C 2^{-j(L-|\alpha|)} \sum_{(\ell', \ell_n) \in \mathbb{Z}^{n-1} \times \mathbb{Z}} \frac{1}{(1 + |\ell'| + |\ell_n|)^M (1 + |x' - \ell'| + |x_n - \ell_n|)^M} \tag{6} \\ &= C 2^{-j(L-|\alpha|)} \sum_{(\ell', \ell_n) \in \mathbb{Z}^{n-1} \times \mathbb{Z}} \int_{\mathcal{R}} \frac{dy' dy_n}{(1 + |\ell'| + |\ell_n|)^M (1 + |x' - \ell'| + |x_n - \ell_n|)^M}, \end{aligned}$$

where $\mathcal{R} = I_1 \times J_1 \in \mathbb{Z}^{n-1} \times \mathbb{Z}$ with unit side-length.

Note that if $y' \in I_1$ and $y_n \in J_1$, then $1 + |x' - \ell'| \sim 1 + |x' - \ell_n|, 1 + |\ell'| \sim 1 + |y'|, 1 + |x_n - \ell'| \sim 1 + |x_n - y_n|$, and $1 + |\ell'| \sim 1 + |y_n|$. The simple calculation gives

$$\frac{1}{(1 + |x' - \ell'| + |x_n - \ell_n|)^M} \sim \frac{1}{(1 + |x' - y'| + |x_n - y_n|)^M}.$$

Similarly,

$$\frac{1}{(1 + |\ell'| + |\ell_n|)^M} \sim \frac{1}{(1 + |y'| + |y_n|)^M}.$$

This implies that the last term in (6) is dominated by

$$\begin{aligned} & C 2^{-j(L-|\alpha|)} \int_{\mathbb{R}^{n-1}} \frac{dy' dy_n}{(1 + |y'| + |y_n|)^M (1 + |x' - y'| + |x_n - y_n|)^M} \\ & \leq C \frac{2^{-j(L-|\alpha|)}}{(1 + |x'| + |x_n|)^{\frac{M}{2}}}. \end{aligned}$$

Thus, the series in (3) converges to zero as N_1 tend to infinity.

We can get the similar result for the case $j = 0, k \in \mathbb{Z}_+$ in (4). Here we omit its proof.

Finally, we will consider the case $j \in \mathbb{Z}_+, k \in \mathbb{Z}_+$ in (5). For the sake of convenience, we denote $x_I = 2^{-(j \wedge k)} \ell'$ and $x_J = 2^{-(j \wedge 2k)} \ell_n$. Let I be dyadic cubes in \mathbb{R}^{n-1} and J be dyadic intervals in \mathbb{R} with side-length $\ell(I) = 2^{-(j \wedge k)}$ and $\ell(J) = 2^{-(j \wedge 2k)}$, and the left lower corners of I and the left end points of J are x_I and x_J , respectively. Then the above limit will follow from the following estimates: for any fixed j, k and any given integer $M > 0, |\alpha| \geq 0$, there exists a constant $C = C(M, \alpha) > 0$ which is independent of j and k such that

$$\left| \sum_{I \times J} |I| |J| (\psi_{j,k} * f)(x_I, x_J) (D^\alpha \psi_{j,k})(x' - x_I, x_n - x_J) \right| \leq C 2^{-j} 2^{-k} (1 + |x'| + |x_n|)^{-M}.$$

Now we apply the classical almost orthogonality argument. To be more precise, for any given positive integers L_1 and L_2 , there exists a constant $C = C(L_1, L_2) > 0$ such that

$$|\psi_j^{(1)} * \psi_{j'}^{(1)}(x', x_n)| \leq C \frac{2^{-|j-j'|L_1} 2^{(j \wedge j')n}}{(1 + 2^{(j \wedge j')} |x'| + 2^{(j \wedge j')} |x_n|)^{L_2}}$$

and

$$|\psi_k^{(2)} * \psi_{k'}^{(2)}(x', x_n)| \leq C \frac{2^{-|k-k'|L_1} 2^{(k \wedge k')(n+1)}}{(1 + 2^{(k \wedge k')} |x'| + 2^{(k \wedge k')} |x_n|)^{L_2}}.$$

Applying the classical almost orthogonality argument with $\psi_0^{(2)} = f, L_1 = L + 2M + n + 1$ and $L_2 = n$, where L and M are any fixed positive integers, we obtain

$$\begin{aligned} |(\psi_k^{(2)} * f)(x', x_n)| & \leq C 2^{-|k|(L+2M+n+1)} \frac{2^{-(k \wedge 0)(n+1)}}{(1 + 2^{(k \wedge 0)} |x'| + 2^{(2k \wedge 0)} |x_n|)^M} \\ & \leq C 2^{-kL} \frac{1}{(1 + |x'| + |x_n|)^M}. \end{aligned}$$

Note that $\psi_k^{(2)} * f \in \mathcal{S}(\mathbb{R}^n)$. Similarly, we have that

$$|(\psi_j^{(1)} * (\psi_k^{(2)} * f))(u', u_n)| \leq C 2^{-kL} 2^{-jL} \frac{1}{(1 + |u'| + |u_n|)^M}.$$

From the size conditions of the functions $\psi^{(1)}$ and $\psi^{(2)}$, we have that for any fixed large M ,

$$\begin{aligned} |D^\alpha \psi_{j,k}(u', u_n)| &= |D^\alpha(\psi_j^{(1)} * \psi_k^{(2)})(u', u_n)| \\ &\leq C 2^{j|\alpha|+2k|\alpha|} \int_{\mathbb{R}^{n-1} \times \mathbb{R}} \frac{2^{jn}}{(1+2^j|u'-v'|+2^j|u_n-v_n|)^M} \frac{2^{k(n+1)}}{(1+2^k|v'|+2^{2k}|v_n|)^M} dv' dv_n \\ &\leq C 2^{j|\alpha|+2k|\alpha|} \frac{2^{(j \wedge k)(n-1)} 2^{(j \wedge 2k)}}{(1+2^{j \wedge k}|u'|+2^{j \wedge 2k}|u_n|)^M} \\ &\leq C 2^{j(M+n+|\alpha|)} 2^{k(2M+2+2|\alpha|)} \frac{1}{(1+|u'|+|u_n|)^M}. \end{aligned}$$

Estimates above yield

$$\begin{aligned} &\left| \sum_{I \times J} |I| |J| (D^\alpha \psi_{j,k})(x' - x_I, x_n - x_J) (\psi_{j,k} * f)(x_I, x_J) \right| \\ &\leq C 2^{-k(L-2M-2|\alpha|-2)} 2^{-|j|(L-M-n-|\alpha|)} \\ &\quad \times \sum_{I \times J} |I| |J| \frac{1}{(1+|x_I|+|x_J|)^M (1+|x'-x_I|+|x_n-x_J|)^M} \\ &= C 2^{-k(L-2M-2|\alpha|-2)} 2^{-j(L-M-n-|\alpha|)} \\ &\quad \times \sum_{I \times J} \int_{I \times J} \frac{dy' dy_n}{(1+|x_I|+|x_J|)^M (1+|x'-x_I|+|x_n-x_J|)^M}. \end{aligned}$$

Note that if $y' \in I$ and $y_n \in J$, then $\ell(I) + |x' - x_I| \sim \ell(I) + |x' - y'|$, $\ell(I) + |x_I| \sim \ell(I) + |y'|$, $\ell(J) + |x_n - x_J| \sim \ell(J) + |x_n - y_n|$, and $\ell(J) + |x_J| \sim \ell(J) + |y_n|$. The simple calculation gives

$$\begin{aligned} \frac{1}{(1+|x'-x_I|+|x_n-x_J|)^M} &\leq C \frac{2^{2jM} 2^{|k|3M}}{(\ell(I) + \ell(J) + |x' - x_I| + |x_n - x_J|)^M} \\ &\leq C \frac{2^{4jM} 2^{6kM}}{(1+|x'-y'|+|x_n-y_n|)^M}. \end{aligned}$$

Similarly,

$$\frac{1}{(1+|x_I|+|x_J|)^M} \leq C \frac{2^{4jM} 2^{6kM}}{(1+|y'|+|y_n|)^M}.$$

Then

$$\begin{aligned} &C 2^{-k(L-20M-2|\alpha|-2)} 2^{-j(L-20M-n-|\alpha|)} \int_{\mathbb{R}^{n-1} \times \mathbb{R}} \frac{dy' dy_n}{(1+|x'-y'|+|x_n-y_n|)^M (1+|y'|+|y_n|)^M} \\ &\leq C 2^{-k(L-20M-2|\alpha|-2)} 2^{-j(L-20M-n-|\alpha|)} \frac{1}{(1+|x'|+|x_n|)^M}. \end{aligned}$$

Choosing $L = 20M + 2|\alpha| + n + 3$, we derive the estimates above and hence the series in (5) converges to zero as N_3 and N_4 tend to infinity. Therefore, the series in the

inhomogeneous discrete Calderón identity converges in $\mathcal{S}'(\mathbb{R}^n)$. By the duality argument, we can also obtain the series in the inhomogeneous discrete Calderón identity converges in $\mathcal{S}'(\mathbb{R}^n)$. The proof of Theorem 1.1 is concluded.

3. Proof of Theorem 1.2

To prove Theorem 1.2, we need the following lemmas.

LEMMA 3.1. (Almost orthogonality estimates) *Suppose that $\psi_{j,k}$ and $\varphi_{j',k'}$ satisfy the same conditions as above. Then for any given integers L and M , there exists a constant $C = C(L, M) > 0$ such that*

$$|\psi_{j,k} * \varphi_{j',k'}(x', x_n)| \leq C 2^{-|j-j'|L} 2^{-|k-k'|L} \frac{2^{(j \wedge j' \wedge k \wedge k')(n-1)}}{(1 + 2^{j \wedge j' \wedge k \wedge k'} |x'|)^{(M+n-1)}} \frac{2^{j \wedge j' \wedge 2(k \wedge k')}}{(1 + 2^{j \wedge j' \wedge 2(k \wedge k')} |x_n|)^{(M+1)}}.$$

Proof. When $j, k, j', k' > 0$, the proof is just [4, Lemma 3.1]. Here we only consider the case: $j, j', k' \in \mathbb{Z}_+$ and $k = 0$ and the proofs of other cases are similar. We first write

$$(\psi_{j,0} * \varphi_{j',k'})(x', x_n) = \int_{\mathbb{R}^{n-1} \times \mathbb{R}} (\psi_j^{(1)} * \varphi_{j'}^{(1)})(x' - y', x_n - y_n) (\psi_0^{(2)} * \varphi_{k'}^{(2)})(y', y_n) dy' dy_n.$$

By the classical almost orthogonal estimates, we have

$$|\psi_j^{(1)} * \varphi_{j'}^{(1)}(u', u_n)| \leq C \frac{2^{(j \wedge j')n} 2^{-|j-j'|L}}{(1 + 2^{(j \wedge j')} |u'|)^{(M+n-1)} (1 + 2^{(j \wedge j')} |u_n|)^{(M+1)}}. \tag{7}$$

Moreover, we claim that

$$|\psi_0^{(2)} * \varphi_{k'}^{(2)}(y', y_n)| \leq C \frac{2^{-k'L}}{(1 + |y'|)^{(M+n-1)} (1 + |y_n|)^{(M+1)}}. \tag{8}$$

In fact, for any $\beta > 0$ and $M > 0$ we have

$$D^\beta \psi_0^{(2)}(y - z) \leq C(1 + |y - z|_h)^{-M},$$

$$\varphi_{k'}^{(2)}(z) \leq C(1 + 2^{k'} |z|_h)^{-M}$$

because of $\psi_0^{(2)}, \varphi_{k'}^{(2)} \in \mathcal{S}$.

To prove this claim, we subtract the Taylor polynomial of $L - 1$ of $\psi_0^{(2)}$ at the point z from the function $\psi_0^{(2)}$ using the cancellation of $\varphi_{k'}^{(2)}$. Then we write

$$\left| \int_{\mathbb{R}^{n-1} \times \mathbb{R}} \psi_0^{(2)}(y - z) \varphi_{k'}^{(2)}(z) dz \right|$$

$$\begin{aligned}
 &= \left| \int_{\mathbb{R}^{n-1} \times \mathbb{R}} \left[\psi_0^{(2)}(y-z) - \sum_{|\gamma| \leq L-1} \frac{D^\gamma \psi_0^{(2)}(y)}{\gamma!} (z^\gamma) \right] \phi_{k'}^{(2)}(z) dz \right| \\
 &= \left| \int_{\mathbb{R}^{n-1} \times \mathbb{R}} \sum_{|\beta|=L} \frac{D^\beta \psi_0^{(2)}(y-\theta z)}{\beta!} (z^\beta) \phi_{k'}^{(2)}(z) dz \right| \\
 &\leq C \sum_{|\beta|=L} \frac{1}{\beta!} \left| \int_{\mathbb{R}^{n-1} \times \mathbb{R}} \frac{|z|^L}{(1+|y-\theta z|_h)^M} \frac{1}{(1+2^{k'}|z|_h)^N} dz \right| \\
 &\leq C \sum_{|\beta|=L} \frac{1}{\beta!} \left| \int_{\mathbb{R}^{n-1} \times \mathbb{R}} \frac{2^{-k'L}}{(1+|y-\theta z|_h)^M} \frac{1}{(1+2^{k'}|z|_h)^{N-L}} dz \right|,
 \end{aligned}$$

where $\theta \in (0, 1)$.

By the triangle inequality and $k' \geq 0$,

$$\begin{aligned}
 1 + |y|_h &\leq 1 + |y - \theta z|_h + |\theta z|_h \\
 &\leq 1 + |y - \theta z|_h + 2^{k'} |z|_h \\
 &\leq (1 + |y - \theta z|_h)(1 + 2^{k'} |z|_h).
 \end{aligned}$$

Hence, we obtain the estimate

$$\begin{aligned}
 &\left| \int_{\mathbb{R}^{n-1} \times \mathbb{R}} \psi_0^{(2)}(y-z) \phi_{k'}^{(2)}(z) dz \right| \\
 &\leq C \frac{2^{-k'L}}{(1+|y|_h)^M} \sum_{|\beta|=L} \frac{1}{\beta!} \left| \int_{\mathbb{R}^{n-1} \times \mathbb{R}} \frac{1}{(1+2^{k'}|z|_h)^{N-L-M}} dz \right| \\
 &\leq C \frac{2^{-k'L}}{(1+|y|_h)^M} \\
 &\leq C \frac{2^{-k'L}}{(1+|y'|)^{(M_1+n-1)} (1+|y_n|)^{(M_1+1)}},
 \end{aligned}$$

where $M = 3M_1 + n + 1$, $N > L + M + n$.

The estimates in (7) and (8) imply that

$$|(\psi_{j,0} * \phi_{j',k'})'(x', x_n)| \leq C 2^{-|j-j'|L} 2^{-k'L} AB,$$

where

$$A = \int_{\mathbb{R}} \frac{2^{(j \wedge j')}}{(1 + 2^{(j \wedge j')} |y_n|)^{(M+1)}} \frac{1}{(1 + |x_n - y_n|)^{(M+1)}} dy_n \leq C \frac{1}{(1 + |x_n|)^{(M+n-1)}}$$

and

$$B = \int_{\mathbb{R}^{n-1}} \frac{2^{(j \wedge j')(n-1)}}{(1 + 2^{(j \wedge j')} |y'|)^{(M+n-1)}} \frac{1}{(1 + |x' - y'|)^{(M+n-1)}} dy' \leq C \frac{1}{(1 + |x'|)^{(M+1)}}.$$

This implies the proof of Lemma 3.1. \square

LEMMA 3.2. [4] Let I, I' be dyadic cubes in \mathbb{R}^{n-1} and J, J' be dyadic intervals in \mathbb{R} with the side lengths $l(I) = 2^{-(j \wedge k)}, l(I') = 2^{-(j' \wedge k')}$ and $l(J) = 2^{-(j \wedge 2k)}, l(J') = 2^{-(j' \wedge 2k')}$, and the left lower corners of I, I' and the left end points of J, J' are $2^{-(j \wedge k)}l', 2^{-(j' \wedge k')}l'', 2^{-(j \wedge 2k)}l_n$ and $2^{-(j' \wedge 2k')}l'_n$, respectively. Then for any $u', v' \in I, u_n, v_n \in J$, and any $\frac{n-1}{M+n-1} < \delta \leq 1$,

$$\begin{aligned} & \sum_{(I'', I''_n) \in \mathbb{Z}^{n-1} \times \mathbb{Z}} \frac{2^{(n-1)(j \wedge j' \wedge k \wedge k')} 2^{j \wedge j' \wedge 2k \wedge 2k'} 2^{-(n-1)(j' \wedge k')} 2^{-(j' \wedge 2k')}}{(1 + 2^{j \wedge j' \wedge k \wedge k'} |u' - 2^{-(j' \wedge k')} l''|)^{(M+n-1)}} \\ & \quad \times \frac{|(\varphi_{j', k'} * f)(2^{-(j' \wedge k')} l'', 2^{-(j' \wedge 2k')} l'_n)|}{(1 + 2^{j \wedge j' \wedge 2k \wedge 2k'} |u_n - 2^{-(j' \wedge 2k')} l'_n|)^{(M+1)}} \\ \leq & C \left\{ \mathcal{M}_s \left[\left(\sum_{(I'', I''_n) \in \mathbb{Z}^{n-1} \times \mathbb{Z}} |(\varphi_{j', k'} * f)(2^{-(j' \wedge k')} l'', 2^{-(j' \wedge 2k')} l'_n)|^2 \chi_{I'} \chi_{J'} \right)^{\delta/2} \right] \right\}^{1/\delta} (v', v_n), \end{aligned}$$

where $C_1 = C 2^{(n-1)(\frac{1}{\delta}-1)(j' \wedge k' - j \wedge k)_+} 2^{(\frac{1}{\delta}-1)(j' \wedge 2k' - j \wedge 2k)_+}$, here $(a - b)_+ = \max\{a - b, 0\}$, and \mathcal{M}_s is the strong maximal function.

Now we return the proof Theorem 1.2.

Proof. Let $f \in \mathcal{S}'(\mathbb{R}^n)$. For any $(x', x_n) \in \mathbb{R}^n$, we have that

$$\begin{aligned} & (\Psi_{j,k} * f)(x', x_n) \\ = & \sum_{j'=0}^{\infty} \sum_{k'=0}^{\infty} \sum_{(I'', I''_n) \in \mathbb{Z}^{n-1} \times \mathbb{Z}} 2^{-(n-1)(j' \wedge k')} 2^{-(j' \wedge 2k')} (\varphi_{j', k'} * f)(2^{-(j' \wedge k')} l'', 2^{-(j' \wedge 2k')} l'_n) \\ & \quad \times (\varphi_{j', k'} * \Psi_{j,k})(x' - 2^{-(j' \wedge k')} l'', x_n - 2^{-(j' \wedge 2k')} l'_n). \end{aligned}$$

First we will prove that

$$\|f\|_{\mathcal{F}_{p,q}^s}^\Psi \sim \|f\|_{\mathcal{F}_{p,q}^s}^\Phi.$$

We only need to prove that

$$\begin{aligned} & \left\| \left(\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{(I', I'_n) \in \mathbb{Z}^{n-1} \times \mathbb{Z}} 2^{js_1 q} 2^{ks_2 q} |(\Psi_{j,k} * f)(2^{-(j \wedge k)} l', 2^{-(j \wedge 2k)} l_n)|^q \chi_I \chi_J \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)} \\ \leq & C \|f\|_{\mathcal{F}_{p,q}^s}^\Phi. \end{aligned}$$

We denote $x_I = 2^{-(j \wedge k)} l', x_J = 2^{-(j \wedge 2k)} l_n, x_{I'} = 2^{-(j' \wedge k')} l''$ and $x_{J'} = 2^{-(j' \wedge 2k')} l'_n$. Discrete Calderón’s identity on $\mathcal{S}'(\mathbb{R}^n)$ and the almost orthogonality estimates yield that for $\frac{n-1}{M+n-1} < \delta < p \leq 1$ and any $v' \in I, v_n \in J$,

$$|(\Psi_{j,k} * f)(x_I, x_J)|$$

$$\leq C \sum_{j',k' \geq 0} 2^{-|j-j'|L} 2^{-|k-k'|L} \left\{ \mathcal{M}_s \left[\left(\sum_{(\ell'', \ell'_n)} |(\varphi_{j',k'} * f)(x_{j'}, x_{j'})|^2 \chi_{I'} \chi_{J'} \right)^{\delta/2} \right] (v', v_n) \right\}^{1/\delta}.$$

When $q < 1$, by a simple calculation we have

$$\begin{aligned} & \left\| \left(\sum_{j,k=0}^{\infty} \sum_{(l', l_n)} 2^{js_1q} 2^{ks_2q} \right) \sum_{j',k'=0}^{\infty} 2^{-|j-j'|L} 2^{-|k-k'|L} \right. \\ & \times \left. \left\{ \mathcal{M}_s \left[\left(\sum_{(l'', l'_n)} |(\varphi_{j',k'} * f)(2^{-(j' \wedge k')} l'', 2^{-(j' \wedge 2k')} l'_n)|^2 \chi_{I'} \chi_{J'} \right)^{\delta/2} \right] \right\}^{1/\delta} \right|^{1/q} \chi_{I'} \chi_{J'} \Big\|_{L^p} \\ & \leq \left\| \left(\sum_{j,k=0}^{\infty} \sum_{(l', l_n)} \right) \sum_{j',k'=0}^{\infty} 2^{(j-j')s_1q} 2^{(k-k')s_2q} 2^{-|j-j'|Lq} 2^{-|k-k'|Lq} 2^{j's_1q} 2^{k's_2q} \right. \\ & \times \left. \left\{ \mathcal{M}_s \left[\left(\sum_{(l'', l'_n)} |(\varphi_{j',k'} * f)(2^{-(j' \wedge k')} l'', 2^{-(j' \wedge 2k')} l'_n)|^2 \chi_{I'} \chi_{J'} \right)^{\delta/2} \right] \right\}^{q/\delta} \right|^{1/q} \chi_{I'} \chi_{J'} \Big\|_{L^p} \\ & \leq \left\| \left(\sum_{j',k'=0}^{\infty} 2^{j's_1q} 2^{k's_2q} \right) \right. \\ & \times \left. \left\{ \mathcal{M}_s \left[\left(\sum_{(l'', l'_n)} |(\varphi_{j',k'} * f)(2^{-(j' \wedge k')} l'', 2^{-(j' \wedge 2k')} l'_n)|^2 \chi_{I'} \chi_{J'} \right)^{\delta/2} \right] \right\}^{q/\delta} \right|^{1/q} \Big\|_{L^p}. \end{aligned}$$

Here notice that $\sum_{j,k=0}^{\infty} 2^{(j-j')s_1q} 2^{(k-k')s_2q} 2^{-|j-j'|Lq} 2^{-|k-k'|Lq} < \infty$, when L is large enough.

When $q > 1$, applying the Hölder inequality with indices q and q' , we get that

$$\begin{aligned} & \left\| \left(\sum_{j,k=0}^{\infty} \sum_{(l', l_n)} 2^{js_1q} 2^{ks_2q} \right) \sum_{j',k'=0}^{\infty} 2^{-|j-j'|L} 2^{-|k-k'|L} \right. \\ & \times \left. \left\{ \mathcal{M}_s \left[\left(\sum_{(l'', l'_n)} |(\varphi_{j',k'} * f)(2^{-(j' \wedge k')} l'', 2^{-(j' \wedge 2k')} l'_n)|^2 \chi_{I'} \chi_{J'} \right)^{\delta/2} \right] \right\}^{1/\delta} \right|^{1/q} \chi_{I'} \chi_{J'} \Big\|_{L^p} \\ & \leq \left\| \left(\sum_{j,k=0}^{\infty} \sum_{(l', l_n)} \right) \left(\sum_{j',k'=0}^{\infty} \left(2^{(j-j')s_1} 2^{(k-k')s_2} 2^{-|j-j'|L} 2^{-|k-k'|L} \right)^{q/q'} \right)^{1/q} \right. \\ & \times \left(\sum_{j',k'=0}^{\infty} 2^{(j-j')s_1} 2^{(k-k')s_2} 2^{-|j-j'|L} 2^{-|k-k'|L} 2^{j's_1q} 2^{k's_2q} \right. \\ & \times \left. \left. \left\{ \mathcal{M}_s \left[\left(\sum_{(l'', l'_n)} |(\varphi_{j',k'} * f)(2^{-(j' \wedge k')} l'', 2^{-(j' \wedge 2k')} l'_n)|^2 \chi_{I'} \chi_{J'} \right)^{\delta/2} \right] \right\}^{q/\delta} \right)^{1/q} \right|^{1/q} \chi_{I'} \chi_{J'} \Big\|_{L^p} \\ & \leq \left\| \left(\sum_{j',k'=0}^{\infty} 2^{j's_1q} 2^{k's_2q} \right) \right. \\ & \times \left. \left\{ \mathcal{M}_s \left[\left(\sum_{(l'', l'_n)} |(\varphi_{j',k'} * f)(2^{-(j' \wedge k')} l'', 2^{-(j' \wedge 2k')} l'_n)|^2 \chi_{I'} \chi_{J'} \right)^{\delta/2} \right] \right\}^{q/\delta} \right|^{1/q} \Big\|_{L^p}. \end{aligned}$$

Thus, applying the boundedness of strong maximal function on $L^{p/\delta}(l^{q/\delta})$ yields

$$\begin{aligned} & \left\| \left(\sum_{j',k'=0}^{\infty} 2^{j's_1q} 2^{k's_2q} \right. \right. \\ & \times \left. \left. \left\{ \mathcal{M}_s \left[\left(\sum_{(l'',l'_n)} |(\varphi_{j',k'} * f)(2^{-(j' \wedge k')} l'', 2^{-(j' \wedge 2k')} l'_n)|^2 \chi_{l'} \chi_{j'} \right)^{\delta/2} \right] \right\}^{q/\delta} \right)^{\delta/q} \right\|_{L^{p/\delta}}^{1/\delta} \\ & \leq C \left\| \left(\sum_{j',k'=0}^{\infty} 2^{j's_1q} 2^{k's_2q} \left(\sum_{(l'',l'_n)} |(\varphi_{j',k'} * f)(2^{-(j' \wedge k')} l'', 2^{-(j' \wedge 2k')} l'_n)|^2 \chi_{l'} \chi_{j'} \right)^{q/2} \right)^{1/q} \right\|_{L^p} \\ & \leq C \left\| \left(\sum_{j',k'=0}^{\infty} \sum_{(l'',l'_n)} 2^{j's_1q} 2^{k's_2q} |(\varphi_{j',k'} * f)(2^{-(j' \wedge k')} l'', 2^{-(j' \wedge 2k')} l'_n)|^q \chi_{l'} \chi_{j'} \right)^{1/q} \right\|_{L^p} \\ & \leq C \|f\|_{\mathcal{F}_{p,q}^s}^\varphi. \end{aligned}$$

Thus it yields the desired inequality.

Now we will prove

$$\|f\|_{\mathcal{B}_{p,q}^s}^\Psi \sim \|f\|_{\mathcal{B}_{p,q}^s}^\varphi.$$

For symmetry, we only need to prove that

$$\begin{aligned} & \left(\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} 2^{js_1q} 2^{ks_2q} \left\| \sum_{(l',l_n) \in \mathbb{Z}^{n-1} \times \mathbb{Z}} (\psi_{j,k} * f)(2^{-(j \wedge k)} l', 2^{-(j \wedge 2k)} l_n) \chi_{l'} \chi_{j'} \right\|_{L^p(\mathbb{R}^n)}^q \right)^{1/q} \\ & \leq \|f\|_{\mathcal{B}_{p,q}^s}^\varphi. \end{aligned}$$

Here we claim that

$$\begin{aligned} & \left\| \sum_{(l',l_n) \in \mathbb{Z}^{n-1} \times \mathbb{Z}} |(\psi_{j,k} * f)(2^{-(j \wedge k)} l', 2^{-(j \wedge 2k)} l_n)| \chi_{l'} \chi_{j'} \right\|_{L^p(\mathbb{R}^n)}^q \\ & \leq C \sum_{j'=0}^{\infty} \sum_{k'=0}^{\infty} 2^{-|j-j'|[L_1-n(\frac{1}{8}-1)-\varepsilon]} 2^{-|k-k'|[L_2-(n+1)(\frac{1}{8}-1)-\varepsilon]} \\ & \times \left\| \sum_{(l'',l'_n) \in \mathbb{Z}^{n-1} \times \mathbb{Z}} |(\varphi_{j',k'} * f)(2^{-(j' \wedge k')} l'', 2^{-(j' \wedge 2k')} l'_n)| \chi_{l'} \chi_{j'} \right\|_{L^p(\mathbb{R}^n)}^q, \end{aligned} \tag{9}$$

where $\left(\frac{n}{n+L_1} \vee \frac{n+1}{n+L_2+1} \vee \frac{n-1}{M+n-1}\right) \leq \left(\frac{n}{n+L_1-|s_1|} \vee \frac{n+1}{n+L_2-|s_2|+1} \vee \frac{n-1}{M+n-1}\right) < p < \infty$.

To prove (9), we will consider the following two case where $0 < p \leq 1$ and $p > 1$.

(i) When $0 < p \leq 1$, we have

$$\begin{aligned} & \left\| \sum_{(l', l_n) \in \mathbb{Z}^{n-1} \times \mathbb{Z}} |(\psi_{j,k} * f)(2^{-(j \wedge k)} l', 2^{-(j \wedge 2k)} l_n)| \chi_{l'}(x') \chi_J(x_n) \right\|_{L^p(\mathbb{R}^n)}^p \\ & \leq C \sum_{j'=0}^{\infty} \sum_{k'=0}^{\infty} 2^{-|j-j'|L_1} 2^{-|k-k'|L_2} 2^{p(n-1)(\frac{1}{8}-1)} (j' \wedge k' - j \wedge k)_+ 2^{p(\frac{1}{8}-1)(j' \wedge 2k' - j \wedge 2k)_+} \\ & \quad \times \left\| \left\{ \mathcal{M}_s \left[\left(\sum_{(l'', l'_n) \in \mathbb{Z}^{n-1} \times \mathbb{Z}} |(\varphi_{j',k'} * f)(2^{-(j' \wedge k')} l'', 2^{-(j' \wedge 2k')} l'_n)|^2 \chi_{l''} \chi_{J'} \right)^{\delta/2} \right] \right\}^{1/\delta} \right\|_{L^p(\mathbb{R}^n)}^p \\ & \leq C \sum_{j'=0}^{\infty} \sum_{k'=0}^{\infty} 2^{-|j-j'|[L_1 - n(\frac{1}{8}-1)]} 2^{-|k-k'|[L_2 - (n+1)(\frac{1}{8}-1)]} p \\ & \quad \times \left\| \sum_{(l'', l'_n) \in \mathbb{Z}^{n-1} \times \mathbb{Z}} |(\varphi_{j',k'} * f)(2^{-(j' \wedge k')} l'', 2^{-(j' \wedge 2k')} l'_n)| \chi_{l''} \chi_{J'} \right\|_{L^p(\mathbb{R}^n)}^p, \end{aligned}$$

where in the last inequality we use the fact that $(j' \wedge k' - j \wedge k)_+ \leq |j - j'| + |k - k'|$, $(j' \wedge 2k' - j \wedge 2k)_+ \leq |j - j'| + 2|k - k'|$ and $(\frac{n}{n+L_1-|s|} \vee \frac{n+1}{n+L_2-|s_2|+1} \vee \frac{n-1}{M+n-1}) < \delta < (p \wedge 1) < \infty$.

If $0 < q \leq p$ using the inequality $(\sum_l a_l)^{(q/p)} \leq \sum_l a_l^{(q/p)}$ and if $q > p$ using the Hölder inequality yield (9).

(ii) When $p \geq 1$, applying the Hölder inequality and the $L^{p/\delta}$ boundedness of \mathcal{M}_s ,

$$\begin{aligned} & \left\| \sum_{(l', l_n) \in \mathbb{Z}^{n-1} \times \mathbb{Z}} |(\psi_{j,k} * f)(2^{-(j \wedge k)} l', 2^{-(j \wedge 2k)} l_n)| \chi_{l'}(x') \chi_J(x_n) \right\|_{L^p(\mathbb{R}^n)} \\ & \leq C \sum_{j'=0}^{\infty} \sum_{k'=0}^{\infty} 2^{-|j-j'|L_1} 2^{-|k-k'|L_2} 2^{(n-1)(\frac{1}{8}-1)} (j' \wedge k' - j \wedge k)_+ 2^{(\frac{1}{8}-1)(j' \wedge 2k' - j \wedge 2k)_+} \\ & \quad \times \left\| \left\{ \mathcal{M}_s \left[\left(\sum_{(l'', l'_n) \in \mathbb{Z}^{n-1} \times \mathbb{Z}} |(\varphi_{j',k'} * f)(2^{-(j' \wedge k')} l'', 2^{-(j' \wedge 2k')} l'_n)|^2 \chi_{l''} \chi_{J'} \right)^{\delta/2} \right] \right\}^{1/\delta} \right\|_{L^p(\mathbb{R}^n)} \\ & \leq C \sum_{j'=0}^{\infty} \sum_{k'=0}^{\infty} 2^{-|j-j'|[L_1 - n(\frac{1}{8}-1)]} 2^{-|k-k'|[L_2 - (n+1)(\frac{1}{8}-1)]} \\ & \quad \times \left\| \sum_{(l'', l'_n) \in \mathbb{Z}^{n-1} \times \mathbb{Z}} |(\varphi_{j',k'} * f)(2^{-(j' \wedge k')} l'', 2^{-(j' \wedge 2k')} l'_n)| \chi_{l''} \chi_{J'} \right\|_{L^p(\mathbb{R}^n)}. \end{aligned}$$

When $0 < q \leq 1$ using the inequality $(\sum_l a_l)^q \leq \sum_l a_l^q$ and if $q > 1$ using the Hölder inequality yield (9).

By (9), we get that

$$\begin{aligned} & \left(\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} 2^{js_1q} 2^{ks_2q} \left\| \sum_{(l', l_n) \in \mathbb{Z}^{n-1} \times \mathbb{Z}} |(\psi_{j,k} * f)(2^{-(j \wedge k)} l', 2^{-(j \wedge 2k)} l_n)| \chi_{l'} \chi_{l_n} \right\|_{L^p(\mathbb{R}^n)}^q \right)^{1/q} \\ & \leq C \left(\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} 2^{js_1q} 2^{ks_2q} \sum_{j'=0}^{\infty} \sum_{k'=0}^{\infty} 2^{-|j-j'|[L_1-n(\frac{1}{8}-1)-\varepsilon]q} 2^{-|k-k'|[L_2-(n+1)(\frac{1}{8}-1)-\varepsilon]q} \right. \\ & \quad \times \left. \left\| \sum_{(l'', l_n) \in \mathbb{Z}^{n-1} \times \mathbb{Z}} |(\varphi_{j',k'} * f)(2^{-(j' \wedge k')} l'', 2^{-(j' \wedge 2k')} l_n')| \chi_{l''} \chi_{l_n'} \right\|_{L^p(\mathbb{R}^n)}^q \right)^{1/q} \\ & \leq C \left(\sum_{j'=0}^{\infty} \sum_{k'=0}^{\infty} 2^{j's_1q} 2^{k's_2q} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} 2^{-|j-j'|[L_1-s_1-n(\frac{1}{8}-1)-\varepsilon]q} 2^{-|k-k'|[L_2-s_2-(n+1)(\frac{1}{8}-1)-\varepsilon]q} \right. \\ & \quad \times \left. \left\| \sum_{(l'', l_n) \in \mathbb{Z}^{n-1} \times \mathbb{Z}} |(\varphi_{j',k'} * f)(2^{-(j' \wedge k')} l'', 2^{-(j' \wedge 2k')} l_n')| \chi_{l''} \chi_{l_n'} \right\|_{L^p(\mathbb{R}^n)}^q \right)^{1/q}, \end{aligned}$$

where $(\frac{n}{n+L_1-|s_1|} \vee \frac{n+1}{n+L_2-|s_2|+1} \vee \frac{n-1}{M+n-1}) < \delta < p < \infty$ yields that

$$\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} 2^{-|j-j'|[L_1-|s_1|-n(\frac{1}{8}-1)-\varepsilon]q} 2^{-|k-k'|[L_2-|s_2|-(n+1)(\frac{1}{8}-1)-\varepsilon]q} < \infty.$$

Therefore, we have that

$$\begin{aligned} & \|f\|_{\mathcal{B}_{p,q}^s} \\ & \sim \left(\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} 2^{js_1q} 2^{ks_2q} \left\| \sum_{(l', l_n) \in \mathbb{Z}^{n-1} \times \mathbb{Z}} |(\psi_{j,k} * f)(2^{-(j \wedge k)} l', 2^{-(j \wedge 2k)} l_n)| \chi_{l'} \chi_{l_n} \right\|_{L^p(\mathbb{R}^n)}^q \right)^{1/q} \\ & \leq C \left(\sum_{j'=0}^{\infty} \sum_{k'=0}^{\infty} 2^{j's_1q} 2^{k's_2q} \right. \\ & \quad \times \left. \left\| \sum_{(l'', l_n) \in \mathbb{Z}^{n-1} \times \mathbb{Z}} |(\varphi_{j',k'} * f)(2^{-(j' \wedge k')} l'', 2^{-(j' \wedge 2k')} l_n')| \chi_{l''} \chi_{l_n'} \right\|_{L^p(\mathbb{R}^n)}^q \right)^{1/q} \\ & \leq C \|f\|_{\mathcal{B}_{p,q}^s}. \end{aligned}$$

So we have proved Theorem 1.2. \square

As a consequence of Theorem 1.2, $L^2 \cap \mathcal{X}_{p,q}^s$ is dense in $\mathcal{X}_{p,q}^s$, where $\mathcal{X}_{p,q}^s$ is $\mathcal{B}_{p,q}^s$ or $\mathcal{F}_{p,q}^s$. Indeed we have the following

COROLLARY 3.1. \mathcal{S} is dense in $\mathcal{X}_{p,q}^s$.

Proof. Let $f \in \mathcal{X}_{p,q}^s$. For any fixed $N > 0$, denote

$$E = \{(j, k, \ell', \ell_n) : 0 \leq j \leq N, 0 \leq k \leq N, 0 \leq \ell' \leq N, 0 \leq \ell_n \leq N\},$$

and

$$f_N(x', x_n) := \sum_{(j,k,\ell',\ell_n) \in E} 2^{-(n-1)(j \wedge k)} 2^{-(j \wedge 2k)} (\psi_{j,k} * f)(2^{-(j \wedge k)} \ell', 2^{-(j \wedge 2k)} \ell_n) \\ \times \psi_{j,k}(x' - 2^{-(j \wedge k)} \ell', x_n - 2^{-(j \wedge 2k)} \ell_n),$$

where $\psi_{j,k}$ is the same as in Theorem 1.1.

Since $\psi_{j,k} \in \mathcal{S}$, we obviously have $f_N \in \mathcal{S}$. Repeating the proof of Theorem 1.2, we can conclude that $\|f_N\|_{\mathcal{X}_{p,q}^s} \leq C \|f\|_{\mathcal{X}_{p,q}^s}$. To see that f_N tends to f in $\mathcal{X}_{p,q}^s$, by the discrete inhomogeneous Calderón’s identity in \mathcal{S}' in Theorem 1.1,

$$(f - f_N)(x', x_n) = \sum_{(j,k,\ell',\ell_n) \in E^c} 2^{-(n-1)(j \wedge k)} 2^{-(j \wedge 2k)} (\psi_{j,k} * f)(2^{-(j \wedge k)} \ell', 2^{-(j \wedge 2k)} \ell_n) \\ \times \psi_{j,k}(x' - 2^{-(j \wedge k)} \ell', x_n - 2^{-(j \wedge 2k)} \ell_n),$$

where the series converges in \mathcal{S}' .

Therefore, repeating the proof of Theorem 1.2, $\|f - f_N\|_{\mathcal{X}_{p,q}^s}$ tends to 0 as N tends to infinity. This implies that f_N tends to f in the $\mathcal{X}_{p,q}^s$ norm as N tend to infinity. Thus, we have proved Corollary 3.1. \square

4. Proof of Theorem 1.3

In order to prove Theorem 1.3, we need an inhomogeneous discrete Calderón-type identity on $L^2 \cap \mathcal{B}_{p,q}^s$, which has its own interests. To do this, let $\phi \in \mathcal{S}$ with $\text{supp } \phi \subseteq B(0, 1)$ and $\widehat{\Phi}^{(1)}$ with

$$|\widehat{\Phi}^{(1)}(\xi)| \geq C > 0, \quad \text{supp } \Phi^{(1)} \subset \{|\xi| \leq 2\}$$

satisfy

$$|\widehat{\Phi}^{(1)}(\xi)|^2 + \sum_{j \geq 1} |\widehat{\phi}^{(1)}(2^{-j} \xi)|^2 = 1 \quad \text{for all } \xi \in \mathbb{R}^n,$$

and

$$\int_{\mathbb{R}^n} \phi^{(1)}(x) x^\alpha dx = 0 \quad \text{for all } |\alpha| \leq 10M,$$

where M is a fixed large positive integer depending on p, q and s .

We also let $\phi^{(2)} \in \mathcal{S}$ with $\text{supp } \phi^{(2)} \subseteq B(0, 1)$ and $\widehat{\Phi}^{(2)}$ with

$$|\widehat{\Phi}^{(2)}(\xi)| \geq C > 0, \quad \text{supp } \Phi^{(2)} \subset \{|\xi| \leq 2\}$$

satisfy

$$|\widehat{\Phi}^{(2)}(\xi)|^2 + \sum_{k \geq 1} |\widehat{\phi}^{(2)}(2^{-k} \xi', 2^{-2k} \xi_n)|^2 = 1 \quad \text{for all } (\xi', \xi_n) \in \mathbb{R}^{n-1} \times \mathbb{R},$$

and

$$\int_{\mathbb{R}^n} \phi^{(2)}(x)x^\beta dx = 0 \quad \text{for all } |\beta| \leq 10M,$$

where M is a fixed large positive integer depending on p, q and s .

We denote $\Phi^{(i)} =: \phi_0^{(i)}$ and $\phi_j^{(i)}(x) = 2^{jn} \phi^{(i)}(2^j x)$, $i = 1, 2$ and set $\phi_{j,k} = \phi_j^{(1)} * \phi_k^{(2)}$, where $\phi_j^{(1)}(x) = 2^{jn} \phi^{(1)}(2^j x)$ and $\phi_k^{(2)}(x', x_n) = 2^{k(n+1)} \phi^{(2)}(2^k x', 2^{2k} x_n)$.

The discrete Calderón-type identity is given by the following

PROPOSITION 4.1. *Let $\phi^{(1)}$ and $\phi^{(2)}$ satisfy conditions above. Then for any $f \in L^2 \cap \mathcal{B}_{p,q}^s$, there exists $h \in L^2 \cap \mathcal{B}_{p,q}^s$ such that for a sufficiently large $N \in \mathbb{N}$,*

$$f(x', x_n) = \sum_{j,k \in \mathbb{Z}} \sum_{(\ell', \ell_n) \in \mathbb{Z}^{n-1} \times \mathbb{Z}} |I| |J| \phi_{j,k}(x' - 2^{-(j \wedge k) - N} \ell', x_n - 2^{-(j \wedge 2k) - N} \ell_n) \times (\phi_{j,k} * h)(2^{-(j \wedge k) - N} \ell', 2^{-(j \wedge 2k) - N} \ell_n),$$

where the series converges in L^2 , I are dyadic cubes in \mathbb{R}^{n-1} and J are dyadic intervals in \mathbb{R} with side-length $\ell(I) = 2^{-(j \wedge k) - N}$ and $\ell(J) = 2^{-(j \wedge 2k) - N}$, and the left lower corners of I and the left end points of J are $2^{-(j \wedge k) - N} \ell'$ and $2^{-(j \wedge 2k) - N} \ell_n$, respectively. Moreover,

$$\|f\|_{L^2} \sim \|h\|_{L^2},$$

and

$$\|f\|_{\mathcal{B}_{p,q}^s} \sim \|h\|_{\mathcal{B}_{p,q}^s}.$$

Following the analogous argument to Proposition 4.1 in [4], we can prove this proposition. Here we leave the details to the readers. We point out that the main difference between the discrete Calderón-type identity above and the discrete Calderón’s identity given in Theorem 1.1 is that for any fixed $j, k \in \mathbb{Z}$, $\phi_{j,k}(x', x_n)$ above have compact supports but $\psi_{j,k}(x', x_n)$ in Theorem 1.1 don’t. Being of compact support allows to use the orthogonality argument in the proof of Theorem 1.3.

Now we prove Theorem 1.3.

Proof of Theorem 1.3. First we prove the T_2 is a bounded operator on $\mathcal{F}_{p,q}^s$. For $f \in L^2 \cap \mathcal{F}_{p,q}^s$, by the L^2 boundedness of T_2 and applying discrete Calderón’s identity of f on $L^2 \cap \mathcal{F}_{p,q}^s$, we conclude

$$\begin{aligned} & \|T_2 f\|_{\mathcal{F}_{p,q}^s} \\ & \sim \left\| \left(\sum_{j,k=0}^{\infty} \sum_{(\ell', \ell_n)} 2^{js_1 q} 2^{ks_2 q} \left| (\phi_{j,k} * T_2 f)(2^{-(j \wedge k)} \ell', 2^{-(j \wedge 2k)} \ell_n) \right|^q \chi_I \chi_J \right)^{1/q} \right\|_{L^p} \\ & \sim \left\| \left(\sum_{j,k=0}^{\infty} \sum_{(\ell', \ell_n)} 2^{js_1 q} 2^{ks_2 q} \right. \right. \\ & \quad \times \left. \left. \sum_{j',k'=0}^{\infty} \sum_{(\ell'', \ell'_n) \in \mathbb{Z}^{n-1} \times \mathbb{Z}} 2^{-(n-1)(j' \wedge k')} 2^{-(j' \wedge 2k')} (\phi_{j',k'} * f)(2^{-(j' \wedge k')} \ell'', 2^{-(j' \wedge 2k')} \ell'_n) \right. \right. \end{aligned}$$

$$\times (\phi_{j,k} * K_2 * \phi_{j',k'})(2^{-(j \wedge k)} l' - 2^{-(j \wedge k)} l'', 2^{-(j \wedge 2k)} l_n - 2^{-(j \wedge 2k)} l'_n) \Bigg| \chi_l \chi_j \Bigg|_{L^p}^q.$$

Here we claim that

$$\begin{aligned} |\phi_{j,k} * K_2 * \phi_{j',k'}(x', x_n)| &\leq C \frac{2^{-|j-j'|L} 2^{-|k-k'|L}}{[2^{-(j \wedge j' \wedge k \wedge k')} + |x'|]^{n+\sigma'_3-1}} \\ &\quad \times \frac{2^{-(j \wedge j' \wedge k \wedge k')} \sigma_3 2^{-(j \wedge j' \wedge 2k \wedge 2k')} \sigma_4}{[2^{-(j \wedge j' \wedge 2k \wedge 2k')} + |x_n|]^{1+\sigma'_4}}, \end{aligned}$$

where $\sigma'_3 + 2\sigma'_4 = \sigma'$, $\sigma' = \delta$ when k or $k' = 0$, otherwise $\sigma' = \varepsilon$.

To prove this, we will consider the cases where $k > 0$ and $k = 0$.

Case 1. $|x|_h \leq 2 \cdot 2^{-k}$.

In this case, $2^k |x'| \leq 2$ and $2^{2k} |x_n| \leq 4$, which imply that

$$1 + 2^k |x'| \sim 1 \quad \text{and} \quad 1 + 2^{2k} |x_n| \sim 1.$$

By the fact $\text{supp } \phi_k^{(2)} \subset \{x : |x|_h \leq 2^{-k}\}$ and the cancellation condition in (viii), $K_2 * \phi_k^{(2)}(x)$ is bounded by

$$\begin{aligned} &|K_2 * \phi_k^{(2)}(x)| \\ &= \left| \lim_{\varepsilon \rightarrow 0} \int_{\varepsilon \leq |x-y|_h \leq 3 \cdot 2^{-k}} K_2(x-y) \phi_k^{(2)}(y) dy \right| \\ &= \left| \lim_{\varepsilon \rightarrow 0} \int_{\varepsilon \leq |x-y|_h \leq 3 \cdot 2^{-k}} K_2(x-y) [\phi_k^{(2)}(y) - \phi_k^{(2)}(x)] dy \right| \\ &\leq C 2^{k(n+1)} \int_{|x'-y'| \leq 3} (|x' - y'|)^{-(n-1)+1} dy' \int_{|x_n - y_n| \leq 9} |x_n - y_n|^{-2+2} dy_n \\ &\leq C 2^{k(n+1)} \leq C \frac{2^{k(n+1)}}{(1 + 2^k |x'|)^{\sigma'_3+n-1} (1 + 2^{2k} |x_n|)^{\sigma'_4+1}} \\ &= C \frac{2^{-k\sigma'}}{[2^{-k} + |x'|]^{n+\sigma'_3-1} [2^{-2k} + |x_n|]^{1+\sigma'_4}}. \end{aligned}$$

Case 2. $|x|_h > 2 \cdot 2^{-k}$:

In this case, $2^k |x'| > 2$ or $2^{2k} |x_n| > 4$, which imply that

$$1 + 2^k |x'| \sim 2^k |x'| \quad \text{or} \quad 1 + 2^{2k} |x_n| \sim 2^{2k} |x_n|.$$

By the cancellation condition of $\phi^{(2)}$ and the smoothness condition of K_2 in (vii),

When $k > 0$, we get that

$$|K_2 * \phi_k^{(2)}(x)| = \left| \int_{|y|_h \leq 2^{-k}} [K_2(x-y) - K_2(x)] \phi_k^{(2)}(y) dy \right|$$

$$\begin{aligned}
&\leq C \int_{|y|_h \leq 2^{-k}} \frac{(|y|_h)^\varepsilon}{(|x|_h)^{n+1+\varepsilon}} |\phi_k^{(2)}(y)| dy \\
&\leq C \frac{2^{k(n+1)}}{(1+2^k|x'|)^{\sigma'_3+n-1} (1+2^{2k}|x_n|)^{\sigma'_4+1}} \\
&= C \frac{2^{-k\sigma'}}{[2^{-k} + |x'|]^{n+\sigma'_3-1} [2^{-2k} + |x_n|]^{1+\sigma'_4}}.
\end{aligned}$$

When $k = 0$, using the size condition on the kernel K_2 when $|x|_h > 1$, we have

$$\begin{aligned}
|K_2 * \phi_0^{(2)}(x)| &= \left| \int_{|x-y|_h \leq 1} K_2(x) \phi_0^{(2)}(x-y) dy \right| \\
&\leq C \int_{|y|_h \leq 1} \frac{1}{(|x|_h)^{n+\delta+1}} |\phi_k^{(2)}(y)| dy \\
&\leq C \frac{1}{(1+|x'|)^{\sigma'_3+n-1} (1+|x_n|)^{\sigma'_4+1}} \\
&= C \frac{1}{[1+|x'|]^{n+\sigma'_3-1} [1+|x_n|]^{1+\sigma'_4}}.
\end{aligned}$$

By the classical orthogonality argument, for any fixed L and M ,

$$|\phi_j^{(1)} * \phi_{j'}^{(1)}(x', x_n)| \leq C \frac{2^{-|j-j'|L} 2^{n(j \wedge j')}}{(1+2^{(j \wedge j')}|x'|)^{(M+n-1)} (1+2^{(j \wedge j')}|x_n|)^{(M+1)}},$$

and

$$|\phi_k^{(2)} * \phi_{k'}^{(2)}(x', x_n)| \leq C \frac{2^{-|k-k'|L} 2^{(k \wedge k')(n+1)}}{(1+2^{(k \wedge k')}|x'|)^{(M+n-1)} (1+2^{2(k \wedge k')}|x_n|)^{(M+1)}}.$$

Therefore,

$$\begin{aligned}
|K_2 * \phi_{j,k} * \phi_{j',k'}(x', x_n)| &= |[\phi_j^{(1)} * \phi_{j'}^{(1)}] * [K_2 * \psi_k^{(2)} * \psi_{k'}^{(2)}](x', x_n)| \\
&\leq C \frac{2^{-|j-j'|L} 2^{-|k-k'|L} 2^{(j \wedge j' \wedge k \wedge k')(n-1)} 2^{j \wedge j' \wedge 2k \wedge 2k'}}{(1+2^{j \wedge j' \wedge k \wedge k'}|x'|)^{(\sigma'_3+n-1)} (1+2^{j \wedge j' \wedge 2k \wedge 2k'}|x_n|)^{(\sigma'_4+1)}}.
\end{aligned}$$

Thus, the claim follows.

Then by Lemma 3.2, for any $u', v' \in I$, $u_n, v_n \in J$, and any $(\frac{n-1}{\sigma'_3+n-1} \vee \frac{1}{\sigma'_4+1}) < \delta \leq (p \wedge q \wedge 1)$,

$$\begin{aligned}
&\sum_{(l'', l'_n) \in \mathbb{Z}^{n-1} \times \mathbb{Z}} \frac{2^{(n-1)(j \wedge j' \wedge k \wedge k')} 2^{j \wedge j' \wedge 2k \wedge 2k'} 2^{-(n-1)(j' \wedge k')} 2^{-(j' \wedge 2k')}}{(1+2^{j \wedge j' \wedge k \wedge k'}|u' - 2^{-(j' \wedge k')}l''|)^{(\sigma'_3+n-1)}} \\
&\quad \times \frac{|(\phi_{j',k'} * f)(2^{-(j' \wedge k')}l'', 2^{-(j' \wedge 2k')}l'_n)|}{(1+2^{j \wedge j' \wedge 2k \wedge 2k'}|u_n - 2^{-(j' \wedge 2k')}l'_n|)^{(\sigma'_4+1)}}
\end{aligned}$$

$$\leq C \left\{ \mathcal{M}_s \left[\left(\sum_{(l'', l'_n) \in \mathbb{Z}^{n-1} \times \mathbb{Z}} |(\phi_{j', k'} * f)(2^{-(j' \wedge k')} l'', 2^{-(j' \wedge 2k')} l'_n)|^2 \chi_{l'} \chi_{j'} \right)^{\delta/2} \right] \right\}^{1/\delta} (v', v_n),$$

where $C_1 = C 2^{(n-1)(\frac{1}{\delta}-1)(j' \wedge k' - j \wedge k)_+} 2^{(\frac{1}{\delta}-1)(j' \wedge 2k' - j \wedge 2k)_+}$, here $(a - b)_+ = \max\{a - b, 0\}$, and \mathcal{M}_s is the strong maximal function.

$$\begin{aligned} & \left\| \left(\sum_{j,k=0}^{\infty} \sum_{(l', l_n)} 2^{js_1 q} 2^{ks_2 q} \right. \right. \\ & \times \left. \left| \sum_{j', k'=0}^{\infty} \sum_{(l'', l'_n)} 2^{-(n-1)(j' \wedge k')} 2^{-(j' \wedge 2k')} (\phi_{j', k'} * f)(2^{-(j' \wedge k')} l'', 2^{-(j' \wedge 2k')} l'_n) \right. \right. \\ & \times \left. \left. (\phi_{j,k} * K_2 * \phi_{j', k'}) (2^{-(j \wedge k)} l', 2^{-(j \wedge 2k)} l_n, 2^{-(j \wedge k)} l'', -2^{-(j \wedge 2k)} l'_n) \right|^q \chi_{l'} \chi_{j'} \right)^{1/q} \Big\|_{L^p} \\ & \leq \left\| \left(\sum_{j,k=0}^{\infty} \sum_{(l', l_n)} 2^{js_1 q} 2^{ks_2 q} \left| \sum_{j', k'=0}^{\infty} 2^{-|j-j'|L} 2^{-|k-k'|L} \right. \right. \right. \\ & \times \left. \left. \left\{ \mathcal{M}_s \left[\left(\sum_{(l'', l'_n)} |(\phi_{j', k'} * f)(2^{-(j' \wedge k')} l'', 2^{-(j' \wedge 2k')} l'_n)|^2 \chi_{l'} \chi_{j'} \right)^{\delta/2} \right] \right\}^{1/\delta} \right|^q \chi_{l'} \chi_{j'} \right)^{1/q} \Big\|_{L^p}. \end{aligned}$$

Repeating the similar argument in Theorem 1.2 and applying the fact that $L^2 \cap \mathcal{F}_{p,q}^s$ is dense in $\mathcal{F}_{p,q}^s$, we can get the desired result.

Similarly, we can also prove that

$$\|T_1 f\|_{\mathcal{F}_{p,q}^s} \leq C \|f\|_{\mathcal{F}_{p,q}^s},$$

where we need to observe that $(\frac{n-1}{\sigma_1'+n-1} \vee \frac{1}{\sigma_2'+1}) \leq (p \wedge q)$ and $\sigma_1' + \sigma_2' = \sigma'$.

Therefore, we have that

$$\|Tf\|_{\mathcal{F}_{p,q}^s} = \|T_1 \circ T_2 f\|_{\mathcal{F}_{p,q}^s} \leq C \|T_2 f\|_{\mathcal{F}_{p,q}^s} \leq C \|f\|_{\mathcal{F}_{p,q}^s},$$

where $(\frac{n-1}{\sigma_1'+n-1} \vee \frac{1}{\sigma_2'+1} \vee \frac{n-1}{\sigma_3'+n-1} \vee \frac{1}{\sigma_4'+1}) \leq (p \wedge q)$.

The proof of the case $f \in \mathcal{B}_{p,q}^s$ is similar. For brevity, we omit its details. So we conclude the proof of Theorem 1.3. \square

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