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MAPPING PROPERTIES OF MAXIMAL OPERATORS ON INFINITE CONNECTED GRAPHS

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Abstract. In this paper, we introduce Morrey, Hölder, Lipschitz and Campanato spaces on infinite connected graphs. We establish the boundedness of the Hardy-Littlewood maximal operator and its fractional variants on the above function spaces under certain conditions on graphs. The relations between Hölder spaces and Campanato spaces will be also investigated.

1. Introduction

It is well known that the Hardy-Littlewood maximal function and fractional maximal functions play key roles in partial differential equations, potential theory and harmonic analysis. For example, see [5, 20, 21, 32] for the Hardy-Littlewood maximal function and [1, 2, 3] for fractional maximal functions. Over the last several years the mapping properties for the fractional maximal operators on various of function spaces have been studied by many authors in the Euclidean setting (see [11, 10, 16, 17, 19, 24, 25]) and in the metric setting (see [9, 13, 14, 15, 28, 29, 33]). The main motivation of this paper is to introduce Morrey, Hölder, Lipschitz and Campanato spaces on connected graphs, as well as establish the boundedness for the Hardy-Littlewood maximal operator on graphs and its fractional variants on the above function spaces.

Let $G=(V_G,E_G)$ be an undirected combinatorial graph with the set of vertices V_G and the set of edges E_G. Two vertices $x,y\in V_G$ are called neighbors if they are connected by an edge $x\sim y\in E_G$. For any $v\in V_G$ we denote by $N_G(v)$ the set of neighbors of v. We say that G is locally finite if for any $v\in V_G$, the cardinality $|N_G(v)|<\infty$. The graph G is called connected if for any distinct $x,y\in V_G$, there is a finite sequence of vertices $\{x_i\}_{i=0}^k$, $k\in\mathbb{N}\setminus\{0\}$, such that $x=x_0\sim x_1\sim\cdots\sim x_k=y$. Here $\mathbb{N}=\{0,1,\ldots\}$. We say that G is infinite if $|V_G|=+\infty$.

In what follows, we always assume that the graph $G=(V_G,E_G)$ is an infinite connected graph. Let d_G be the metric induced by the edges in E_G. That is, given

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$u,v \in V_G$, the distance $d_G(u,v)$ is the number of edges in a shortest path connecting u and v. Let $B_G(v,r)$ be the ball centered at v, with radius r on the graph G, i.e.

$$B_G(v,r) = \{u \in V_G : d_G(u,v) \leqslant r\}.$$

For example,

$$B_G(v,r) = \begin{cases} \{v\}, & \text{if } 0 \leqslant r < 1; \\ \{v\} \cup N_G(v), & \text{if } 1 \leqslant r < 2. \end{cases}$$

For any subset $A \subset V_G$, we denote by $|A|$ the cardinality of A.

We now introduce the definitions of fractional maximal operators on graphs.

DEFINITION 1.1. (*Fractional maximal operators*) (see [23]). Let $0 \leqslant \alpha < 1$. For a function $f : V_G \to \mathbb{R}$, the (centered) fractional maximal operator on G is given by

$$M_{\alpha,G}f(v) = \sup_{r \geqslant 0} \frac{1}{|B_G(v,r)|^{1-\alpha}} \sum_{w \in B_G(v,r)} |f(w)|.$$

Another version is defined as

$$\widetilde{M}_{\alpha,G}f(v) = \sup_{r>0} \frac{r^\alpha}{|B_G(v,r)|} \sum_{w \in B_G(v,r)} |f(w)|.$$

Since the distance d_G only takes natural numbers as values, the fractional maximal operator $M_{\alpha,G}$ can be redefined by

$$M_{\alpha,G}f(v) = \sup_{r \in \mathbb{N}} \frac{1}{|B_G(v,r)|^{1-\alpha}} \sum_{w \in B_G(v,r)} |f(w)|.$$

When $\alpha = 0$, the operators $M_{\alpha,G}$ and $\widetilde{M}_{\alpha,G}$ reduce to the centered Hardy-Littlewood maximal operator on G, which is denoted by M_G.

The operators M_G and $M_{\alpha,G}$ have their roots in the discrete harmonic analysis (see [6, 7, 8]). More precisely, let $G_1 = (V_{G_1}, E_{G_1})$, where $V_{G_1} = \mathbb{Z}$ and $E_{G_1} = \{i \sim i+1 : \ i \in \mathbb{Z}\}$. The operator M_{G_1} is just the classical one-dimensional discrete centered Hardy-Littlewood maximal operator M, i.e.

$$Mf(n) = \sup_{r \in \mathbb{N}} \frac{1}{2r+1} \sum_{k=-r}^{r} |f(n+k)|, \ \ n \in \mathbb{Z}.$$

Then the operator M_{α,G_1} is the usual one-dimensional discrete centered fractional maximal operator M_α, i.e.

$$M_\alpha f(n) = \sup_{r \in \mathbb{N}} \frac{1}{(2r+1)^{1-\alpha}} \sum_{k=-r}^{r} |f(n+k)|, \ \ n \in \mathbb{Z}.$$

Over the last twenty years the Hardy-Littlewood maximal operator on graphs has also been studied by many authors (see [4, 12, 18, 26, 30, 31]). This type of operator M_G was firstly introduced by Korányi and Picardello [18] who used this operator

to study the boundary behaviour of eigenfunctions of the Laplace operator on trees. Subsequently, Cowling et al. [12] further studied the operator M_G with G being homogeneous trees. In 2010, Naor and Tao studied the weak $L^1(G)$ norm of M_G with G being an infinite rooted regular tree. It would be worth noting that Naor and Tao's result was recently extended to the weighted setting in [27]. In 2012, Badr and Martell [4] established the weighted norm inequalities for the Hardy-Littlewoood maximal operators on infinite graphs. In 2016, Soria and Tradacete [30] studied the best constants for the L^p-norm of the Hardy-Littlewood maximal operators on finite connected graphs. Recently, Soria and Tradacete [31] further investigated some different geometric properties on infinite graphs, related to the weak-type boundedness of the Hardy-Littlewood maximal operator. To be more precise, they illustrated the connections and differences of the doubling condition, finite dilation and overlapping indices, uniformly bounded degree, the equidistant comparison property of a infinite graph G and the weak type $(1,1)$ boundedness of M_G via some non-trivial examples.

Let us introduce some geometric conditions on graphs, which are useful for our aim.

DEFINITION 1.2. ([31]). Let $G=(V_G,E_G)$.

(i) (*Doubling condition*). We say that the graph G is doubling, if

$$\mathscr{D}(G) := \sup_{x\in V_G,\ r\in\mathbb{N}} \frac{|B_G(x,2r)|}{|B_G(x,r)|} < \infty.$$

(ii) (*Overlapping index*). The overlapping index of G is defined as

$$\mathscr{O}(G) := \min\Big\{m\in\mathbb{N}:\ \forall\{B_j\}_{j\in J},\ \ B_j \text{ a ball in } G,\ \exists I\subset J,\\ \bigcup_{j\in J} B_j = \bigcup_{i\in I} B_i \ \text{ and } \ \sum_{i\in I}\chi_{B_i}\leqslant m\Big\}.$$

(iii) (*Uniformly bounded degree*). We say that the graph G satisfies the uniformly bounded degree condition, if

$$\Delta_G := \sup_{v\in V_G} |N_G(v)| < \infty.$$

We remark that there is no relation between $\mathscr{D}(G)$ and Δ_G for any infinite graph G (see [31]). In order to establish our main results, we also introduce other conditions on graph.

DEFINITION 1.3. Let $G=(V_G,E_G)$.

(i) (*Lower bound condition*). We say that the graph G satisfies a lower bound condition, if there exists $Q\geqslant 1$ such that

$$\mathscr{B}_{1,Q} := \inf_{x\in V_G,\, r\in\mathbb{N}\setminus\{0\}} \frac{|B_G(x,r)|}{r^Q} > 0.$$

(ii) (*Upper bound condition*). We say that the graph G satisfies a upper bound condition, if there exists $Q \geqslant 1$ such that

$$\mathscr{B}_{2,Q} := \sup_{x\in V_G,\, r\in\mathbb{N}\setminus\{0\}} \frac{|B_G(x,r)|}{r^Q} < \infty.$$

(iii) (*Annular decay properties*). Let $0 < \delta \leqslant 1$. We say that the graph G satisfies the δ-annular decay property, if

$$\mathscr{B}_{3,\delta} := \sup_{\substack{x\in V_G,\\ h,R\in\mathbb{N}\setminus\{0\},\ 0<h<R}} \frac{|B_G(x,R)| - |B_G(x,R-h)|}{|B_G(x,R)|}\left(\frac{R}{h}\right)^{\delta} < \infty.$$

(iv) (*Inverse doubling condition*). We say that the graph G is inverse doubling, if

$$\widetilde{\mathscr{D}}(G) := \inf_{x\in V_G,\ r\in\mathbb{N}\setminus\{0\}} \frac{|B_G(x,2r)|}{|B_G(x,r)|} > 1.$$

REMARK 1.4. There are some examples for infinite graphs satisfy the above properties appearing in Definitions 1.2 and 1.3. Let $d \in \mathbb{N}\setminus\{0\}$ and $G_d = (V_{G_d}, E_{G_d})$, where $V_{G_d} = \mathbb{Z}^d$ and

$$E_{G_d} = \bigcup_{i=1}^{d} \{(a_1,\dots,a_{i-1},a_i,a_{i+1},\dots,a_d) \sim (a_1,\dots,a_{i-1},a_i+1,a_{i+1},\dots,a_d) : \\ a_j \in \mathbb{Z},\ \ j = 1,2,\dots,d.\}.$$

It was pointed out in [22] that

$$\max\{1, c_d(r-c_1)^d\} \leqslant |B_{G_d}(v,r)| \leqslant c_d(r+c_1)^d,$$

for any $\vec{n} \in \mathbb{Z}^d$ and $r \geqslant 0$, where $c_1 = \sqrt{d}/2$ and $c_d = \frac{2\pi^{d/2}}{\Gamma(d/2)d}$. Let $c_2 = c_1 + c_d^{-1/d}$. One can easily check that $1 \leqslant \mathscr{D}(G_d) \leqslant (5d)^d$, $\mathscr{O}(G_d) = 2^d$, $1 < \Delta_{G_d} \leqslant 3^d - 1$. Taking $Q = d$ and $\delta = 1$, we have $\min\{c_d, 1, c_2^{-d}\} \leqslant \mathscr{B}_{1,Q} < c_d 2^d d^{d/2}$ $1 \leqslant \mathscr{B}_{2,Q} < c_d 2^d d^{d/2}$ and $2/(c_d(2+c_1)^d) \leqslant \mathscr{B}_{3,\delta} \leqslant 2^d d(1+2c_1+2c_2)^d$. Particularly, when $d = 1$, then $\widetilde{\mathscr{D}}(G) = \frac{5}{3}$.

It should be pointed out that the fractional maximal operators $M_{\alpha,G}$ and $\widetilde{M}_{\alpha,G}$ were first introduced by Liu and Zhang [23] who investigated the Lebesgue space boundedness for the above maximal operators and the regularity properties for the above maximal operators on the endpoint Sobolev spaces and Hajłasz-Sobolev spaces on G under certain geometric conditions on G. In this paper we shall introduce Morrey, Hölder, Lipschitz and Campanato spaces on infinite connect graphs, and establish the boundedness of $M_{\alpha,G}$ and $\widetilde{M}_{\alpha,G}$ on the above function spaces, which is the main motivation of this work.

This paper will be organized as follows. Section 2 will be devoted to introducing the Morrey spaces on graphs and studying the boundedness of the Hardy-Littlewood

maximal operator and its fractional variants on Morrey spaces. In Section 3, we introduce the Hölder spaces and Lipschitz spaces on graphs and study the action of the above operators on the above function spaces. Finally, we introduce the Campanato spaces on graphs and study the behaviors of the above operators on Campanato spaces in Section 4. We would like to remark that the main ideas employed in the proofs of main Theorems are motivated by [9, 13, 14], but our methods and techniques are more refine and simple than those in [9, 13, 14]. Particularly, some new techniques will be explored in the graph setting.

Throughout this paper, we use the symbol $C_{\alpha,\beta,\cdots}$ to denote positive constants that depend on parameters $\alpha,\beta,\cdots$ appeared in the statements of the theorems and other conclusions, but they are independent of the essential variables. In what follows, given a graph $G=(V_G,E_G)$, we denote $f_B=\frac{1}{|B|}\sum_{v\in B}f(v)$ for any arbitrary function $f:V_G\to\mathbb{R}$ and any subset B of G.

2. Boundedness on Morrey spaces

This section is devoted to establishing the boundedness of the Hardy-Littlewood maximal operator and its fractional variants on Morrey spaces. Let us introduce the Morrey spaces on graphs.

DEFINITION 2.1. (*Morrey spaces*) Let $1\leqslant p<\infty$, $\beta\in\mathbb{R}$ and $G=(V_G,E_G)$. A locally integrable function $f:G\to\mathbb{R}$ belongs to the Morrey space $L^{p,\beta}(G)$, if

$$\|f\|_{L^{p,\beta}(G)}:=\sup_{x\in V_G,r\in\mathbb{N}}|B_G(x,r)|^{\beta}\Big(\frac{1}{|B_G(x,r)|}\sum_{v\in B_G(x,r)}|f(v)|^p\Big)^{1/p}<\infty.$$

Another way to define Morrey space is the following

$$\widetilde{L}^{p,\beta}(G):=\Big\{f\in L^1_{\mathrm{loc}}(G):\ \ \|f\|_{\widetilde{L}^{p,\beta}(G)}<\infty\Big\},$$

where

$$\|f\|_{\widetilde{L}^{p,\beta}(G)}:=\sup_{x\in V_G,r>0}r^{\beta}\Big(\frac{1}{|B_G(x,r)|}\sum_{v\in B_G(x,r)}|f(v)|^p\Big)^{1/p},\ \text{ if } \beta\geqslant 0;$$

$$\|f\|_{\widetilde{L}^{p,\beta}(G)}:=\sup_{x\in V_G,r\geqslant 1}r^{\beta}\Big(\frac{1}{|B_G(x,r)|}\sum_{v\in B_G(x,r)}|f(v)|^p\Big)^{1/p},\ \text{ if } \beta<0.$$

REMARK 2.2. (i) It is clear that $L^{p,1/p}(G)=L^p(G)$ for all $1\leqslant p<\infty$. Also, $L^{q,\beta}(G)\subset L^{p,\beta}(G)$ and $\widetilde{L}^{q,\beta}(G)\subset\widetilde{L}^{p,\beta}(G)$ for $1\leqslant p<q<\infty$ and $\beta\in\mathbb{R}$.

(ii) The definition of $\widetilde{L}^{p,\beta}(G)$ for the case $\beta<0$ is reasonable. Actually, unlike the definitions of classical Morrey spaces on the n-dimensional Euclidean spaces $\mathbb{R}^n$ and more general metric measure spaces, the supremum about $r\geqslant 1$ in the definition of $\widetilde{L}^{p,\beta}(G)$-norm can't replaced by $r>0$. The reason is as follows:

$$\sup_{x\in V_G,0<r<1}r^{\beta}\Big(\frac{1}{|B_G(x,r)|}\sum_{v\in B_G(x,r)}|f(v)|^p\Big)^{1/p}=\sup_{x\in V_G,0<r<1}r^{\beta}|f(x)|=+\infty,$$

when $\beta < 0$ and $\|f\|_{L^\infty(G)} > 0$.

(iii) It is clear that

$$\|f\|_{L^\infty(G)} \leqslant \|f\|_{L^{p,\beta}(G)}, \quad \text{for } 1 \leqslant p < \infty, \ \ \beta \in \mathbb{R},$$

and

$$\|f\|_{L^\infty(G)} \leqslant \|f\|_{\widetilde{L}^{p,\beta}(G)}, \quad \text{for } 1 \leqslant p < \infty, \ \ \beta \geqslant 0.$$

We shall establish the following result.

THEOREM 2.1. *Let $1 < p < \infty$ and $G = (V_G, E_G)$. Assume that $\mathscr{D}(G) < \infty$ and $1 < \widetilde{\mathscr{D}}(G) < \infty$. Then*

(i) *Let $\beta > 0$. Then the map $M_G : L^{p,\beta}(G) \to L^{p,\beta}(G)$ is bounded. Moreover,*

$$\|M_G f\|_{L^{p,\beta}(G)} \leqslant C_{p,\beta,\mathscr{D}(G),\widetilde{\mathscr{D}}(G)} \|f\|_{L^{p,\beta}(G)}, \quad \forall f \in L^{p,\beta}(G).$$

(ii) *Let $0 < \alpha < 1$, $\beta > \alpha$ and $q = p\beta/(\beta - \alpha)$. Then the map $M_{\alpha,G} : L^{p,\beta}(G) \to L^{q,\beta-\alpha}(G)$ is bounded. Moreover,*

$$\|M_{\alpha,G} f\|_{L^{q,\beta-\alpha}(G)} \leqslant C_{p,\beta,\alpha,\mathscr{D}(G),\widetilde{\mathscr{D}}(G)} \|f\|_{L^{p,\beta}(G)}, \quad \forall f \in L^{p,\beta}(G).$$

THEOREM 2.2. *Let $1 < p < \infty$ and $G = (V_G, E_G)$. Assume that $\mathscr{D}(G) < \infty$ and $1 < \widetilde{\mathscr{D}}(G) < \infty$. Let $0 < \alpha < 1$, $\beta > \alpha$ and $q = p\beta/(\beta - \alpha)$. Then $\widetilde{M}_{\alpha,G}$ is bounded from $\widetilde{L}^{p,\beta}(G)$ to $\widetilde{L}^{q,\beta-\alpha}(G)$. Moreover,*

$$\|\widetilde{M}_{\alpha,G} f\|_{\widetilde{L}^{q,\beta-\alpha}(G)} \leqslant C_{p,\beta,\alpha,\mathscr{D}(G),\widetilde{\mathscr{D}}(G)} \|f\|_{\widetilde{L}^{p,\beta}(G)}, \quad f \in \widetilde{L}^{p,\beta}(G).$$

REMARK 2.3. The corresponding results in Theorems 2.1 and 2.2 hold for the graph in $\{G_d\}_{d\in\mathbb{N}\setminus\{0\}}$, where $\{G_d\}_{d\in\mathbb{N}\setminus\{0\}}$ are given as in Remark 1.4. Moreover, we see that the conclusions in Theorems 2.1 and 2.2 also hold for the discrete centered fractional maximal operator

$$\mathrm{M}_{\alpha,d} f(\vec{n}) = \sup_{r\geqslant 0} \frac{1}{N(B(\vec{n},r))^{1-\alpha}} \sum_{\vec{m}\in B(\vec{n},r)\cap\mathbb{Z}^d} |f(\vec{n}+\vec{m})|, \quad \text{for } \vec{n} \in \mathbb{Z}^d,$$

where $B(\vec{n},r)$ is the open ball centered at $\vec{n}$ with radius r and $N(B(\vec{n},r))$ is the number of the lattice points in the set $B(\vec{n},r)$. The above claim follows from the following

$$M_{\alpha,G_d} f(\vec{n}) \leqslant \mathrm{M}_{\alpha,d} f(\vec{n}) \leqslant C_{\alpha,d} M_{\alpha,G_d} f(\vec{n}), \quad \forall \vec{n} \in \mathbb{R}^d.$$

We now present the proofs of Theorems 2.1 and 2.2.

Proof of Theorem 2.1. Fix $x \in V_G$ and $r \in \mathbb{N}\setminus\{0\}$, let $B_1 = B_G(x,4r)$ and $S_j = B_G(x,2^j r)\setminus B_G(x,2^{j-1}r)$ for $j \geqslant 3$. We can write $f = f_1 + g$, where $f_1 = f\chi_{B_1}$, $g = \sum_{j=3}^{\infty} f_j$ and $f_j = f\chi_{S_j}$ for all $j \geqslant 3$. By the sublinearity of M_G, one has

$$\sum_{v\in B_G(x,r)} |M_G f(v)|^p \leqslant 2^{p-1}\Big(\sum_{v\in B_G(x,r)} |M_G f_1(v)|^p + \sum_{v\in B_G(x,r)} |M_G g(v)|^p\Big). \tag{2.1}$$

It was shown in [23] that M_G is bounded on $L^p(G)$ for $1<p<\infty$. This together with the fact that $1<\widetilde{\mathscr{D}}(G)<\infty$ implies

$$\begin{aligned}\frac{1}{|B_G(x,r)|}\sum_{v\in B_G(x,r)}|M_Gf_1(v)|^p &\leqslant C_{p,\mathscr{D}(G)}\frac{1}{|B_G(x,r)|}\sum_{v\in V_G}|f_1(v)|^p\\ &\leqslant C_{p,\mathscr{D}(G)}\frac{1}{|B_G(x,r)|}\sum_{v\in B_G(x,4r)}|f(v)|^p\\ &= C_{p,\mathscr{D}(G)}\frac{|B_G(x,4r)|}{|B_G(x,r)|}\frac{1}{|B_G(x,4r)|}\sum_{v\in B_G(x,4r)}|f(v)|^p\\ &\leqslant C_{p,\mathscr{D}(G)}|B_G(x,4r)|^{-p\beta}\|f\|^p_{L^{p,\beta}(G)}\\ &\leqslant C_{p,\beta,\mathscr{D}(G),\widetilde{\mathscr{D}}(G)}|B_G(x,r)|^{-p\beta}\|f\|^p_{L^{p,\beta}(G)}.\end{aligned}\tag{2.2}$$

On the other hand, let us fix $j\geqslant 3$, $v\in B_G(x,r)$ and $t\in\mathbb{N}$. Note that $t\in[(2^{j-1}-1)r,(2^j+1)r]$ when $B_G(v,t)\cap S_j\neq\emptyset$. Moreover, $B_G(x,2^{j-2}r)\subset B_G(x,(2^{j-1}-1)r)\subset B_G(v,2^{j-1}r)$. These facts together with the fact that $1<\widetilde{\mathscr{D}}(G)<\infty$ and Hölder's inequality will imply

$$\begin{aligned}\frac{1}{|B_G(v,t)|}\sum_{w\in B_G(v,t)}|f_j(w)| &\leqslant \Big(\frac{1}{|B_G(v,t)|}\sum_{w\in B_G(v,t)}|f_j(w)|^p\Big)^{1/p}\\ &= \Big(\frac{1}{|B_G(v,t)|}\sum_{w\in B_G(v,t)\cap S_j}|f(w)|^p\Big)^{1/p}\\ &\leqslant \sup_{(2^{j-1}-1)r\leqslant s\leqslant(2^j+1)r}\Big(\frac{1}{|B_G(v,s)|}\sum_{w\in B_G(v,s)}|f(w)|^p\Big)^{1/p}\\ &\leqslant \sup_{(2^{j-1}-1)r\leqslant s\leqslant(2^j+1)r}\|f\|_{L^{p,\beta}(G)}|B_G(v,s)|^{-\beta}\\ &\leqslant |B_G(v,2^{j-1}r)|^{-\beta}\|f\|_{L^{p,\beta}(G)}\\ &\leqslant |B_G(x,2^{j-2}r)|^{-\beta}\|f\|_{L^{p,\beta}(G)}\\ &\leqslant \widetilde{\mathscr{D}}(G)^{-(j-2)\beta}|B_G(x,r)|^{-\beta}\|f\|_{L^{p,\beta}(G)}.\end{aligned}$$

It follows that

$$\begin{aligned}M_Gg(v) &\leqslant \sum_{j=3}^{\infty}M_Gf_j(v)\leqslant\sum_{j=3}^{\infty}|B_G(x,r)|^{-\beta}\widetilde{\mathscr{D}}(G)^{-(j-2)\beta}\|f\|_{L^{p,\beta}(G)}\\ &\leqslant C_{\beta,\widetilde{\mathscr{D}}(G)}|B_G(x,r)|^{-\beta}\|f\|_{L^{p,\beta}(G)}.\end{aligned}\tag{2.3}$$

Then we get from (2.1)–(2.3) that

$$\begin{aligned}&\frac{|B_G(x,r)|^{p\beta}}{|B_G(x,r)|}\sum_{v\in B_G(x,r)}|M_Gf(v)|^p\\ &\leqslant 2^{p-1}|B_G(x,r)|^{p\beta}\Big(\frac{1}{|B_G(x,r)|}\sum_{v\in B_G(x,r)}|M_Gf_1(v)|^p\\ &\quad+\frac{1}{|B_G(x,r)|}\sum_{v\in B_G(x,r)}|M_Gg(v)|^p\Big)\leqslant C_{p,\beta,\mathscr{D}(G),\widetilde{\mathscr{D}}(G)}\|f\|^p_{L^{p,\beta}(G)},\end{aligned}$$

which shows

$$\|M_G f\|_{L^{p,\beta}(G)} \leqslant C_{p,\beta,\mathscr{D}(G)} \|f\|_{L^{p,\beta}(G)}.$$

This proves part (i).

We now prove part (ii). Let $x \in V_G$ and $r \in \mathbb{N} \setminus \{0\}$. By Hölder's inequality, we get

$$\begin{aligned}
&\frac{1}{|B_G(x,r)|^{1-\alpha}} \sum_{v \in B_G(x,r)} |f(v)| \\
&= \Big(\frac{1}{|B_G(x,r)|} \sum_{v \in B_G(x,r)} |f(v)| \Big)^{\alpha/\beta} |B_G(x,r)|^{\alpha} \Big(\frac{1}{|B_G(x,r)|} \sum_{v \in B_G(x,r)} |f(v)| \Big)^{1-\alpha/\beta} \\
&\leqslant \Big(\frac{1}{|B_G(x,r)|} \sum_{v \in B_G(x,r)} |f(v)|^p \Big)^{\alpha/(p\beta)} \Big(\frac{1}{|B_G(x,r)|^{1-\alpha\beta/(\beta-\alpha)}} \sum_{v \in B_G(x,r)} |f(v)| \Big)^{1-\alpha/\beta} \\
&\leqslant \|f\|_{L^{p,\beta}(G)}^{\alpha/\beta} \Big(\frac{1}{|B_G(x,r)|} \sum_{v \in B_G(x,r)} |f(v)| \Big)^{1-\alpha/\beta} \\
&\leqslant \|f\|_{L^{p,\beta}(G)}^{\alpha/\beta} (M_G f(x))^{1-\alpha/\beta}.
\end{aligned}$$

When $r = 0$, by the fact that $\|f\|_{L^{\infty}(G)} \leqslant \|f\|_{L^{p,\beta}(G)}$, one has

$$\begin{aligned}
&\frac{1}{|B_G(x,r)|^{1-\alpha}} \sum_{v \in B_G(x,r)} |f(v)| \\
&= |f(x)| = |f(x)|^{\alpha/\beta} |f(x)|^{1-\alpha/\beta} \leqslant \|f\|_{L^{p,\beta}(G)}^{\alpha/\beta} (M_G f(x))^{1-\alpha/\beta}.
\end{aligned}$$

Hence, we have

$$M_{\alpha,G} f(x) \leqslant \|f\|_{L^{p,\beta}(G)}^{\alpha/\beta} (M_G f(x))^{1-\alpha/\beta}, \quad \forall x \in V_G. \tag{2.4}$$

Fix $x \in V_G$ and $r \in \mathbb{N}$. By (2.4) and part (i), we get

$$\begin{aligned}
&\Big(\frac{1}{|B_G(x,r)|} \sum_{v \in B_G(x,r)} |M_{\alpha,G} f(v)|^{p\beta/(\beta-\alpha)} \Big)^{(\beta-\alpha)/(p\beta)} \\
&\leqslant \|f\|_{L^{p,\beta}(G)}^{\alpha/\beta} \Big(\frac{1}{|B_G(x,r)|} \sum_{v \in B_G(x,r)} |M_G f(v)|^p \Big)^{(\beta-\alpha)/(p\beta)} \\
&\leqslant \|f\|_{L^{p,\beta}(G)}^{\alpha/\beta} |B_G(x,r)|^{-(\beta-\alpha)} \|M_G f\|_{L^{p,\beta}(G)}^{(\beta-\alpha)/\beta} \\
&\leqslant C_{p,\alpha,\beta,\mathscr{D}(G),\widetilde{\mathscr{D}}(G)} |B_G(x,r)|^{-(\beta-\alpha)} \|f\|_{L^{p,\beta}(G)},
\end{aligned}$$

which gives

$$\|M_\alpha f\|_{L^{p\beta/(\beta-\alpha),\beta-\alpha}(G)} \leqslant C_{p,\alpha,\beta,\mathscr{D}(G),\widetilde{\mathscr{D}}(G)} \|f\|_{L^{p,\beta}(G)}.$$

This yields the conclusion of part (ii) and finishes the proof of Theorem 2.1. □

Proof of Theorem 2.2. By the arguments similar to those used in deriving part (ii) of Theorem 2.1, we can get the desired conclusion of Theorem 2.2. The details are omitted. □

3. Boundedness on Hölder and Lipschitz spaces

In this section, we investigate the boundedness of the Hardy-Littlewood maximal operator and its fractional variants on Hölder and Lipschitz spaces. Let us introduce Hölder spaces and Lipschitz spaces on graphs.

DEFINITION 3.1. (*Hölder spaces*) Let $\beta \geqslant 0$ and $G = (V_G, E_G)$. A locally integrable function $f : G \to \mathbb{R}$ belongs to the space of Hölder continuous functions $\mathscr{C}^{0,\beta}(G)$, if

$$\|f\|_{\mathscr{C}^{0,\beta}(G)} := \sup_{\substack{x,y\in V_G,\\ x\neq y}} \frac{|f(x)-f(y)|}{d_G(x,y)^\beta} < \infty.$$

DEFINITION 3.2. (*Lipschitz spaces*) Let $\beta \in \mathbb{R}$ and $G = (V_G, E_G)$. A locally integrable function $f : G \to \mathbb{R}$ belongs to the space of Lipschitz continuous functions $\mathrm{Lip}_\beta(G)$, if

$$\|f\|_{\mathrm{Lip}_\beta(G)} := \|f\|_{L^\infty(G)} + \|f\|_{\mathscr{C}^{0,\beta}(G)} < \infty.$$

REMARK 3.3.

(i) It is clear that $\|\cdot\|_{\mathscr{C}^{0,\beta}(G)}$ is a seminorm and $\mathrm{Lip}_\beta(G)$ is a Banach space.

(ii) Note that

$$\mathrm{Lip}_\beta(G) = \mathscr{C}^{0,\beta}(G), \quad \text{for } \beta = 0. \tag{3.1}$$

To see (3.1), it is clear that $\mathrm{Lip}_\beta(G) \subset \mathscr{C}^{0,\beta}(G)$. It suffices to check that $\mathscr{C}^{0,\beta}(G) \subset L^\infty(G)$ when $\beta = 0$. Let $G = (V_G, E_G)$ and fix $f \in \mathscr{C}^{0,\beta}(G)$ with $\beta = 0$. There exists a vertex $v \in V_G$ such that $|f(v)| < \infty$. It is clear that

$$|f(u)| \leqslant |f(u) - f(v)| + |f(v)| \leqslant \|f\|_{\mathscr{C}^{0,0}(G)} + |f(v)| < \infty, \quad \forall u \in V_G.$$

This yields $f \in L^\infty(G)$ and gives the above claim.

(iii) We have that

$$\mathrm{Lip}_\beta(G) \subsetneq \mathscr{C}^{0,\beta}(G), \quad \forall \beta > 0, \tag{3.2}$$

which is a proper inclusion.

To see the above proper inclusion (3.2), let us consider the Graph $G = (V_G, E_G)$ with $V_G = \mathbb{N}$ and $E_G = \{i \sim i+1; i \in \mathbb{N}\}$ and discuss the following two cases:

(a) When $\beta \geqslant 1$, let us take $f(n) = n$, $n \in \mathbb{N}$. It is clear that

$$\|f\|_{\mathscr{C}^{0,\beta}(G)} = \sup_{m,n\in\mathbb{N}} \frac{|f(m)-f(n)|}{d_G(m,n)^\beta} \leqslant 1,$$

which gives that $f \in \mathscr{C}^{0,\beta}(G)$. However $f \notin L^\infty(G)$.

(b) When $\beta \in (0,1)$, let us consider the function $f(n) = n^\beta$, $n \in \mathbb{N}$. It is clear that $f \notin L^\infty(G)$. However,

$$\begin{aligned}\|f\|_{\mathscr{C}^{0,\beta}(G)} &= \sup_{m,n\in\mathbb{N}} \frac{|f(m)-f(n)|}{d_G(m,n)^\beta} \\ &\leqslant \sup_{m,n\in\mathbb{N}} \frac{|m^\beta - n^\beta|}{|m-n|^\beta} \leqslant 1,\end{aligned}$$

since $\beta \in (0,1)$.

Our main results of this section can be listed as follows:

THEOREM 3.1. *Let $0 < \beta \leqslant \delta \leqslant 1$ and $0 \leqslant \alpha \leqslant \delta - \beta$. Let $G = (V_G, E_G)$ satisfy the δ-annular decay property. Then the map $M_G : \mathscr{C}^{0,\beta}(G) \to \mathscr{C}^{0,\beta}(G)$ is bounded. More precisely,*

$$\|\widetilde{M}_{\alpha,G} f\|_{\mathscr{C}^{0,\alpha+\beta}(G)} \leqslant 5(\mathscr{B}_{3,\delta}+1)\|f\|_{\mathscr{C}^{0,\beta}(G)}, \tag{3.3}$$

holds for all $f \in \mathscr{C}^{0,\beta}(G)$ with $\|f\|_{\mathscr{C}^{0,\beta}(G)} > 0$.

Proof. The proof is motivated by the idea in the proof in [9, Theorem 1.1]. Let $f \in \mathscr{C}^{0,\beta}(G)$. If $\|f\|_{\mathscr{C}^{0,\beta}(G)} = 0$, then we have $f(v) \equiv C_1$ for some constant $C_1 \in \mathbb{R}$ and all $v \in V_G$. In this case we have $M_G f(v) \equiv 1$ for all $v \in V_G$ and then (3.3) is trivial since $\|M_G f\|_{\mathscr{C}^{0,\beta}(G)} = 0$. Hence, the boundedness of $M_G : \mathscr{C}^{0,\beta}(G) \to \mathscr{C}^{0,\beta}(G)$ follows from (3.3). Without loss of generality we shall prove (3.3) under the restrictive conditions $\|f\|_{\mathscr{C}^{0,\beta}(G)} = 1$ and $f \geqslant 0$ since $\widetilde{M}_{\alpha,G} f = \widetilde{M}_{\alpha,G}|f|$ and $\||f|\|_{\mathscr{C}^{0,\beta}(G)} \leqslant \|f\|_{\mathscr{C}^{0,\beta}(G)}$.

Fix $x, y \in V_G$ with $x \neq y$ and let $a = d_G(x,y)$. It suffices to show that

$$|\widetilde{M}_{\alpha,G} f(x) - \widetilde{M}_{\alpha,G} f(y)| \leqslant 5(\mathscr{B}_{3,\delta}+1)a^{\alpha+\beta}. \tag{3.4}$$

Without loss of generality we may assume that $\widetilde{M}_{\alpha,G} f(x) > \widetilde{M}_{\alpha,G} f(y)$. By the definition of $\widetilde{M}_{\alpha,G} f(x)$, there exists $r > 0$ such that

$$\widetilde{M}_{\alpha,G} f(x) \leqslant r^\alpha f_{B_G(x,r)} + a^{\alpha+\beta}.$$

It follows that

$$\begin{aligned}|\widetilde{M}_{\alpha,G} f(x) - \widetilde{M}_{\alpha,G} f(y)| &\leqslant r^\alpha f_{B_G(x,r)} - (r+a)^\alpha f_{B_G(y,r+a)} + a^{\alpha+\beta} \\ &\leqslant r^\alpha (f_{B_G(x,r)} - f_{B_G(y,r+a)}) + a^{\alpha+\beta}.\end{aligned}$$

Therefore, inequality (3.4) reduces to the following

$$r^\alpha (f_{B_G(x,r)} - f_{B_G(y,r+a)}) \leqslant 4(\mathscr{B}_{3,\delta}+1)a^{\alpha+\beta}. \tag{3.5}$$

We consider the following two cases:

(i) ($r \leqslant a$). In this case we have

$$|f(u)-f(v)| \leqslant d_G(u,v)^\beta \leqslant (d_G(u,y)+d_G(v,y))^\beta \leqslant (4a)^\beta, \quad \forall u, v \in B_G(y, r+a).$$

This together with the fact that $B_G(x,r) \subset B_G(y,r+a)$ implies

$$f_{B_G(x,r)} - f_{B_G(y,r+a)} \leqslant f(\omega_1) - f(\omega_2) \leqslant (4a)^\beta,$$

where $f(\omega_1) = \max\limits_{u \in B_G(y,r+a)} f(u)$ and $f(\omega_2) = \min\limits_{u \in B_G(y,r+a)} f(u)$. Therefore,

$$r^\alpha (f_{B_G(x,r)} - f_{B_G(y,r+a)}) \leqslant r^\alpha (4a)^\beta \leqslant 4a^{\alpha+\beta}.$$

This proves (3.5) in this case.

(ii) ($r > a$). Without loss of generality we may assume that $r \in \mathbb{N} \setminus \{0\}$. Set $m = \min_{w \in B_G(x,r)} f(w)$. One can easily check that $0 \leqslant f(z) - m \leqslant (2r)^\beta$ for all $z \in B_G(x,r)$. By the δ-annular decay property of G and $\beta \leqslant \delta < 1$, we have

$$\begin{aligned} f_{B_G(x,r)} - f_{B_G(y,r+a)} &= (f-m)_{B_G(x,r)} - (f-m)_{B_G(y,r+a)} \\ &\leqslant \Big(\frac{1}{|B_G(x,r)|} - \frac{1}{|B_G(y,r+a)|}\Big) \sum_{u \in B_G(x,r)} (f(u)-m) \\ &\leqslant \Big(\frac{|B_G(y,r+a)| - |B_G(x,r)|}{|B_G(y,r+a)|}\Big)(f-m)_{B_G(x,r)} \\ &\leqslant (2r)^\beta \Big(\frac{|B_G(y,r+a)| - |B_G(y,r-a)|}{|B_G(y,r+a)|}\Big) \\ &\leqslant \mathscr{B}_{3,\delta}(2r)^\beta \Big(\frac{2a}{r+a}\Big)^\delta. \end{aligned} \tag{3.6}$$

Combining (3.6) with the fact that $\alpha + \beta \leqslant \delta$ implies that

$$\begin{aligned} r^\alpha (f_{B_G(x,r)} - f_{B_G(y,r+a)}) &\leqslant \mathscr{B}_{3,\delta} 2^{\beta+\delta} r^{\alpha+\beta} \Big(\frac{a}{r+a}\Big)^\delta \\ &\leqslant 4\mathscr{B}_{3,\delta} a^{\alpha+\beta} \Big(\tfrac{r}{a}\Big)^{\alpha+\beta-\delta} \\ &\leqslant 4\mathscr{B}_{3,\delta} a^{\alpha+\beta}. \end{aligned}$$

This proves (3.5) in this case. Theorem 3.1 is proved. □

THEOREM 3.2. *Let $Q \geqslant 1$, $0 < \beta < \delta \leqslant 1$ and $0 < \alpha \leqslant (\delta - \beta)/Q$. Assume that the graph $G = (V_G, E_G)$ has δ-annular decay property and a upper bound Q condition. Then*

$$\|M_{\alpha,G} f\|_{\mathscr{C}^{0,Q\alpha+\beta}(G)} \leqslant (4\mathscr{B}_{2,Q}^\alpha (\mathscr{B}_{3,\delta}+1)+1)\|f\|_{\mathscr{C}^{0,\beta}(G)} \tag{3.7}$$

holds for all $f \in \mathscr{C}^{0,\beta}(G)$ with $\|f\|_{\mathscr{C}^{0,\beta}(G)} > 0$.

Proof. The proof is similar as the proof of Theorem 3.1. It suffices to prove (3.7) for all $f \in \mathscr{C}^{0,\beta}(G)$ with $\|f\|_{\mathscr{C}^{0,\beta}(G)} > 0$ and $f \geqslant 0$. Let us fix a nonnegative function $f : G \to \mathbb{R}$ with $\|f\|_{\mathscr{C}^{0,\beta}(G)} = 1$. Fix $x, y \in V_G$ with $x \neq y$ and let $a = d_G(x,y)$. We want to show that

$$|M_{\alpha,G}f(x) - M_{\alpha,G}f(y)| \leqslant (4\mathscr{B}^{\alpha}_{2,Q}(\mathscr{B}_{3,\delta}+1)+1)a^{Q\alpha+\beta}. \tag{3.8}$$

We may assume, without loss of generality that $M_{\alpha,G}f(x) > M_{\alpha,G}f(y)$. By the definition of $M_{\alpha,G}f(x)$, there exists $r \in \mathbb{N}$ such that

$$M_{\alpha,G}f(x) \leqslant |B_G(x,r)|^{\alpha} f_{B_G(x,r)} + a^{Q\alpha+\beta}.$$

This together with the fact $B_G(x,r) \subset B_G(y,r+a)$ and the upper bound condition implies

$$\begin{aligned} M_{\alpha,G}f(x) - M_{\alpha,G}f(y) &\leqslant |B_G(x,r)|^{\alpha} f_{B_G(x,r)} - |B_G(y,r+a)|^{\alpha} f_{B_G(y,r+a)} + a^{Q\alpha+\beta} \\ &\leqslant |B_G(x,r)|^{\alpha}(f_{B_G(x,r)} - f_{B_G(y,r+a)}) + a^{Q\alpha+\beta} \\ &\leqslant \mathscr{B}^{\alpha}_{2,Q} r^{Q\alpha}(f_{B_G(x,r)} - f_{B_G(y,r+a)}) + a^{Q\alpha+\beta}. \end{aligned}$$

Therefore, to prove (3.8), it is enough to prove that

$$r^{Q\alpha}(f_{B_G(x,r)} - f_{B_G(y,r+a)}) \leqslant 4(\mathscr{B}_{3,\delta}+1)a^{Q\alpha+\beta}. \tag{3.9}$$

When $r \leqslant a$. It was shown in the proof of Theorem 3.1 that

$$f_{B_G(x,r)} - f_{B_G(y,r+a)} \leqslant (4a)^{\beta},$$

which together with $r \leqslant a$ gives (3.9) in this case.

When $r > a$, we get from (3.6) that

$$\begin{aligned} r^{Q\alpha}(f_{B_G(x,r)} - f_{B_G(y,r+a)}) &\leqslant 2^{\beta}\mathscr{B}_{3,\delta} r^{Q\alpha+\beta}\left(\frac{2a}{r+a}\right)^{\delta} \\ &\leqslant 2^{\beta+\delta} a^{Q\alpha+\beta}\left(\frac{r}{a}\right)^{Q\alpha+\beta}\left(\frac{a}{r+a}\right)^{\delta} \\ &\leqslant 4\mathscr{B}_{3,\delta} a^{Q\alpha+\beta}, \end{aligned}$$

since $\beta + Q\alpha \leqslant \delta$. This proves (3.9) in this case. □

It is clear that the map $M_{\alpha,G} : \mathrm{Lip}_{\beta}(G) \to \mathrm{Lip}_{\beta}(G)$ is unbounded because of the lack of the boundedness of $M_{\alpha,G} : L^{\infty}(G) \to L^{\infty}(G)$ when $\alpha \in (0,1)$. However, the boundedness for $M_G : \mathscr{C}^{0,\beta}(G) \to \mathscr{C}^{0,\beta}(G)$ and the fact that $\|M_G f\|_{L^{\infty}(G)} \leqslant \|f\|_{L^{\infty}(G)}$ yields the following result.

COROLLARY 3.3. *Let $0 < \beta \leqslant \delta \leqslant 1$ and the graph $G = (V_G, E_G)$ has the δ-annular decay property. Then the map $M_G : \mathrm{Lip}_{\beta}(G) \to \mathrm{Lip}_{\beta}(G)$ is bounded.*

REMARK 3.4. The corresponding results in Theorems 3.1 and 3.2 and Corollary 3.3 hold for the graph in $\{G_d\}_{d\in\mathbb{N}\setminus\{0\}}$. Here $\{G_d\}_{d\in\mathbb{N}\setminus\{0\}}$ are given as in Remark 1.4.

4. Boundedness on Campanato spaces

In this section we study the behaviors of the Hardy-Littlewood maximal operator and its fractional variants on Campanato spaces.

4.1. Some definitions and lemmas

Let us introduce the Campanato spaces on graphs.

DEFINITION 4.1. (*Campanato spaces*) Let $G=(V_G,E_G)$, $1\leqslant p<\infty$ and $\beta\in\mathbb{R}$. A locally integrable function $f:G\to\mathbb{R}$ belongs to the Campanato space $\mathscr{L}^{p,\beta}(G)$, if

$$\|f\|_{\mathscr{L}^{p,\beta}(G)}:=\sup_{x\in V_G,\,r\in\mathbb{N}\setminus\{0\}}|B_G(x,r)|^{\beta}\Big(\frac{1}{|B_G(x,r)|}\sum_{v\in B_G(x,r)}|f(v)-f_{B_G(x,r)}|^p\Big)^{1/p}<\infty.$$

Another way to define the Campanato spaces is the following

$$\widetilde{\mathscr{L}}^{p,\beta}(G):=\Big\{f\in L^1_{\mathrm{loc}}(G):\ \ \|f\|_{\widetilde{\mathscr{L}}^{p,\beta}(G)}<\infty\Big\}.$$

where

$$\|f\|_{\widetilde{\mathscr{L}}^{p,\beta}(G)}:=\sup_{x\in V_G,\,r\geqslant 1}r^{\beta}\Big(\frac{1}{|B_G(x,r)|}\sum_{v\in B_G(x,r)}|f(v)-f_{B_G(x,r)}|^p\Big)^{1/p}.$$

REMARK 4.2. In contrast to what happens in for Euclidean spaces and more general metric measure spaces it suffices to give the definitions for $\mathscr{L}^{p,\beta}(G)$ and $\widetilde{\mathscr{L}}^{p,\beta}(G)$ for $r\geqslant 1$. The reason is the fact that

$$\Big(\frac{1}{|B_G(x,r)|}\sum_{v\in B_G(x,r)}|f(v)-f_{B_G(x,r)}|^p\Big)^{1/p}=0,\ \ \forall x\in V_G,\ 1\leqslant p<\infty,\ 0<r<1.$$

Applying Minkowski's inequality and Hölder's inequality, one finds

$$\|f\|_{\mathscr{L}^{p,\beta}(G)}\leqslant 2\|f\|_{L^{p,\beta}(G)},\quad \|f\|_{\widetilde{\mathscr{L}}^{p,\beta}(G)}\leqslant 2\|f\|_{\widetilde{L}^{p,\beta}(G)},\ \ \forall 1\leqslant p<\infty,\ \beta\in\mathbb{R}. \tag{4.1}$$

LEMMA 4.1. *Let $\beta\in\mathbb{R}$ and $f\in\widetilde{\mathscr{L}}^{p,\beta}(G)$ for $1\leqslant p<\infty$. Let $r>0$, $x,y\in V_G$, $x\neq y$ and $R\geqslant\max\{r,d_G(x,y)\}$. Then we have*

$$|f_{B_G(x,r)}-f_{B_G(y,R)}|\leqslant C_G\Big(\ln\frac{R}{r}+1\Big)\|f\|_{\widetilde{\mathscr{L}}^{p,\beta}(G)},\ \ \text{if } \beta=0;$$

$$|f_{B_G(x,r)}-f_{B_G(y,R)}|\leqslant C_G r^{-\beta}\|f\|_{\widetilde{\mathscr{L}}^{p,\beta}(G)},\ \ \text{if } \beta>0;$$

$$|f_{B_G(x,r)}-f_{B_G(y,R)}|\leqslant C_G R^{-\beta}\|f\|_{\widetilde{\mathscr{L}}^{p,\beta}(G)},\ \ \text{if } \beta<0.$$

Here

$$C_G=\begin{cases}C_\beta(\mathscr{D}(G)^4+\Delta_G), & \text{if } 0<r<1;\\ C_\beta\mathscr{D}(G)^4, & \text{if } r\geqslant 1,\end{cases}$$

where $C_\beta>0$ depends only on β.

Proof. Note that $R/r \geqslant 1$. There exists an integer $k_0 \in \mathbb{N}$ such that $2^{k_0} \leqslant R/r < 2^{k_0+1}$. Clearly, $B_G(y,R) \subset B_G(x,2R) \subset B_G(x,2^{k_0+2}r)$. By Hölder's inequality, one has

$$\begin{aligned}
&|f_{B_G(x,r)} - f_{B_G(y,R)}| \\
&\leqslant |f_{B_G(x,r)} - f_{B_G(x,2^{k_0+2}r)}| + |f_{B_G(x,2^{k_0+2}r)} - f_{B_G(y,R)}| \\
&\leqslant \sum_{i=1}^{k_0+2} |f_{B_G(x,2^{i-1}r)} - f_{B_G(x,2^i r)}| + \frac{1}{|B_G(y,R)|} \sum_{u\in B_G(y,R)} |f(u) - f_{B_G(x,2^{k_0+2}r)}| \\
&\leqslant \sum_{i=1}^{k_0+2} \frac{1}{|B_G(x,2^{i-1}r)|} \sum_{u\in B_G(x,2^{i-1}r)} |f(u) - f_{B_G(x,2^i r)}| \\
&\quad + \frac{|B_G(x,2^{k_0+2}r)|}{|B_G(y,R)|} \frac{1}{|B_G(x,2^{k_0+2}r)|} \sum_{u\in B_G(x,2^{k_0+2}r)} |f(u) - f_{B_G(x,2^{k_0+2}r)}| \\
&\leqslant \sum_{i=1}^{k_0+2} \frac{|B_G(x,2^i r)|}{|B_G(x,2^{i-1}r)|} \Big(\frac{1}{|B_G(x,2^i r)|} \sum_{u\in B_G(x,2^i r)} |f(u) - f_{B_G(x,2^i r)}|^p \Big)^{1/p} \\
&\quad + \frac{|B_G(x,2^{k_0+2}r)|}{|B_G(y,R)|} \Big(\frac{1}{|B_G(x,2^{k_0+2}r)|} \sum_{u\in B_G(x,2^{k_0+2}r)} |f(u) - f_{B_G(x,2^{k_0+2}r)}|^p \Big)^{1/p} \\
&\leqslant \Big(\sum_{i=1}^{k_0+2} \frac{|B_G(x,2^i r)|}{|B_G(x,2^{i-1}r)|} (2^i r)^{-\beta} + \frac{|B_G(x,2^{k_0+2}r)|}{|B_G(y,R)|} (2^{k_0+2}r)^{-\beta} \Big) \|f\|_{\widetilde{\mathscr{L}}^{p,\beta}(G)}.
\end{aligned}$$

Note that $R \geqslant d_G(x,y) \geqslant 1$. Then

$$\frac{|B_G(x,2^{k_0+2}r)|}{|B_G(y,R)|} \leqslant \frac{|B_G(y,5R)|}{|B_G(y,R)|} = \frac{|B_G(y,[5R])|}{|B_G(y,[R])|} \leqslant \frac{|B_G(y,10[R])|}{|B_G(y,[R])|} \leqslant \mathscr{D}(G)^4$$

and

$$\frac{|B_G(y,2s)|}{|B_G(y,s)|} \leqslant \begin{cases} 1, & \text{if } s \in (0,1/2); \\ \Delta_G, & \text{if } s \in [1/2,1); \\ \mathscr{D}(G), & \text{if } s \geqslant 1. \end{cases}$$

Here $[x] = \max\{k \in \mathbb{Z}; k \leqslant x\}$ for any $x \in \mathbb{R}$. Therefore,

$$|f_{B_G(x,r)} - f_{B_G(y,R)}| \leqslant 2\Big(\sum_{i=1}^{k_0+2} 2^{-\beta i} \Big) \mathscr{D}(G)^4 r^{-\beta} \|f\|_{\widetilde{\mathscr{L}}^{p,\beta}(G)}, \quad \text{if } r \geqslant 1.$$

$$|f_{B_G(x,r)} - f_{B_G(y,R)}| \leqslant 2\Big(\sum_{i=1}^{k_0+2} 2^{-\beta i} \Big) (\mathscr{D}(G)^4 + \Delta_G) r^{-\beta} \|f\|_{\widetilde{\mathscr{L}}^{p,\beta}(G)}, \quad \text{if } r \in (0,1).$$

When $\beta = 0$, one has

$$\sum_{i=1}^{k_0+2} 2^{-\beta i} = k_0 + 2 \leqslant 2\Big(\ln \frac{R}{r} + 1 \Big).$$

When $\beta > 0$, one gets

$$\sum_{i=1}^{k_0+2} 2^{-\beta i} \leqslant \sum_{i=1}^{\infty} 2^{-\beta i} = \frac{1}{2^\beta - 1}.$$

When $\beta < 0$, we have

$$\sum_{i=1}^{k_0+2} 2^{-\beta i} \leqslant \int_1^{k_0+3} (2^{-\beta})^x dx \leqslant \frac{1}{-\beta \ln 2} 2^{-\beta(k_0+3)} \leqslant \frac{8^{-\beta}}{-\beta \ln 2} \left(\frac{R}{r}\right)^{-\beta}.$$

Then the conclusions of Lemma 4.1 follow from the above estimates. □

Applying Lemma 4.1, we have some relationships between Hölder's spaces and Campanato spaces.

LEMMA 4.2. *Let $G = (V_G, E_G)$, $\beta < 0$ and $1 \leqslant p < \infty$. Then*

(i) *For any $f \in \widetilde{L}^{p,\beta}(G)$, there exists a constant $C_\beta > 0$ depending only on β such that*

$$\|f\|_{\mathscr{C}^{0,-\beta}(G)} \leqslant C_\beta (\mathscr{D}(G)^4 + \Delta_G) \|f\|_{\widetilde{\mathscr{L}}^{p,\beta}(G)}. \tag{4.2}$$

(ii) *For any $f \in \mathscr{C}^{0,-\beta}(G)$, we have*

$$\|f\|_{\widetilde{\mathscr{L}}^{p,\beta}(G)} \leqslant \|f\|_{\mathscr{C}^{0,-\beta}(G)}. \tag{4.3}$$

Proof. First we prove (i). Let $f \in \widetilde{\mathscr{L}}^{p,\beta}(G)$ and fix $x, y \in V_G$ with $x \neq y$. It is clear that $d_G(x,y) \geqslant 1$. Let $r \in (1/2, 1)$. Then we have

$$\begin{aligned} &|f(x) - f(y)| \\ &= |f_{B_G(x,r)} - f_{B_G(y,r)}| \\ &\leqslant |f_{B_G(x,r)} - f_{B_G(y,d_G(x,y))}| + |f_{B_G(y,d_G(x,y))} - f_{B_G(x,r+d_G(x,y))}| \\ &\quad + |f_{B_G(y,r)} - f_{B_G(x,r+d_G(x,y))}|. \end{aligned} \tag{4.4}$$

Invoking Lemma 4.1, we obtain

$$|f_{B_G(x,r)} - f_{B_G(y,d_G(x,y))}| \leqslant C_\beta (\mathscr{D}(G)^4 + \Delta_G) d_G(x,y)^{-\beta} \|f\|_{\widetilde{\mathscr{L}}^{p,\beta}(G)}, \tag{4.5}$$

$$\begin{aligned} |f_{B_G(y,d_G(x,y))} - f_{B_G(x,r+d_G(x,y))}| &\leqslant C_\beta \mathscr{D}(G)^4 (r + d_G(x,y))^{-\beta} \|f\|_{\widetilde{\mathscr{L}}^{p,\beta}(G)} \\ &\leqslant C_\beta \mathscr{D}(G)^4 d_G(x,y)^{-\beta} \|f\|_{\widetilde{\mathscr{L}}^{p,\beta}(G)}, \end{aligned} \tag{4.6}$$

$$\begin{aligned} &|f_{B_G(y,r)} - f_{B_G(x,r+d_G(x,y))}| \\ &\leqslant C_\beta (\mathscr{D}(G)^4 + \Delta_G)(r + d_G(x,y))^{-\beta} \|f\|_{\widetilde{\mathscr{L}}^{p,\beta}(G)} \\ &\leqslant C_\beta (\mathscr{D}(G)^4 + \Delta_G) d_G(x,y)^{-\beta} \|f\|_{\widetilde{\mathscr{L}}^{p,\beta}(G)}. \end{aligned} \tag{4.7}$$

Inequality (4.4) together with (4.5)–(4.7) implies

$$|f(x) - f(y)| \leqslant C_\beta (\mathscr{D}(G)^4 + \Delta_G) d_G(x,y)^{-\beta} \|f\|_{\widetilde{\mathscr{L}}^{p,\beta}(G)},$$

which gives (4.2).

Now we prove (ii). Let $f \in \mathscr{C}^{0,-\beta}(G)$. Fix $r \geqslant 1$ and $x \in V_G$, one has

$$\begin{aligned} |f(v) - f_{B_G(x,r)}| &\leqslant \frac{1}{|B_G(x,r)|} \sum_{u \in B_G(x,r)} |f(v) - f(u)| \\ &\leqslant \sup_{u \in B_G(x,r)} d_G(u,v)^{-\beta} \|f\|_{\mathscr{C}^{0,-\beta}(G)} \leqslant r^{-\beta} \|f\|_{\mathscr{C}^{0,-\beta}(G)} \end{aligned}$$

for all $v \in B(x,r)$. It follows that

$$r^{\beta}\Big(\frac{1}{|B_G(x,r)|}\sum_{v\in B_G(x,r)}|f(v)-f_{B_G(x,r)}|^p\Big)^{1/p} \leqslant \|f\|_{\mathscr{C}^{0,-\beta}(G)},$$

which leads to (4.3) and completes the proof. □

As an application of Lemma 4.2, we have

COROLLARY 4.3. *Let $G=(V_G,E_G)$, $Q\geqslant 1$, $\beta<0$ and $1\leqslant p<\infty$.*
(i) *If G has a lower bound Q condition, then*

$$\|f\|_{\mathscr{L}^{p,\beta}(G)} \leqslant C_{\beta,Q,\mathscr{B}_{1,Q}}\|f\|_{\mathscr{C}^{0,-\beta Q}(G)}.$$

(ii) *If G has a upper bound Q condition, then*

$$\|f\|_{\mathscr{C}^{0,-\beta Q}(G)} \leqslant C_{\beta,Q,\mathscr{B}_{2,Q}}(\mathscr{D}_2(G)^4+\Delta_G)\|f\|_{\mathscr{L}^{p,\beta}(G)}.$$

4.2. Main results and proofs

The main results of this section can be stated as follows:

THEOREM 4.4. *Let $G=(V_G,E_G)$ and $\delta\in(0,1]$. Let $\mathscr{D}(G)<\infty$, $\Delta_G<\infty$, $\mathscr{O}(G)<\infty$ and G satisfy the δ-annular decay property. Assume that one of the following conditions holds:*
(i) *$0<\alpha<\delta$ and $\beta=0$;*
(ii) *$0\leqslant\alpha\leqslant\delta$, $0\leqslant\alpha-\beta\leqslant\delta$ and $\beta\neq 0$.*
Then the map $\widetilde{M}_{\alpha,G}:\widetilde{\mathscr{L}}^{p,\beta}(G)\to\mathscr{C}^{0,\alpha-\beta}(G)$ is bounded. More precisely,

$$\|\widetilde{M}_{\alpha,G}f\|_{\mathscr{C}^{0,\alpha-\beta}(G)} \leqslant C_{\beta,\mathscr{D}(G),\mathscr{O}(G),\Delta_G,\mathscr{B}_{3,\delta}}\|f\|_{\widetilde{\mathscr{L}}^{p,\beta}(G)}, \quad \forall f\in\widetilde{\mathscr{L}}^{p,\beta}(G).$$

Proof. Let $f\in\widetilde{\mathscr{L}}^{p,\beta}(G)$. Fix $x,y\in V_G$ with $x\neq y$ and let $a=d_G(x,y)$. It suffices to show that

$$|\widetilde{M}_{\alpha,G}f(x)-\widetilde{M}_{\alpha,G}f(y)| \leqslant C_{\beta,\mathscr{D}(G),\Delta_G,\mathscr{B}_{3,\delta}}a^{\alpha-\beta}\|f\|_{\widetilde{\mathscr{L}}^{p,\beta}(G)}. \tag{4.8}$$

Without loss of generality we may assume that $\widetilde{M}_{\alpha,G}f(x)>\widetilde{M}_{\alpha,G}f(y)$. By the definition of $\widetilde{M}_{\alpha,G}f(x)$, there exists $r>0$ such that

$$\widetilde{M}_{\alpha,G}f(x)\leqslant r^{\alpha}f_{B_G(x,r)}+a^{\alpha-\beta}\|f\|_{\widetilde{\mathscr{L}}^{p,\beta}(G)}.$$

It follows that

$$\begin{aligned}|\widetilde{M}_{\alpha,G}f(x)-\widetilde{M}_{\alpha,G}f(y)| &\leqslant r^{\alpha}f_{B_G(x,r)}-(r+a)^{\alpha}f_{B_G(y,r+a)}+a^{\alpha-\beta}\|f\|_{\widetilde{\mathscr{L}}^{p,\beta}(G)}\\ &\leqslant r^{\alpha}(f_{B_G(x,r)}-f_{B_G(y,r+a)})+a^{\alpha-\beta}\|f\|_{\widetilde{\mathscr{L}}^{p,\beta}(G)}.\end{aligned}$$

Therefore, inequality (4.8) reduces to the following

$$r^{\alpha}(f_{B_G(x,r)} - f_{B_G(y,r+a)}) \leqslant C_{\beta,\mathscr{D}(G),\mathscr{O}(G),\Delta_G,\mathscr{B}_{3,\delta}} a^{\alpha-\beta} \|f\|_{\widetilde{\mathscr{L}}^{p,\beta}(G)}. \tag{4.9}$$

We consider the different cases:

Case 1: $(r \leqslant a)$. By Lemma 4.1, we have

$$|f_{B_G(x,r)} - f_{B_G(y,r+a)}| \leqslant C_{\beta,\mathscr{D}(G),\Delta_G} r^{-\beta} \|f\|_{\widetilde{\mathscr{L}}^{p,\beta}(G)}, \quad \text{if } \beta > 0;$$

$$\begin{aligned} |f_{B_G(x,r)} - f_{B_G(y,r+a)}| &\leqslant C_{\beta,\mathscr{D}(G),\Delta_G} (r+a)^{-\beta} \|f\|_{\widetilde{\mathscr{L}}^{p,\beta}(G)} \\ &\leqslant C_{\beta,\mathscr{D}(G),\Delta_G} a^{-\beta} \|f\|_{\widetilde{\mathscr{L}}^{p,\beta}(G)}, \quad \text{if } \beta < 0; \end{aligned}$$

$$|f_{B_G(x,r)} - f_{B_G(y,r+a)}| \leqslant C_{\beta,\mathscr{D}(G),\Delta_G} \Big(\ln \frac{r+a}{r} + 1\Big) \|f\|_{\widetilde{\mathscr{L}}^{p,\beta}(G)}, \quad \text{if } \beta = 0.$$

It follows that

$$\begin{aligned} r^{\alpha}(f_{B_G(x,r)} - f_{B_G(y,r+a)}) &\leqslant C_{\beta,\mathscr{D}(G),\Delta_G} r^{\alpha-\beta} \|f\|_{\widetilde{\mathscr{L}}^{p,\beta}(G)} \\ &\leqslant C_{\beta,\mathscr{D}(G),\Delta_G} a^{\alpha-\beta} \|f\|_{\widetilde{\mathscr{L}}^{p,\beta}(G)}, \quad \text{if } \beta > 0, \end{aligned}$$

since $\alpha - \beta \geqslant 0$ and $r \leqslant a$. Moreover,

$$\begin{aligned} r^{\alpha}(f_{B_G(x,r)} - f_{B_G(y,r+a)}) &\leqslant C_{\beta,\mathscr{D}(G),\Delta_G} r^{\alpha} a^{-\beta} \|f\|_{\widetilde{\mathscr{L}}^{p,\beta}(G)} \\ &\leqslant C_{\beta,\mathscr{D}(G),\Delta_G} a^{\alpha-\beta} \|f\|_{\widetilde{\mathscr{L}}^{p,\beta}(G)}, \quad \text{if } \beta < 0; \end{aligned}$$

$$\begin{aligned} r^{\alpha}(f_{B_G(x,r)} - f_{B_G(y,r+a)}) &\leqslant C_{\mathscr{D}(G),\Delta_G} r^{\alpha} \Big(\ln \frac{r+a}{r} + 1\Big) \|f\|_{\widetilde{\mathscr{L}}^{p,\beta}(G)} \\ &\leqslant C_{\mathscr{D}(G),\Delta_G} a^{\alpha} \Big(\Big(\frac{2a}{r}\Big)^{-\alpha} \ln \frac{2a}{r} + 1\Big) \|f\|_{\widetilde{\mathscr{L}}^{p,\beta}(G)} \\ &\leqslant C_{\mathscr{D}(G),\Delta_G} a^{\alpha} \|f\|_{\widetilde{\mathscr{L}}^{p,\beta}(G)}, \quad \text{if } \beta = 0. \end{aligned}$$

Here we used the fact that the function $t^{-\alpha} \ln t \leqslant \frac{1}{\alpha e}$ for all $t \geqslant 1$. This proves (4.9) in this case.

Case 2: $(r > a)$. We write

$$\begin{aligned} & f_{B_G(x,r)} - f_{B_G(y,r+a)} \\ &= \frac{1}{|B_G(x,r)|} \Big(\sum_{u \in B_G(x,r)} f(u) - \frac{|B_G(x,r)|}{|B_G(y,r+a)|} \sum_{v \in B_G(y,r+a)} f(v) \Big) \\ &= \frac{1}{|B_G(x,r)|} \Big(\sum_{u \in B_G(x,r)} f(u) - \sum_{u \in B_G(y,r+a)} f(u) \\ &\quad + \frac{|B_G(y,r+a)| - |B_G(x,r)|}{|B_G(y,r+a)|} \sum_{v \in B_G(y,r+a)} f(v) \Big) \\ &= \frac{1}{|B_G(x,r)|} \Big(- \sum_{u \in B_G(y,r+a) \setminus B_G(x,r)} f(u) + (|B_G(y,r+a)| - |B_G(x,r)|) f_{B_G(y,r+a)} \Big) \\ &= \frac{1}{|B_G(x,r)|} \sum_{u \in B_G(y,r+a) \setminus B_G(x,r)} (f_{B_G(y,r+a)} - f(u)). \end{aligned}$$

It follows that

$$|f_{B_G(x,r)} - f_{B_G(y,r+a)}| \leqslant \frac{1}{|B_G(x,r)|} \sum_{u\in B_G(y,r+a)\setminus B_G(x,r)} |f(u) - f_{B_G(y,r+a)}|.$$

Note that $a \in \mathbb{N}\setminus\{0\}$. Without loss of generality we may assume that $r \in \mathbb{N}\setminus\{0\}$. By the δ-annular decay property of G,

$$\begin{aligned}|B_G(y,r+a)\setminus B_G(x,r)| &= |B_G(y,r+a)| - |B_G(x,r)| \\ &\leqslant |B_G(x,r+2a)| - |B_G(x,r)| \\ &\leqslant \mathscr{B}_{3,\delta}\Big(\frac{2a}{r+2a}\Big)^{\delta} |B_G(x,r+2a)|.\end{aligned}$$

When $\beta < 0$, by Lemma 4.2 (i), we have

$$\begin{aligned}&|f(u) - f_{B_G(y,r+a)}| \\ &= \frac{1}{|B_G(y,r+a)|} \sum_{v\in B_G(y,r+a)} |f(v) - f(u)| \\ &\leqslant \sup_{v\in B_G(y,r+a)} |f(v) - f(u)| \leqslant \sup_{v\in B_G(y,r+a)} d(u,v)^{-\beta} \|f\|_{\mathscr{C}^{0,-\beta}(G)} \\ &\leqslant C_\beta(\mathscr{D}(G)^4 + \Delta_G)(r+a)^{-\beta} \|f\|_{\widetilde{\mathscr{L}}^{p,\beta}(G)}, \quad \forall u \in B_G(y,r+a).\end{aligned}$$

Hence, we get

$$\begin{aligned}&r^\alpha |f_{B_G(x,r)} - f_{B_G(y,r+a)}| \\ &\leqslant C_{\beta,\mathscr{D}(G),\Delta_G,\mathscr{B}_{3,\delta}} (r+a)^{-\beta} \Big(\frac{2a}{r+2a}\Big)^{\delta} \frac{|B_G(x,r+2a)|}{|B_G(x,r)|} \|f\|_{\widetilde{\mathscr{L}}^{p,\beta}(G)} \\ &\leqslant C_{\beta,\mathscr{D}(G),\Delta_G,\mathscr{B}_{3,\delta}} r^\alpha (r+a)^{-\beta} \Big(\frac{2a}{r+2a}\Big)^{\delta} \|f\|_{\widetilde{\mathscr{L}}^{p,\beta}(G)} \\ &\leqslant C_{\beta,\mathscr{D}(G),\Delta_G,\mathscr{B}_{3,\delta}} a^{\alpha-\beta} \Big(\frac{r}{a}\Big)^{\alpha-\beta-\delta} \|f\|_{\widetilde{\mathscr{L}}^{p,\beta}(G)} \\ &\leqslant C_{\beta,\mathscr{D}(G),\Delta_G,\mathscr{B}_{3,\delta}} a^{\alpha-\beta} \|f\|_{\widetilde{\mathscr{L}}^{p,\beta}(G)},\end{aligned}$$

since $\alpha - \beta - \delta \leqslant 0$. This proves (4.9) in this case.

When $\beta \geqslant 0$. Let $F := B_G(y,r+a)\setminus B_G(x,r)$. It is clear that $F \subset \bigcup_{v\in F} B_G(v,a)$. Since the graph G satisfies $\mathscr{O}(G) < \infty$, then there exists a set $E \subset F$ such that $\sum_{v\in E} \chi_{B_G(v,a)} \leqslant \mathscr{O}(G)$ and $\bigcup_{v\in F} B_G(v,a) = \bigcup_{v\in E} B_G(v,a)$. We write

$$\begin{aligned}&|f_{B_G(x,r)} - f_{B_G(y,r+a)}| \\ &\leqslant \frac{1}{|B_G(x,r)|} \sum_{u\in B_G(y,r+a)\setminus B_G(x,r)} |f(u) - f_{B_G(y,r+a)}| \\ &\leqslant \frac{1}{|B_G(x,r)|} \sum_{v\in E} \sum_{u\in B_G(v,a)} |f(u) - f_{B_G(y,r+a)}| \\ &= \frac{1}{|B_G(x,r)|} \sum_{v\in E} |B_G(v,a)| \frac{1}{|B_G(v,a)|} \sum_{u\in B_G(v,a)} |f(u) - f_{B_G(y,r+a)}|.\end{aligned} \tag{4.10}$$

One can easily check that

$$\bigcup_{v\in E} B_G(v,a) \subset B_G(y,r+2a)\setminus B_G(x,r-a) \subset B_G(x,r+3a)\setminus B_G(x,r-a).$$

This together with the δ-annular decay property of G gives that

$$\begin{aligned}
&\sum_{v\in E}|B_G(v,a)| \leqslant \mathscr{O}(G)\Big|\bigcup_{v\in E} B_G(v,a)\Big| \\
&\leqslant \mathscr{O}(G)|B_G(x,r+3a)\setminus B_G(x,r-a)| \\
&\leqslant \mathscr{O}(G)(|B_G(x,r+3a)|-|B_G(x,r-a)|) \\
&\leqslant \mathscr{O}(G)\mathscr{B}_{3,\delta}\Big(\frac{4a}{r+3a}\Big)^{\delta}|B_G(x,r+3a)| \\
&\leqslant 4\mathscr{O}(G)\mathscr{B}_{3,\delta}\Big(\frac{a}{r}\Big)^{\delta}|B_G(x,r+3a)|.
\end{aligned} \tag{4.11}$$

On the other hand, for a fixed $v\in E$, we have

$$\begin{aligned}
&\frac{1}{|B_G(v,a)|}\sum_{u\in B_G(v,a)}|f_{B_G(y,r+a)}-f(u)| \\
&\leqslant \frac{1}{|B_G(v,a)|}\sum_{u\in B_G(v,a)}|f(u)-f_{B_G(v,a)}|+|f_{B_G(v,a)}-f_{B_G(y,r+a)}|.
\end{aligned}$$

Since $v\in B_G(y,r+a)\setminus B_G(x,r)$, invoking Lemma 4.1, we get

$$|f_{B_G(v,a)}-f_{B_G(y,r+a)}| \leqslant C_\beta \mathscr{D}(G)^4 a^{-\beta}\|f\|_{\widetilde{\mathscr{L}}^{p,\beta}(G)}, \quad \text{if } \beta>0;$$

$$|f_{B_G(v,a)}-f_{B_G(y,r+a)}| \leqslant C_\beta \mathscr{D}(G)^4\Big(\ln\frac{r+a}{2a}+1\Big)\|f\|_{\widetilde{\mathscr{L}}^{p,\beta}(G)}, \quad \text{if } \beta=0.$$

By Hölder's inequality, we get

$$\begin{aligned}
&\frac{1}{|B_G(v,a)|}\sum_{u\in B_G(v,a)}|f(u)-f_{B_G(v,a)}| \\
&\leqslant \Big(\frac{1}{|B_G(v,a)|}\sum_{u\in B_G(v,a)}|f(u)-f_{B_G(v,a)}|^p\Big)^{1/p} \leqslant a^{-\beta}\|f\|_{\widetilde{\mathscr{L}}^{p,\beta}(G)}.
\end{aligned}$$

It follows that

$$\frac{1}{|B_G(v,a)|}\sum_{u\in B_G(v,a)}|f(u)-f_{B_G(y,r+a)}| \leqslant C_\beta \mathscr{D}(G)^4 a^{-\beta}\|f\|_{\widetilde{\mathscr{L}}^{p,\beta}(G)}, \quad \text{if } \beta>0; \tag{4.12}$$

$$\begin{aligned}
&\frac{1}{|B_G(v,a)|}\sum_{u\in B_G(v,a)}|f(u)-f_{B_G(y,r+a)}| \\
&\leqslant C_\beta \mathscr{D}(G)^4\Big(\ln\frac{r+a}{2a}+1\Big)\|f\|_{\widetilde{\mathscr{L}}^{p,\beta}(G)}, \quad \text{if } \beta=0.
\end{aligned} \tag{4.13}$$

When $\beta > 0$, we get from (4.10)–(4.12) that

$$\begin{aligned}
&r^{\alpha}|f_{B_G(x,r)} - f_{B_G(y,r+a)}| \\
&\leqslant C_{\beta,\mathscr{D}(G),\mathscr{O}(G),\mathscr{B}_{3,\delta}} r^{\alpha} a^{-\beta} \left(\frac{a}{r}\right)^{\delta} \frac{|B_G(x,r+3a)|}{|B_G(x,r)|} \|f\|_{\widetilde{\mathscr{L}}^{p,\beta}(G)} \\
&\leqslant C_{\beta,\mathscr{D}(G),\mathscr{O}(G),\mathscr{B}_{3,\delta}} a^{\alpha-\beta} \left(\frac{r}{a}\right)^{\alpha-\delta} \|f\|_{\widetilde{\mathscr{L}}^{p,\beta}(G)} \leqslant C_{\beta,\mathscr{D}(G),\mathscr{O}(G),\mathscr{B}_{3,\delta}} a^{\alpha-\beta} \|f\|_{\widetilde{\mathscr{L}}^{p,\beta}(G)}.
\end{aligned}$$

since $\alpha \leqslant \delta$. This proves (4.9) in this case.

When $\beta = 0$, it follows from (4.10), (4.11) and (4.13) that

$$\begin{aligned}
&r^{\alpha}|f_{B_G(x,r)} - f_{B_G(y,r+a)}| \\
&\leqslant C_{\beta,\mathscr{D}(G),\mathscr{O}(G),\mathscr{B}_{3,\delta}} r^{\alpha} \left(\ln \frac{r+a}{2a} + 1\right) \left(\frac{a}{r}\right)^{\delta} \frac{|B_G(x,r+3a)|}{|B_G(x,r)|} \|f\|_{\widetilde{\mathscr{L}}^{p,\beta}(G)} \\
&\leqslant C_{\beta,\mathscr{D}(G),\mathscr{O}(G),\mathscr{B}_{3,\delta}} a^{\alpha} \left(\frac{r}{a}\right)^{-(\delta-\alpha)} \left(\ln \frac{r}{a} + 1\right) \|f\|_{\widetilde{\mathscr{L}}^{p,\beta}(G)} \\
&\leqslant C_{\beta,\mathscr{D}(G),\mathscr{O}(G),\mathscr{B}_{3,\delta}} a^{\alpha} \|f\|_{\widetilde{\mathscr{L}}^{p,\beta}(G)},
\end{aligned}$$

since $\alpha < \delta$ and the function $g(t) = t^{-(\delta-\alpha)} \log t \leqslant \frac{1}{(\delta-\alpha)e}$ for all $t \geqslant 1$. This proves (4.9) in this case and completes the proof of Theorem 4.4. $\square$

THEOREM 4.5. *Let $G = (V_G, E_G)$ and $\delta \in (0,1]$. Let $\mathscr{D}(G) < \infty$, $\mathscr{O}(G) < \infty$, $\Delta_G < \infty$ and G satisfy the δ-annular decay property. Assume that one of the following conditions holds:*

(i) *G satisfies the upper bound condition, $0 < \alpha < \delta/Q$ and $\beta = 0$;*
(ii) *G satisfies the upper bound condition, $0 \leqslant \alpha \leqslant \delta/Q + \beta$ and $\beta < 0$;*
(ii) *G satisfies the lower and upper bound conditions, $0 < \beta \leqslant \alpha \leqslant \delta/Q$.*

Then the map $M_{\alpha,G} : \mathscr{L}^{p,\beta}(G) \to \mathscr{C}^{0,(\alpha-\beta)Q}(G)$ is bounded. More precisely,

$$\|M_{\alpha,G} f\|_{\mathscr{C}^{0,(\alpha-\beta)Q}(G)} \leqslant C_{\beta,\mathscr{D}(G),\mathscr{O}(G),\Delta_G,\mathscr{B}_{2,Q},\mathscr{B}_{3,\delta}} \|f\|_{\mathscr{L}^{p,\beta}(G)}, \quad \forall f \in \mathscr{L}^{p,\beta}(G).$$

Proof. We assume that $\|f\|_{\mathscr{L}^{p,\beta}(G)} = 1$. Fix $x, y \in V_G$ with $x \neq y$ and let $a = d_G(x,y)$. We want to show that

$$|M_{\alpha,G} f(x) - M_{\alpha,G} f(y)| \leqslant C_{\beta,\mathscr{D}(G),\Delta_G,\mathscr{B}_{2,Q},\mathscr{B}_{3,\delta}} a^{(\alpha-\beta)Q}. \tag{4.14}$$

Without loss of generality we may assume that $M_{\alpha,G} f(x) > M_{\alpha,G} f(y)$. By the definition of $M_{\alpha,G} f(x)$, there exists $r \in \mathbb{N}$ such that

$$M_{\alpha,G} f(x) \leqslant |B_G(x,r)|^{\alpha} f_{B_G(x,r)} + a^{(\alpha-\beta)Q}.$$

By the upper bound condition on G,

$$\begin{aligned}
&M_{\alpha,G} f(x) - M_{\alpha,G} f(y)| \\
&\leqslant |B_G(x,r)|^{\alpha} f_{B_G(x,r)} - |B_G(y,r+a)|^{\alpha} f_{B_G(y,r+a)} + a^{(\alpha-\beta)Q} \\
&\leqslant |B_G(x,r)|^{\alpha} (f_{B_G(x,r)} - f_{B_G(y,r+a)}) + a^{(\alpha-\beta)Q} \\
&\leqslant \mathscr{B}_{2,Q}^{\alpha} r^{\alpha Q} (f_{B_G(x,r)} - f_{B_G(y,r+a)}) + a^{(\alpha-\beta)Q},
\end{aligned} \tag{4.15}$$

$$\|f\|_{\widetilde{\mathscr{L}}^{p,\beta Q}(G)} \leqslant C_{\beta,\mathscr{B}_{2,Q}}\|f\|_{\mathscr{L}^{p,\beta}(G)}, \text{ if } \beta < 0. \tag{4.16}$$

By the lower bound condition on G,

$$\|f\|_{\widetilde{\mathscr{L}}^{p,\beta Q}(G)} \leqslant C_{\beta,\mathscr{B}_{1,Q}}\|f\|_{\mathscr{L}^{p,\beta}(G)}, \text{ if } \beta > 0. \tag{4.17}$$

It is clear that

$$\|f\|_{\widetilde{\mathscr{L}}^{p,0}(G)} = \|f\|_{\mathscr{L}^{p,0}(G)}. \tag{4.18}$$

Similar arguments to those in getting (4.9) will imply that

$$r^{\alpha Q}(f_{B_G(x,r)} - f_{B_G(y,r+a)}) \leqslant C_{\beta,Q,\mathscr{D}(G),\mathscr{O}(G),\Delta_G,\mathscr{B}_{3,\delta}} a^{(\alpha-\beta)Q}\|f\|_{\widetilde{\mathscr{L}}^{p,\beta Q}(G)}, \tag{4.19}$$

provided that one of the following conditions holds:

(i) $0 < \alpha < \delta/Q$ and $\beta = 0$;

(ii) $0 \leqslant \alpha \leqslant \delta/Q$, $0 \leqslant \alpha - \beta \leqslant \delta/Q$ and $\beta \neq 0$.

Then (4.14) follows from (4.15)–(4.19). Theorem 4.5 is proved. □

4.3. Some applications

As applications of Theorems 2.1 and 2.2 and (4.1), the following theorem is valid.

COROLLARY 4.6. *Let $1 < p < \infty$ and $G = (V_G, E_G)$. Assume that $\mathscr{D}(G) < \infty$ and $1 < \widetilde{\mathscr{D}}(G) < \infty$. Then*

(i) *Let $\beta > 0$. Then the map $M_G : L^{p,\beta}(G) \to \mathscr{L}^{p,\beta}(G)$ is bounded. Moreover,*

$$\|M_G f\|_{\mathscr{L}^{p,\beta}(G)} \leqslant C_{p,\beta,\mathscr{D}(G),\widetilde{\mathscr{D}}(G)}\|f\|_{L^{p,\beta}(G)}, \quad \forall f \in L^{p,\beta}(G).$$

(ii) *Let $0 < \alpha < 1$, $\beta > \alpha$ and $q = p\beta/(\beta-\alpha)$. Then the map $M_{\alpha,G} : L^{p,\beta}(G) \to \mathscr{L}^{q,\beta-\alpha}(G)$ is bounded. Moreover,*

$$\|M_{\alpha,G} f\|_{\mathscr{L}^{q,\beta-\alpha}(G)} \leqslant C_{p,\beta,\alpha,\mathscr{D}(G),\widetilde{\mathscr{D}}(G)}\|f\|_{L^{p,\beta}(G)}, \quad \forall f \in L^{p,\beta}(G).$$

(iii) *Let $0 < \alpha < 1$, $\beta > \alpha$ and $q = p\beta/(\beta-\alpha)$. Then the map $\widetilde{M}_{\alpha,G} : \widetilde{L}^{p,\beta}(G) \to \widetilde{\mathscr{L}}^{q,\beta-\alpha}(G)$ is bounded. Moreover,*

$$\|\widetilde{M}_{\alpha,G} f\|_{\widetilde{\mathscr{L}}^{q,\beta-\alpha}(G)} \leqslant C_{p,\beta,\alpha,\mathscr{D}(G),\widetilde{\mathscr{D}}(G)}\|f\|_{\widetilde{L}^{p,\beta}(G)}, \quad \forall f \in \widetilde{L}^{p,\beta}(G).$$

Applying Theorem 4.4 and Lemma 4.2, we have

COROLLARY 4.7. *Let $G = (V_G, E_G)$, $\delta \in (0,1]$ and $\beta \in [-\delta, 0)$. Let $\mathscr{D}(G) < \infty$, $\mathscr{O}(G) < \infty$, $\Delta_G < \infty$ and G satisfy the δ-annular decay property. Then the map $M_G : \widetilde{\mathscr{L}}^{p,\beta}(G) \to \widetilde{\mathscr{L}}^{p,\beta}(G)$ is bounded. More precisely,*

$$\|M_G f\|_{\widetilde{\mathscr{L}}^{p,\beta}(G)} \leqslant C_{\beta,\mathscr{D}(G),\mathscr{O}(G),\Delta_G,\mathscr{B}_{3,\delta}}\|f\|_{\widetilde{\mathscr{L}}^{p,\beta}(G)}, \quad \forall f \in \widetilde{\mathscr{L}}^{p,\beta}(G).$$

COROLLARY 4.8. *Let $G=(V_G,E_G)$ and $\delta\in(0,1]$. Let $\mathscr{D}(G)<\infty$, $\mathscr{O}(G)<\infty$, $\Delta_G<\infty$ and G satisfy the δ-annular decay property. Assume that one of the following conditions holds:*

(i) $0<\alpha<\delta$ *and* $\beta=0$;

(ii) $0<\alpha\leqslant\delta$, $0<\alpha-\beta\leqslant\delta$ *and* $\beta\neq 0$.

Then the map $\widetilde{M}_{\alpha,G}:\widetilde{\mathscr{L}}^{p,\beta}(G)\to\widetilde{\mathscr{L}}^{p,\beta-\alpha}(G)$ is bounded. More precisely,

$$\|\widetilde{M}_{\alpha,G}f\|_{\widetilde{\mathscr{L}}^{p,\beta-\alpha}(G)}\leqslant C_{\beta,\mathscr{D}(G),\mathscr{O}(G),\Delta_G,\mathscr{B}_{3,\delta}}\|f\|_{\widetilde{\mathscr{L}}^{p,\beta}(G)},\quad\forall f\in\widetilde{\mathscr{L}}^{p,\beta}(G).$$

Applying Theorem 4.5 and Corollary 4.3, we have

COROLLARY 4.9. *Let $G=(V_G,E_G)$, $\delta\in(0,1]$, $Q\geqslant 1$ and $\beta\in[-\delta/Q,0)$. Let $\mathscr{D}(G)<\infty$, $\mathscr{O}(G)<\infty$, $\Delta_G<\infty$ and G satisfy the δ-annular decay property and the lower and upper bound Q conditions. Then the map $M_G:\mathscr{L}^{p,\beta}(G)\to\mathscr{L}^{p,\beta}(G)$ is bounded. More precisely,*

$$\|M_Gf\|_{\mathscr{L}^{p,\beta}(G)}\leqslant C_{\beta,\mathscr{D}(G),\mathscr{O}(G),\Delta_G,\mathscr{B}_{1,Q},\mathscr{B}_{2,Q},\mathscr{B}_{3,\delta}}\|f\|_{\mathscr{L}^{p,\beta}(G)},\quad\forall f\in\mathscr{L}^{p,\beta}(G).$$

COROLLARY 4.10. *Let $G=(V_G,E_G)$ and $\delta\in(0,1]$. Let $\mathscr{D}(G)<\infty$, $\mathscr{O}(G)<\infty$, $\Delta_G<\infty$ and G satisfy the δ-annular decay property. Assume that the graph G satisfies the lower and upper bound Q conditions. $\beta<\alpha$. Suppose that one of the following conditions holds:*

(i) $0<\alpha<\delta/Q$ *and* $\beta=0$;

(ii) $0<\alpha\leqslant\delta/Q+\beta$ *and* $\beta<0$;

(ii) $0<\beta<\alpha\leqslant\delta/Q$.

Then the map $M_{\alpha,G}:\mathscr{L}^{p,\beta}(G)\to\mathscr{L}^{p,\beta-\alpha}(G)$ is bounded. More precisely,

$$\|M_{\alpha,G}f\|_{\mathscr{L}^{p,\beta-\alpha}(G)}\leqslant C_{\beta,\mathscr{D}(G),\mathscr{O}(G),\Delta_G,\mathscr{B}_{1,Q},\mathscr{B}_{2,Q},\mathscr{B}_{3,\delta}}\|f\|_{\mathscr{L}^{p,\beta}(G)},\quad\forall f\in\mathscr{L}^{p,\beta}(G).$$

REMARK 4.3. The corresponding results in Theorems 4.4 and 4.5 and Corollaries 4.7-4.10 hold for the graph in $\{G_d\}_{d\in\mathbb{N}\setminus\{0\}}$. Here $\{G_d\}_{d\in\mathbb{N}\setminus\{0\}}$ are given as in Remark 1.4.

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