

SOME GENERALIZED GRONWALL–BELLMAN TYPE DIFFERENCE INEQUALITIES AND APPLICATIONS

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Abstract. We establish some generalized sums-difference inequalities involving a finite sum, which includes three sums and seven sums, respectively. We present the estimation of the inequality is decided by a finite recursion. Using the lemma and difference techniques, we transform the complex difference inequalities into the simple forms of difference inequalities. We apply our results to boundary value problem of a partial difference equation for uniform boundedness, uniqueness and continuous dependence of the solutions.

1. Introduction

Gronwall-Bellman integral inequality is an important tool in the study of existence, uniqueness, boundedness of solutions of differential equations and integral equations. Various generalizations of Gronwall-Bellman type integral inequality [7, 21] and their applications have attracted great attention of many mathematicians. Some recent works of integral inequality with one variable can be found, e.g., in [5, 6, 8, 9, 16, 23, 32, 36, 47, 51] and some references therein. Some works of integral inequality with two variables can be found in [10, 13, 17, 22, 26, 33, 42] and some references therein. Agarwal et al. [5] investigated the inequality

$$u(t) \leq a(t) + \sum_{i=1}^n \int_{b_i(t_0)}^{b_i(t)} g_i(t, s) w_i(u(s)) ds, \quad t_0 \leq t < t_1.$$

Agarwal et al. [6] discussed the retarded integral inequality

$$\varphi(u(t)) \leq c + \sum_{i=1}^n \int_{\alpha_i(t_0)}^{\alpha_i(t)} u^q(s) [f_i(s)\varphi(u(s)) + g_i(s)] ds,$$

where c is a constant. Zhou et al. [51] studied the following retarded integral inequality

$$u(t) \leq a(t) + \sum_{i=1}^n \left\{ \int_{b_i(t_0)}^{b_i(t)} g_i(t, s) w_i(u(s)) ds \right\}^{p_i}, \quad t_0 \leq t < \infty,$$

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where $n \in \mathbb{N}, p_i \geq 1$, and all a, b_i, g_i, w_i and u are nonnegative continuous functions for $i = 1, 2, \dots, n$. Kim [22] studied the retarded integral inequality with two variables

$$\psi(u(x, y)) \leq a(x, y) + c(x, y) \sum_{i=1}^n \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(y_0)}^{\beta_i(y)} u^q(s, t) [f_i(s, t)\psi(u(s, t)) + g_i(s, t)] dt ds,$$

for all $(x, y) \in [x_0, X] \times [y_0, Y]$. Wang et al. [42] considered the two variables of integral inequality

$$\begin{aligned} \psi(u(x, y)) \leq & a(x, y) + \sum_{i=1}^n \left\{ \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(y_0)}^{\beta_i(y)} u^q(s, t) g_i(x, y, s, t) ds dt \right. \\ & \left. + \int_{\delta_i(x_0)}^{\delta_i(x)} \int_{\gamma_i(y_0)}^{\gamma_i(y)} u^q(s, t) f_i(x, y, s, t) \phi_i(u(s, t)) ds dt \right\}. \end{aligned}$$

The inequality includes not only a nonconstant term outside the integrals but also more than one distinct nonlinear integrals without assumption of monotonicity. Using analytical techniques, they obtained an appropriate upper bound estimation.

The theory of discrete inequalities of Gronwall-Bellman type is very important in studying qualitative characteristics of solutions of difference equations. Recently, with the development of the theory of difference equations, more attentions are paid on some discrete versions of Gronwall type inequalities (e.g., [3, 44] for some early works). Some recent works of difference inequality with one variable can be found, e.g., in [1, 2, 4, 18, 19, 25, 34, 35, 37, 38, 39, 45, 46, 50] and some references therein. Some works of difference inequality with two variables can be found in [11, 12, 14, 15, 20, 27, 28, 29, 30, 31, 40, 41, 43, 48, 49]. A fundamental one of those known results is the sum-difference inequality

$$u(n) \leq a(n) + b(n) \sum_{k=k_0}^{n-1} f(k)u(k), \quad n \geq k_0,$$

as shown in [3], the unknown function $u(n)$ is estimated by

$$u(n) \leq a(n) + b(n) \sum_{k=k_0}^{n-1} a(k)f(k)\Pi_{s=k+1}^{n-1}(1 + b(s)f(s)), \quad n \geq k_0.$$

Pachpatte [32, 35] studied the inequality

$$\begin{aligned} u(n) & \leq a(n) + \sum_{s=0}^{n-1} f(s)w(u(s)), \\ u^2(n) & \leq c^2 + 2 \sum_{s=0}^{n-1} [f_1(s)u(s)w(u(s)) + f_2(s)u(s)], \end{aligned}$$

where c is a constant, f, f_1, f_2, w are both real-valued nonnegative functions defined on $\mathbb{N}_0 = \{0, 1, \dots, n\}$. Wu et al. [45] discussed the inequality

$$u(n) \leq a(n) + \sum_{i=1}^m \sum_{b_i(0)}^{b_i(n-1)} f_i(n, s)w_i(u(s)), \quad n \in \mathbb{N}_0,$$

where $a(n)$ is nonnegative, $b_i(n)$ are nondecreasing and $b_i(n) \leq n$, f_i are nonnegative, w_i are continuous and nondecreasing functions on $[0, \infty)$.

Meng et al. [30] studied the inequality with two variables

$$u^p(m, n) \leq a(m, n) + b(m, n) \sum_{s=0}^{m-1} \sum_{t=n+1}^{\infty} [c(s, t)u(s, t) + e(s, t)],$$

for all $m, n \in \mathbb{N}_0$, and $p \geq 1$ is a constant. Chueng et al. [11, 14] discussed the inequalities

$$u(m, n) \leq c + \sum_{s=0}^{m-1} \sum_{t=0}^{n-1} b(s, t)w(u(s, t)),$$

$$u^p(m, n) \leq c + \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} a(s, t)u(s, t) + \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} b(s, t)u(s, t)\varphi(u(s, t)),$$

where $c \geq 0$, and a, b are nonnegative real-valued functions in \mathbb{Z}_+^2 , and φ is a continuous nondecreasing function with $\varphi(r) > 0$, for $r > 0$. Ma and Cheung [27] studied the inequality

$$\psi(u(m, n)) \leq a(m, n) + c(m, n) \sum_{s=0}^{m-1} \sum_{t=n+1}^{\infty} \psi'(u(s, t))[d(s, t)w(u(s, t)) + e(s, t)].$$

Wang et al. [41] investigated the inequality

$$\psi(u(m, n)) \leq c(m, n) + \sum_{i=1}^k \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} f_i(m, n, s, t)\varphi_i(u(s, t)),$$

where ψ is a strictly increasing continuous function on \mathbb{R}_+ , satisfying that $\psi(\infty) = \infty$ and $\psi(u) > 0$ for all $u > 0$, φ_i are continuous and positive functions on \mathbb{R}_+ , $a(m, n) \geq 0$, f_i are nonnegative functions in the domain. Ma [29] considered the power difference inequality with two variables

$$u^p(m, n) \leq a(m, n) + c(m, n) \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} b(s, t)u^q(s, t) + \sum_{s=m_0}^{M-1} \sum_{t=n_0}^{N-1} c(s, t)u^r(s, t),$$

where $p \geq q > 0, p \geq r > 0$. Feng et al. [20] discussed the inequalities including four sums

$$u^p(m, n) \leq c(m, n) + \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} \left[b(s, t, m, n)u^q(s, t) + \sum_{\xi=m_0}^s \sum_{\eta=n_0}^t c(\xi, \eta, m, n)u^r(\xi, \eta) \right]$$

$$+ \sum_{s=m_0}^{M-1} \sum_{t=n_0}^{N-1} \left[d(s, t, m, n)u^h(s, t) + \sum_{\xi=m_0}^s \sum_{\eta=n_0}^t e(\xi, \eta, m, n)u^j(\xi, \eta) \right].$$

Li [24] investigated the five sums difference inequality

$$\begin{aligned} \psi(u(m,n)) \leq & c(m,n) + \sum_{i=1}^k \left(\sum_{s=0}^{m-1} \sum_{t=0}^{n-1} \sum_{j=0}^{s-1} \sum_{l=0}^{t-1} f_i(s,t,j,l)u^p(s,t) \right. \\ & \left. + \sum_{s=0}^{m-1} \sum_{t=0}^{n-1} \sum_{j=0}^{s-1} \sum_{l=0}^{t-1} h_i(s,t,j,l)u^q(s,t)\varphi_i(u(j,l)) \right). \end{aligned}$$

Motivated by the ideas in [5, 14, 24, 35, 41, 45, 51], in this paper, we establish some new more general forms of sums-difference inequalities, and we do not require the monotonicity to of the unknown functions. We apply a technique of monotonization to overcome the difficulty from nonmonotonicity and give the upper bound estimation of unknown function. We apply the obtained results to the bounded and uniqueness and continuous dependence of the solutions of the difference equation.

2. Lemma

Throughout this paper, \mathbb{R} denote the set of all real numbers, let $\mathbb{R}_+ := [0, \infty)$ and $\mathbb{N}_0 := \{0, 1, \dots\}$. $m_i, n_i \in \mathbb{N}_0, (i = 1, 2, \dots, k)$ are given numbers, $I_i := [0, m_i) \cap \mathbb{N}_0$ ($i = 1, 2, \dots, k$) are two fixed lattices of integer points in \mathbb{R} , $\Lambda_i := I_1 \times I_2 \times \dots \times I_i \subset \mathbb{N}_0^i, i = 1, 2, \dots, k$, $\Lambda_{(M_1, N_1, Z_1)} := [0, M_1) \cap \mathbb{N}_0 \times [0, N_1) \cap \mathbb{N}_0 \times [0, Z_1) \cap \mathbb{N}_0$. For any $s \in \Delta_1$, let Δ_s denote the sublattice $[0, s] \cap \Delta_1$ of Δ_1 . For functions $w(m), z(m, n, j), m, n \in \mathbb{N}_0$, let Δ denotes the forward difference operator, i.e. $\Delta w(m) := w(m + 1) - w(m)$ and $\Delta_1 z(m, n, j) := z(m + 1, n, j) - z(m, n, j)$. Obviously, the linear difference equation $\Delta x(m) = b(m)$ with the initial condition $x(0) = 0$ has the solution $\sum_{s=0}^{m-1} b(s)$. For convenience, in the sequel we define that $\sum_{s=0}^{-1} b(s) = 0$.

LEMMA 1. *Suppose w is a continuous and positive functions on \mathbb{R}_+ , f is a nonnegative function on $\Lambda_k \times \Lambda_k$, u is a nonnegative function on Λ_k , then we can obtain the following formula*

$$\begin{aligned} & \sum_{s_1=0}^{m_1-1} \dots \sum_{s_k=0}^{m_k-1} \sum_{j_1=0}^{s_1-1} \dots \sum_{j_k=0}^{s_k-1} f(s_1, \dots, s_k, j_1, \dots, j_k)w(u(j_1, \dots, j_k)) \\ & = \sum_{s_1=0}^{m_1-1} \dots \sum_{s_k=0}^{m_k-1} w(u(s_1, \dots, s_k)) \sum_{j_1=s_1+1}^{m_1-1} \dots \sum_{j_k=s_k+1}^{m_k-1} f(j_1, \dots, j_k, s_1, \dots, s_k). \end{aligned}$$

Proof. We use mathematical induction with respect to $m_i, i = 1, 2, \dots, k$. If $m_1 = m_2 = \dots = m_k = 2$, we obtain

$$\begin{aligned} & \sum_{s_1=0}^1 \dots \sum_{s_k=0}^1 \sum_{j_1=0}^{s_1-1} \dots \sum_{j_k=0}^{s_k-1} f(s_1, \dots, s_k, j_1, \dots, j_k)w(u(j_1, \dots, j_k)) \\ & = f(1, \dots, 1, 0, \dots, 0)w(u(0, \dots, 0)), \end{aligned}$$

$$\begin{aligned} & \sum_{s_1=0}^1 \cdots \sum_{s_k}^1 w(u(s_1, \dots, s_k)) \sum_{j_1=s_1+1}^1 \cdots \sum_{j_k=s_k+1}^1 f(j_1, \dots, j_k, s_1, \dots, s_k) \\ &= w(u(0, \dots, 0))f(1, \dots, 1, 0, \dots, 0). \end{aligned}$$

Thus, we have

$$\begin{aligned} & \sum_{s_1=0}^1 \cdots \sum_{s_k=0}^1 \sum_{j_1=0}^{s_1-1} \cdots \sum_{j_k=0}^{s_k-1} f(s_1, \dots, s_k, j_1, \dots, j_k)w(u(j_1, \dots, j_k)) \\ &= \sum_{s_1=0}^1 \cdots \sum_{s_k=0}^1 w(u(s_1, \dots, s_k)) \sum_{j_1=s_1+1}^1 \cdots \sum_{j_k=s_k+1}^1 f(j_1, \dots, j_k, s_1, \dots, s_k). \end{aligned}$$

It means that the lemma is true for $m_i = 2, i = 1, \dots, k$. Suppose that the lemma is true for $m_i = M_i, i = 1, \dots, k$, that is

$$\begin{aligned} & \sum_{s_1=0}^{M_1-1} \cdots \sum_{s_k=0}^{M_k-1} \sum_{j_1=0}^{s_1-1} \cdots \sum_{j_k=0}^{s_k-1} f(s_1, \dots, s_k, j_1, \dots, j_k)w(u(j_1, \dots, j_k)) \\ &= \sum_{s_1=0}^{M_1-1} \cdots \sum_{s_k=0}^{M_k-1} w(u(s_1, \dots, s_k)) \sum_{j_1=s_1+1}^{M_1-1} \cdots \sum_{j_k=s_k+1}^{M_k-1} f(j_1, \dots, j_k, s_1, \dots, s_k). \end{aligned}$$

Consider $m_i = M_i + 1, i = 1, \dots, k$, then we have

$$\begin{aligned} & \sum_{s_1=0}^{M_1} \cdots \sum_{s_k=0}^{M_k} w(u(s_1, \dots, s_k)) \sum_{j_1=s_1+1}^{M_1} \cdots \sum_{j_k=s_k+1}^{M_k} f(j_1, \dots, j_k, s_1, \dots, s_k) \\ &= \sum_{s_1=0}^{M_1-1} \cdots \sum_{s_k=0}^{M_k-1} w(u(s_1, \dots, s_k)) \sum_{j_1=s_1+1}^{M_1} \cdots \sum_{j_k=s_k+1}^{M_k} f(j_1, \dots, j_k, s_1, \dots, s_k) \\ &= \sum_{s_1=0}^{M_1-1} \cdots \sum_{s_k=0}^{M_k-1} w(u(s_1, \dots, s_k)) \sum_{j_1=s_1+1}^{M_1-1} \cdots \sum_{j_k=s_k+1}^{M_k-1} f(j_1, \dots, j_k, s_1, \dots, s_k) \\ & \quad + \sum_{s_1=0}^{M_1-1} \cdots \sum_{s_k=0}^{M_k-1} w(u(s_1, \dots, s_k))f(M_1, \dots, M_k, s_1, \dots, s_k) \\ &= \sum_{s_1=0}^{M_1-1} \cdots \sum_{s_k=0}^{M_k-1} w(u(s_1, \dots, s_k)) \sum_{j_1=s_1+1}^{M_1-1} \cdots \sum_{j_k=s_k+1}^{M_k-1} f(j_1, \dots, j_k, s_1, \dots, s_k) \\ & \quad + \sum_{j_1=0}^{M_1-1} \cdots \sum_{j_k=0}^{M_k-1} w(u(j_1, \dots, j_k))f(M_1, \dots, M_k, j_1, \dots, j_k) \\ &= \sum_{s_1=0}^{M_1} \cdots \sum_{s_k=0}^{M_k} \sum_{j_1=0}^{s_1-1} \cdots \sum_{j_k=0}^{s_k-1} f(s_1, \dots, s_k, j_1, \dots, j_k)w(u(j_1, \dots, j_k)) \end{aligned}$$

Using the inductive assumption, then, we obtain

$$\begin{aligned} & \sum_{s_1=0}^{M_1} \cdots \sum_{s_k=0}^{M_k} \sum_{j_1=0}^{s_1-1} \cdots \sum_{j_k=0}^{s_k-1} f(s_1, \dots, s_k, j_1, \dots, j_k) w(u(j_1, \dots, j_k)) \\ &= \sum_{s_1=0}^{M_1} \cdots \sum_{s_k=0}^{M_k} w(u(s_1, \dots, s_k)) \sum_{j_1=s_1+1}^{M_1} \cdots \sum_{j_k=s_k+1}^{M_k} f(j_1, \dots, j_k, s_1, \dots, s_k). \end{aligned}$$

It implies that it is true for $m_i = M_i + 1, i = 1, \dots, k$. Therefore, it is true for any natural number $m_i \geq 2, i = 1, \dots, k$. This completes the proof. \square

COROLLARY 1. *Suppose w is a continuous and positive functions on \mathbb{R}_+ , f is a nonnegative function on Λ_2 , u is a nonnegative function on Λ_1 , then we can obtain the following formula*

$$\sum_{s=0}^{m-1} \sum_{j=0}^{s-1} f(s, j) w(u(j)) = \sum_{s=0}^{m-1} w(u(s)) \sum_{j=s+1}^{m-1} f(j, s).$$

COROLLARY 2. *Suppose w is a continuous and positive functions on \mathbb{R}_+ , f is a nonnegative function on Λ_6 , u is a nonnegative function on Λ_3 , then we can obtain the following formula*

$$\begin{aligned} & \sum_{s=0}^{m-1} \sum_{t=0}^{n-1} \sum_{e=0}^{z-1} \sum_{j=0}^{s-1} \sum_{l=0}^{t-1} \sum_{r=0}^{e-1} f(s, t, e, j, l, r) w(u(j, l, r)) \\ &= \sum_{s=0}^{m-1} \sum_{t=0}^{n-1} \sum_{e=0}^{z-1} w(u(s, t, e)) \sum_{j=s+1}^{m-1} \sum_{l=t+1}^{n-1} \sum_{r=e+1}^{z-1} f(j, l, r, s, t, e). \end{aligned}$$

3. Main result

Firstly, we consider the sums-difference inequality

$$\psi(u(m)) \leq c(m) + \sum_{i=1}^k \left[\sum_{s=0}^{m-1} \sum_{j=0}^{s-1} f_i(s, j) u^p(s) + \sum_{s=0}^{m-1} \sum_{j=0}^{s-1} h_i(s, j) u^q(s) \varphi_i(u(j)) \right]. \tag{1}$$

Suppose that

- (H₁) ψ is a strictly increasing continuous function on \mathbb{R}_+ , $\psi(u) > 0$ for all $u > 0$,
- (H₂) all $\varphi_i, (i = 1, 2, \dots, k)$ are continuous functions on \mathbb{R}_+ and positive on $(0, \infty)$,
- (H₃) $c(m) > 0$ on Λ_1 ,
- (H₄) $p > 0, q > 0$ are constants and $p > q$,
- (H₅) all $f_i, h_i (i = 1, 2, \dots, k)$ are nonnegative functions on Λ_2 .

We consider a sequence of functions $w_i(s)$, which can be calculated recursively by

$$\begin{cases} w_1(s) := \max_{\tau \in [0, s]} \varphi_1(\tau) + \varepsilon, & s \in \mathbb{R}_+, \\ w_{i+1}(s) := \max_{\tau \in [0, s]} \left\{ \frac{\varphi_{i+1}(\tau)}{w_i(\tau)} \right\} w_i(s), & i = 1, 2, \dots, k-1, s \in \mathbb{R}_+, \end{cases} \tag{2}$$

where $\varepsilon > 0$ is an arbitrarily small positive number. We define the functions:

$$\Psi_p(u) := \int_0^u \frac{ds}{(\Psi^{-1}(s))^p}, \quad u > 0, \tag{3}$$

$$\Psi_q(u) := \int_0^u \frac{ds}{(\Psi^{-1}(s))^q}, \quad u > 0, \tag{4}$$

$$W_i(u) := \int_0^u \frac{ds}{w_i(\Psi^{-1}(\Psi_p^{-1}(s)))}, \quad i = 1, 2, \dots, k, \quad u > 0, \tag{5}$$

$$\tilde{W}_i(u) := \int_0^u \frac{ds}{w_i(\Psi^{-1}(\Psi_p^{-1}(s)))}, \quad i = 1, 2, \dots, k, \quad u > 0.$$

Then, Ψ_p, Ψ_q and W_i, \tilde{W}_i are strictly increasing and continuous functions, let Ψ_p^{-1}, W_i^{-1} denote Ψ, W_i inverse function, respectively, then both Ψ_p^{-1} and W_i^{-1} are also continuous and increasing functions. Furthermore, let

$$\tilde{c}(m) := \max_{\tau \in [0, m]} c(m) \tag{6}$$

$$\tilde{f}_i(m, s) := \max_{\tau \in [0, m]} f_i(\tau, s), \tag{7}$$

$$\tilde{h}_i(m, s) := \max_{\tau \in [0, m]} h_i(\tau, s), \tag{8}$$

which are nondecreasing in m for each fixed s and satisfies $\tilde{f}_i(m, s) \geq f_i(m, s) \geq 0, \tilde{h}_i(m, s) \geq h_i(m, s) \geq 0$, for all $i = 1, 2, \dots, k$.

THEOREM 1. *Suppose that $(H_1 - H_5)$ hold and $u(m)$ is a nonnegative function on Λ_1 satisfying (1). Then, case one: if $\Psi^{-1}(z(m)) > 1$,*

$$u(m) \leq \Psi^{-1} \left\{ \Psi_p^{-1} \left[W_k^{-1} \left(W_k(E_k(m)) + \sum_{s=0}^{m-1} \tilde{g}_k(m, s) \right) \right] \right\}, \tag{9}$$

for $(m < M_1) \in \Lambda_1$, where $\Psi^{-1}(z(m))$ is defined in (13)

$$E_1(m) := \Psi_p(\tilde{c}(m)) + \sum_{i=1}^k \sum_{s=0}^{m-1} \sum_{j=0}^{s-1} \tilde{f}_i(s, j),$$

$$E_i(m) := W_{i-1}^{-1} \left(W_{i-1}(E_{i-1}(m)) + \sum_{s=0}^{m-1} \tilde{g}_{i-1}(m, s) \right), \quad i = 2, 3, \dots, k,$$

and $M_1 \in \Lambda_1$ is arbitrarily given on the boundary of the lattice

$$\mathcal{R} := \left\{ m \in \Lambda_1 : W_i(E_i(m)) + \sum_{s=0}^{m-1} \tilde{h}_i(m, s) \leq \int_0^\infty \frac{ds}{w_i(\Psi^{-1}(\Psi_p^{-1}(s)))}, \right. \\ \left. W_i^{-1} \left(W_i(E_i(m)) + \sum_{s=0}^{m-1} \tilde{h}_i(m, s) \right) \leq \int_0^\infty \frac{ds}{\Psi^{-1}(s)}, \quad i = 1, 2, \dots, k \right\}.$$

Case two: if $\Psi^{-1}(z(m)) < 1$,

$$u(m) \leq \Psi^{-1} \left\{ \Psi_q^{-1} \left[\tilde{W}_k^{-1}(\tilde{W}_k(E_k(m)) + \sum_{s=0}^{m-1} \tilde{g}_k(m, s)) \right] \right\}, \quad (10)$$

for $(m, n) \in \Lambda_{M_1}$, where

$$E_1(m) := \Psi_q(\tilde{c}(m)) + \sum_{i=1}^k \sum_{s=0}^{m-1} \sum_{j=0}^{s-1} \tilde{f}_i(s, j),$$

$$E_i(m) := \tilde{W}_{i-1}^{-1} \left(\tilde{W}_{i-1}(E_{i-1}(m)) + \sum_{s=0}^{m-1} \tilde{g}_{i-1}(m, s) \right), i = 2, 3, \dots, k,$$

and $M_1 \in \Lambda_1$ is arbitrarily given on the boundary of the lattice

$$\mathcal{R} := \left\{ (m, n) \in \Lambda : \tilde{W}_i(E_i(m)) + \sum_{s=0}^{m-1} \tilde{h}_i(m, s) \leq \int_0^\infty \frac{ds}{\tilde{w}_i(\Psi^{-1}(\Psi_q^{-1}(s)))}, \right.$$

$$\left. \tilde{W}_i^{-1} \left(\tilde{W}_i(E_i(m)) + \sum_{s=0}^{m-1} \tilde{h}_i(m, s) \right) \leq \int_0^\infty \frac{ds}{\Psi^{-1}(s)}, i = 1, 2, \dots, k \right\}.$$

Proof. First of all, we monotinize some given functions φ_i in the sums. Obviously the sequence $w_i(s)$ defined by $\varphi_i(s)$ in (2) are nondecreasing and nonnegative functions and satisfy $w_i(s) \geq \varphi_i(s), i = 1, 2, \dots, k$. Moreover, the ratio $w_{i+1}(s)/w_i(s)$ are also nondecreasing, $i = 1, 2, \dots, k$. By (1), (6), (7), (8), from (2), we have

$$\Psi(u(m)) \leq \tilde{c}(m) + \sum_{i=1}^k \left(\sum_{s=0}^{m-1} \sum_{j=0}^{s-1} \tilde{f}_i(s, j) u^p(s) + \sum_{s=0}^{m-1} \sum_{j=0}^{s-1} \tilde{h}_i(s, j) u^q(s) w_i(u(j)) \right). \quad (11)$$

for all $m \in \Lambda_1$. From (11), we have

$$\Psi(u(m)) \leq \tilde{c}(M) + \sum_{i=1}^k \left(\sum_{s=0}^{m-1} \sum_{j=0}^{s-1} \tilde{f}_i(s, j) u^p(s) + \sum_{s=0}^{m-1} \sum_{j=0}^{s-1} \tilde{h}_i(s, j) u^q(s) w_i(u(j)) \right). \quad (12)$$

for $M \in \Lambda_1$, where $0 \leq M \leq M_1$ is chosen arbitrarily. Let $z(m)$ denote the function on the right-hand side of (12), which is a nonnegative and nondecreasing function on Λ_M and $z(0) = \tilde{c}(M)$. Then we obtain the equivalent form of (12)

$$u(m) \leq \Psi^{-1}(z(m)), \quad \forall m \in \Lambda_M. \quad (13)$$

Since w_i is nondecreasing and satisfy $w_i(u) > 0$, for $u > 0$. By the definition of z and (13), from (12), we have

$$\Delta_1 z(m) = \sum_{i=1}^k \sum_{j=0}^{m-1} \tilde{f}_i(m, j) u^p(m) + \sum_{i=1}^k \sum_{j=0}^{m-1} \tilde{h}_i(m, j) u^q(m) w_i(u(j))$$

$$\leq \sum_{i=1}^k \sum_{j=0}^{m-1} \tilde{f}_i(m, j) (\Psi^{-1}(z(m)))^p + \sum_{i=1}^k \sum_{j=0}^{m-1} \tilde{h}_i(m, j) (\Psi^{-1}(z(m)))^q w_i(\Psi^{-1}(z(j))). \quad (14)$$

Case one: if $\psi^{-1}(z(m)) > 1$. Using the monotonicity of ψ^{-1} and z , from (14), we have

$$\Delta_1 z(m) \leq (\psi^{-1}(z(m)))^p \left(\sum_{i=1}^k \sum_{j=0}^{m-1} \tilde{f}_i(m, j) + \sum_{i=1}^k \sum_{j=0}^{m-1} \tilde{h}_i(m, j) w_i(\psi^{-1}(z(j))) \right). \quad (15)$$

that is

$$\frac{\Delta_1 z(m)}{(\psi^{-1}(z(m)))^p} \leq \sum_{i=1}^k \sum_{j=0}^{m-1} \tilde{f}_i(m, j) + \sum_{i=1}^k \sum_{j=0}^{m-1} \tilde{h}_i(m, j) w_i(\psi^{-1}(z(j))). \quad (16)$$

On the other hand, by the mean-value theorem for integrals, for arbitrarily given $m, m+1 \in \Lambda_m$, in the open interval $(z(m), z(m+1))$, there exists ξ , which satisfies

$$\begin{aligned} \Psi_p(z(m+1)) - \Psi_p(z(m)) &= \int_{z(m)}^{z(m+1)} \frac{ds}{(\psi^{-1}(s))^p} \\ &= \frac{\Delta_1 z(m)}{(\psi^{-1}(\xi))^p} \leq \frac{\Delta_1 z(m)}{(\psi^{-1}(z(m)))^p}. \end{aligned} \quad (17)$$

Using the definition of Ψ_p in (3). From (15) and (17), we obtain

$$\Psi_p(z(m+1)) \leq \Psi_p(z(m)) + \left(\sum_{i=1}^k \sum_{j=0}^{m-1} \tilde{f}_i(m, j) + \sum_{i=1}^k \sum_{j=0}^{m-1} \tilde{h}_i(m, j) w_i(\psi^{-1}(z(j))) \right). \quad (18)$$

Substituting m with s in (18). Then, taking the sums on both sides of (18) over $s = 0, 1, \dots, m-1$, we have

$$\begin{aligned} \Psi_p(z(m)) &\leq \Psi_p(z(0)) + \sum_{i=1}^k \left(\sum_{s=0}^{m-1} \sum_{j=0}^{s-1} \tilde{f}_i(s, j) + \sum_{s=0}^{m-1} \sum_{j=0}^{s-1} \tilde{h}_i(s, j) w_i(\psi^{-1}(z(j))) \right) \\ &\leq \Psi_p(\tilde{c}(M)) + \sum_{i=1}^k \left(\sum_{s=0}^{M-1} \sum_{j=0}^{s-1} \tilde{f}_i(s, j) + \sum_{s=0}^{m-1} \sum_{j=0}^{s-1} \tilde{h}_i(s, j) w_i(\psi^{-1}(z(j))) \right) \\ &= C_k(M) + \sum_{i=1}^k \sum_{s=0}^{m-1} \sum_{j=0}^{s-1} \tilde{h}_i(s, j) w_i(\psi^{-1}(z(j))), \end{aligned} \quad (19)$$

where

$$C_k(M) = \Psi_p(c(M)) + \sum_{i=1}^k \sum_{s=0}^{M-1} \sum_{j=0}^{s-1} \tilde{f}_i(s, j). \quad (20)$$

Let

$$v(m) = \Psi_p(z(m)). \quad (21)$$

From (19), we have

$$v(m) \leq C_k(M) + \sum_{i=1}^k \sum_{s=0}^{m-1} \sum_{j=0}^{s-1} \tilde{h}_i(s, j) w_i(\psi^{-1}(\Psi_p^{-1}(v(j)))), \tag{22}$$

for all $m < M \in \Lambda_1$. Using the Corollary 1, (22) can be written as

$$v(m) \leq C_k(M) + \sum_{i=1}^k \sum_{s=0}^{m-1} \tilde{g}_i(m, s) w_i(\psi^{-1}(\Psi_p^{-1}(v(s)))), \tag{23}$$

where $\tilde{g}_i(m, s) = \sum_{j=s+1}^{m-1} \tilde{h}_i(j, s)$. Obviously, $\tilde{g}_i(m, s)$, $i = 1, 2, \dots, k$ are nondecreasing in m for each fixed s and $\tilde{g}_i(m, s) \geq 0$. Then from (23), we have

$$v(m) \leq C_k(M) + \sum_{i=1}^k \sum_{s=0}^{m-1} \tilde{g}_i(M, s) w_i(\psi^{-1}(\Psi_p^{-1}(v(s)))). \tag{24}$$

From (24), we can conclude that

$$v(m) \leq W_k^{-1} \left(W_k(E_k(m)) + \sum_{s=0}^{m-1} \tilde{g}_k(M, s) \right), \tag{25}$$

for $m < M \in \Lambda_1$, where

$$E_i(M) := W_{i-1}^{-1} \left(W_{i-1}(E_{i-1}(M)) + \sum_{s=0}^{M-1} \tilde{g}_{i-1}(M, s) \right), \quad i = 2, \dots, k, \tag{26}$$

$$E_1(M) := C_1(M).$$

For $k = 1$, let $z_1(m)$ denote the function on the right-hand side of (24), which is a nonnegative and nondecreasing function on Λ_1 , $z_1(0) = C_1(M)$ and $v(m) \leq z_1(m)$. Then we obtain

$$\begin{aligned} \Delta_1 z_1(m) &= \tilde{g}_1(M, m) w_1(\psi^{-1}(\Psi_p^{-1}(v(m)))) \\ &\leq \tilde{g}_1(M, m) w_1(\psi^{-1}(\Psi_p^{-1}(z_1(m)))). \end{aligned} \tag{27}$$

From (27), we have

$$\frac{\Delta_1 z_1(m)}{w_1(\psi^{-1}(\Psi_p^{-1}(z_1(m))))} \leq \tilde{g}_1(M, m). \tag{28}$$

By the mean-value theorem for integrals, there exists ξ in the open interval $(z_1(m), z_1(m+1))$, for arbitrarily given $(m), (m+1) \in \Lambda_1$ such that

$$\begin{aligned} W_1(z_1(m+1)) - W_1(z_1(m)) &= \int_{z_1(m)}^{z_1(m+1)} \frac{ds}{w_1(\psi^{-1}(\Psi_p^{-1}(s)))} \\ &= \frac{\Delta_1 z_1(m)}{w_1(\psi^{-1}(\Psi_p^{-1}(\xi)))} \\ &\leq \frac{\Delta_1 z_1(m)}{w_1(\psi^{-1}(\Psi_p^{-1}(z_1(m))))}. \end{aligned} \tag{29}$$

From (28) and (29), we have

$$W_1(z_1(m+1)) \leq W_1(z_1(m)) + \tilde{g}_1(M, m). \quad (30)$$

Substituting m with s in (30). Then, taking the sums on both sides of (30) over $s = 0, 1, \dots, m-1$, we have

$$\begin{aligned} W_1(z_1(m)) &\leq W_1(z_1(0)) + \sum_{s=0}^{m-1} \tilde{g}_1(M, s) \\ &= W_1(C_1(M)) + \sum_{s=0}^{m-1} \tilde{g}_1(M, s), \end{aligned} \quad (31)$$

Using $v(m) \leq z_1(m)$, from (31), we get

$$v(m) \leq z_1(m) \leq W_1^{-1} \left(W_1(C_1(M)) + \sum_{s=0}^{m-1} \tilde{g}_1(M, s) \right), \quad (32)$$

for all $m < M \in \Lambda_1$. This proves that (25) is true for $k = 1$.

Next, we make the inductive assumption that (25) is true for $k = l$, then

$$v(m) \leq W_l^{-1} \left(W_l(E_l(M)) + \sum_{s=0}^{m-1} \tilde{g}_l(M, s) \right), \quad (33)$$

for all $m \in \Lambda_M$, where

$$\begin{aligned} E_1(M) &:= C_1(M), \\ E_i(M) &:= W_{i-1}^{-1} \left(W_{i-1}(E_{i-1}(M)) + \sum_{s=0}^{M-1} \tilde{g}_{i-1}(M, s) \right), \quad i = 2, 3, \dots, l. \end{aligned}$$

We consider

$$v(m) \leq C_{l+1}(M) + \sum_{i=1}^{l+1} \sum_{s=0}^{m-1} \tilde{g}_i(M, s) w_i(\psi^{-1}(\Psi_p^{-1}(v(s))))), \quad (34)$$

for all $m < M \in \Lambda_1$. Let $z_2(m)$ denote the nonnegative and nondecreasing function of the right-hand of (34), then $z_2(0) = C_{l+1}(M)$ and $v(m) \leq z_2(m)$. Let

$$\phi_i(u) := w_i(u)/w_1(u), \quad i = 1, 2, \dots, l+1. \quad (35)$$

By (2), we conclude that ϕ_i $i = 1, 2, \dots, l$ are nondecreasing functions. From (34), we have

$$\begin{aligned} &\frac{\Delta_1 z_2(m)}{w_1(\psi^{-1}(\Psi_p^{-1}(z_2(m))))} \\ &= \frac{\sum_{i=1}^{l+1} \tilde{g}_i(M, m) w_i(\psi^{-1}(\Psi_p^{-1}(v(m))))}{w_1(\psi^{-1}(\Psi_p^{-1}(z_2(m))))} \end{aligned}$$

$$\begin{aligned}
&\leq \frac{\sum_{i=1}^{l+1} \tilde{g}_i(M, m) w_i(\psi^{-1}(\Psi_p^{-1}(z_2(m))))}{w_1(\psi^{-1}(\Psi_p^{-1}(z_2(m))))} \\
&\leq \tilde{g}_1(M, m) + \sum_{i=2}^{l+1} \tilde{g}_i(M, m) \phi_i(\psi^{-1}(\Psi_p^{-1}(z_2(m)))) \\
&= \tilde{g}_1(M, m) + \sum_{i=1}^l \tilde{g}_{i+1}(M, m) \phi_{i+1}(\psi^{-1}(\Psi_p^{-1}(z_2(m))))). \tag{36}
\end{aligned}$$

By the mean-value theorem for integrals, there exists ξ in the open interval $(z_2(m), z_2(m+1))$, for arbitrarily given $m, m+1 \in \Lambda_1$, we obtain

$$\begin{aligned}
W_1(z_2(m+1)) - W_1(z_2(m)) &= \int_{z_2(m)}^{z_2(m+1)} \frac{ds}{w_1(\psi^{-1}(\Psi_p^{-1}(s)))} \\
&= \frac{\Delta_1 z_2(m)}{w_1(\psi^{-1}(\Psi_p^{-1}(\xi)))} \\
&\leq \frac{\Delta_1 z_2(m)}{w_1(\psi^{-1}(\Psi_p^{-1}(z_2(m))))}. \tag{37}
\end{aligned}$$

From (36) and (37), we get

$$\begin{aligned}
&W_1(z_2(m+1)) - W_1(z_2(m)) \\
&\leq \tilde{g}_1(M, m) + \sum_{i=1}^l \tilde{g}_{i+1}(M, m) \phi_{i+1}(\psi^{-1}(\Psi_p^{-1}(z_2(m)))) \tag{38}
\end{aligned}$$

Substituting m with s in (38), then taking the sums on both sides of (38) over $s = 0, 1, \dots, m-1$, we have

$$\begin{aligned}
W_1(z_2(m)) &\leq W_1(C_{l+1}(M)) + \sum_{s=0}^{m-1} \tilde{g}_1(M, s) \\
&\quad + \sum_{i=1}^l \sum_{s=0}^{m-1} \tilde{g}_{i+1}(M, s) \phi_{i+1}(\psi^{-1}(\Psi_p^{-1}(z_2(s))))), \tag{39}
\end{aligned}$$

for all $m < M \in \Lambda_1$.

Let

$$\theta(m) := W_1(z_2(m)), \tag{40}$$

$$\rho_1(M) := W_1(C_{l+1}(M)) + \sum_{s=0}^{M-1} \tilde{g}_1(M, s). \tag{41}$$

Using (40) and (41), from (39), we have

$$\theta(m) \leq \rho_1(M) + \sum_{i=1}^l \sum_{s=0}^{m-1} \tilde{g}_{i+1}(M, s) \phi_{i+1}[\psi^{-1}(\Psi_p^{-1}(W_1^{-1}(\theta(m))))]. \tag{42}$$

It has the same form as (24). We are ready to use the inductive assumption for (42). Let $\delta(s) := \psi^{-1}(\Psi_p^{-1}(W_1^{-1}(s)))$. Since $\psi^{-1}, \Psi_p^{-1}, W_1^{-1}, \phi_i$ are continuous, nondecreasing and positive on $(0, \infty)$, each $\phi_i(\delta(s))$ is continuous and nondecreasing on $(0, \infty)$. Moreover

$$\frac{\phi_{i+1}(\delta(s))}{\phi_i(\delta(s))} = \frac{w_{i+1}(\delta(s))}{w_i(\delta(s))} = \max_{\tau \in [0, \delta(s)]} \left\{ \frac{\phi_{i+1}(\tau)}{w_i(\tau)} \right\}, \quad i = 1, 2, \dots, l, \tag{43}$$

which is also continuous and nondecreasing and positive on $(0, \infty)$. Therefore, by the inductive assumption in (33), from (42), we have

$$\theta(m) \leq \Phi_l^{-1} \left(\Phi_l(\rho_l(M)) + \sum_{s=0}^{m-1} \tilde{g}_{l+1}(M, s) \right), \tag{44}$$

where

$$\Phi_i(u) := \int_0^u \frac{ds}{\phi_{i+1}(\psi^{-1}(\Psi_p^{-1}(W_1^{-1}(s))))}, \quad u > 0, \quad i = 1, 2, \dots, l, \tag{45}$$

$$\rho_i(M) := \Phi_{i-1}^{-1} \left(\Phi_{i-1}(\rho_{i-1}(M)) + \sum_{s=0}^{M-1} \tilde{g}_i(M, s) \right), \quad i = 2, 3, \dots, l. \tag{46}$$

Note that

$$\begin{aligned} \Phi_i(u) &= \int_0^u \frac{w_1(\psi^{-1}(\Psi_p^{-1}(W_1^{-1}(s)))) ds}{w_{i+1}(\psi^{-1}(\Psi_p^{-1}(W_1^{-1}(s))))}, \\ &= \int_0^{W_1^{-1}(u)} \frac{ds}{w_{i+1}(\psi^{-1}(\Psi_p^{-1}(s)))}, \\ &= W_{i+1}(W_1^{-1}(u)), \quad i = 1, 2, \dots, l. \end{aligned} \tag{47}$$

Thus, from (40), (44) and (47), we have

$$\begin{aligned} v(m) &\leq z_2(m) = W_1^{-1}(\theta(m)) \\ &\leq W_1^{-1} \left(\Phi_l^{-1} \left(\Phi_l(\rho_l(M)) + \sum_{s=0}^{m-1} \tilde{g}_{l+1}(M, s) \right) \right) \\ &= W_{l+1}^{-1} \left(W_{l+1} \left(W_1^{-1}(\rho_l(M)) \right) + \sum_{s=0}^{m-1} \tilde{g}_{l+1}(M, s) \right). \end{aligned} \tag{48}$$

We can prove that the term of $W_1^{-1}(\rho_l(M))$ in (48) is just the same as $E_{l+1}(M)$ defined in (26). Let $\tilde{\rho}_i(M) := W_1^{-1}(\rho_i(M))$. By (41), we have

$$\begin{aligned} \tilde{\rho}_1(M) &= W_1^{-1}(\rho_1(M)) \\ &= W_1^{-1} \left(W_1(C_{l+1}(M)) + \sum_{s=0}^{M-1} \tilde{g}_1(M, s) \right) \\ &= E_2(M). \end{aligned}$$

Then by the mathematical induction for i , using (46) and (47), we get

$$\begin{aligned} \tilde{\rho}_i(M) &= W_1^{-1} \left(\Phi_{i-1}^{-1} \left(\Phi_{i-1}(\rho_{i-1}(M)) + \sum_{s=0}^{M-1} \tilde{g}_i(M, s) \right) \right) \\ &= W_i^{-1} \left[W_i(W_1^{-1}(\rho_{i-1}(M))) + \sum_{s=0}^{M-1} \tilde{g}_i(M, s) \right] \\ &= W_i^{-1} \left[W_i(\tilde{\rho}_{i-1}(M)) + \sum_{s=0}^{M-1} \tilde{g}_i(M, s) \right] \\ &= E_{i+1}(M), i = 2, 3 \dots, l. \end{aligned}$$

This proves that $W_1^{-1}(\rho_l(M))$ in (48) is just the same as $E_{l+1}(M)$ defined in (26). Hence (48) can be equivalently written as

$$v(m) \leq W_{l+1}^{-1} \left(W_{l+1}(E_{l+1}(M)) + \sum_{s=0}^{m-1} \tilde{g}_{l+1}(M, s) \right), \forall m \in \Lambda_M. \tag{49}$$

The estimation (25) of unknown function v in the inequality (22) is proved by induction. By (13), (21), (25) and (49), we have

$$\begin{aligned} u(m) &\leq \psi^{-1}(z(m)) \leq \psi^{-1} \left(\Psi_p^{-1}(v(m)) \right) \\ &\leq \psi^{-1} \left(\Psi_p^{-1} \left(W_k^{-1} \left(W_k(E_k(M)) + \sum_{s=0}^{m-1} \tilde{g}_k(M, s) \right) \right) \right), \end{aligned} \tag{50}$$

for all $m < M \in \Lambda_1$. Let $m = M$, from (50), we have

$$u(M) \leq \psi^{-1} \left(\Psi_p^{-1} \left(W_k^{-1} \left(W_k(E_k(M)) + \sum_{s=0}^{M-1} \tilde{g}_k(M, s) \right) \right) \right).$$

This proves (9), since M and N are chosen arbitrarily.

Case two: if $\psi^{-1}(z(m)) < 1$. Using the monotonicity of ψ^{-1} and z , we can conclude that $(\psi^{-1}(z(m)))^p < (\psi^{-1}(z(m)))^q$, from (14), we have

$$\Delta_1 z(m) \leq (\psi^{-1}(z(m)))^q \left(\sum_{i=1}^k \sum_{j=0}^{m-1} \tilde{f}_i(m, j) + \sum_{i=1}^k \sum_{j=0}^{m-1} \tilde{h}_i(m, j) w_i(\psi^{-1}(z(j))) \right). \tag{51}$$

This follows from (51) by similar arguments as in the proof of Case one, we get

$$u(m) \leq \psi^{-1} \left(\Psi_q^{-1} \left(\tilde{W}_k^{-1} \left(\tilde{W}_k(E_k(m)) + \sum_{s=0}^{m-1} \tilde{g}_k(m, s) \right) \right) \right).$$

This completes the proof. \square

Next, we consider the following sums-difference inequality

$$\begin{aligned} \psi(u(m,n,z)) \leq & c(m,n,z) + \sum_{i=1}^k \sum_{s=0}^{m-1} \sum_{t=0}^{n-1} \sum_{e=0}^{z-1} \sum_{j=0}^{s-1} \sum_{l=0}^{t-1} \sum_{r=0}^{e-1} f_i(s,t,e,j,l,r)u^p(s,t,e) \\ & + \sum_{i=1}^k \sum_{s=0}^{m-1} \sum_{t=0}^{n-1} \sum_{e=0}^{z-1} \sum_{j=0}^{s-1} \sum_{l=0}^{t-1} \sum_{r=0}^{e-1} h_i(s,t,e,j,l,r)u^p(s,t,e)\varphi_i(u(j,l,r)). \end{aligned} \tag{52}$$

Suppose that

- (L₁) ψ is a strictly increasing continuous function on \mathbb{R}_+ , $\psi(u) > 0$ for all $u > 0$,
- (L₂) all φ_i , ($i = 1, 2, \dots, k$) are continuous functions on \mathbb{R}_+ and positive on $(0, \infty)$,
- (L₃) $c(m,n,z) > 0$ on Λ_3 ,
- (L₄) $p > 0$, is a constant,
- (L₅) all f_i, h_i ($i = 1, 2, \dots, k$) are nonnegative functions on Λ_6 .

Let

$$\tilde{c}(m,n,z) := \max_{(\tau,\xi,v) \in [0,m] \times [0,n] \times [0,z]} c(m,n,z), \tag{53}$$

$$\tilde{f}_i(m,n,z,s,t,e) := \max_{(\tau,\xi,v) \in [0,m] \times [0,n] \times [0,z]} f_i(\tau,\xi,v,s,t,e), \tag{54}$$

$$\tilde{h}_i(m,n,z,s,t,e) := \max_{(\tau,\xi,v) \in [0,m] \times [0,n] \times [0,z]} h_i(\tau,\xi,v,s,t,e), \tag{55}$$

which are nondecreasing in m,n,z for each fixed s and t and e and satisfies $\tilde{f}_i(m,n,z,s,t,e) \geq f_i(m,n,z,s,t,e) \geq 0$, $\tilde{h}_i(m,n,z,s,t,e) \geq h_i(m,n,z,s,t,e) \geq 0$, for all $i = 1, 2, \dots, k$.

THEOREM 2. *Suppose that (L₁ – L₅) hold and $u(m,n,z)$ is a nonnegative function on Λ_3 satisfying (52). Then*

$$u(m,n,z) \leq \psi^{-1} \left\{ \Psi_p^{-1} \left[W_k^{-1} \left(W_k(E_k(m,n)) + \sum_{s=0}^{m-1} \sum_{t=0}^{n-1} \sum_{e=0}^{z-1} \tilde{g}_k(m,n,z,s,t,e) \right) \right] \right\}, \tag{56}$$

for $(m,n,z) \in \Lambda_{(M_1,N_1,Z_1)}$, where

$$E_1(m,n,z) := \Psi_p(\tilde{c}(m,n,z)) + \sum_{i=1}^k \sum_{s=0}^{m-1} \sum_{t=0}^{n-1} \sum_{e=0}^{z-1} \sum_{j=0}^{s-1} \sum_{l=0}^{t-1} \sum_{r=0}^{e-1} \tilde{f}_i(s,t,e,j,l,r),$$

$$E_i(m,n,z) := W_{i-1}^{-1} \left(W_{i-1}(E_{i-1}(m,n,z)) + \sum_{s=0}^{m-1} \sum_{t=0}^{n-1} \sum_{e=0}^{z-1} \tilde{g}_{i-1}(m,n,z,s,t,e) \right), i = 2, 3, \dots, k,$$

and $(M_1, N_1, Z_1) \in \Lambda_3$ is arbitrarily given on the boundary of the lattice

$$\begin{aligned} \mathcal{R} := \left\{ (m,n,z) \in \Lambda : W_i(E_i(m,n,z)) + \sum_{s=0}^{m-1} \sum_{t=0}^{n-1} \sum_{e=0}^{z-1} \tilde{h}_i(m,n,z,s,t,e) \leq \int_0^\infty \frac{ds}{w_i(\Psi^{-1}(\Psi_p^{-1}(s)))}, \right. \\ \left. W_i^{-1} \left(W_i(E_i(m,n,z)) + \sum_{s=0}^{m-1} \sum_{t=0}^{n-1} \sum_{e=0}^{z-1} \tilde{h}_i(m,n,z,s,t,e) \right) \leq \int_0^\infty \frac{ds}{\Psi^{-1}(s)}, i = 1, 2, \dots, k \right\}. \end{aligned}$$

Proof. By the similar arguments as in the proof of Theorem 1, we can obtain the estimation (56). \square

4. Corollaries

COROLLARY 3. *Suppose that (H_2, H_3, H_5) hold and $p = q = 1$, $\psi(u(m)) = u(m)$ is a nonnegative function on Λ_1 satisfying*

$$u(m) \leq c(m) + \sum_{i=1}^k \left(\sum_{s=0}^{m-1} \sum_{j=0}^{s-1} f_i(s, j)u(s) + \sum_{s=0}^{m-1} \sum_{j=0}^{s-1} h_i(s, j)u(s)\varphi_i(u(j)) \right).$$

Then

$$u(m, n) \leq \exp \left[\hat{W}_k^{-1} \left(\hat{W}_k(\hat{E}_k(m, n)) + \sum_{s=0}^{m-1} \sum_{t=0}^{n-1} \tilde{g}_k(m, n, s, t) \right) \right],$$

for $m \in \Lambda_{M_1}$, where

$$\hat{W}_i(u) := \int_0^u \frac{ds}{w_i(e^s)}, \quad i = 1, 2, \dots, k, \quad u > 0,$$

$$\hat{E}_1(m) := \ln(\tilde{c}(m)) + \sum_{i=1}^k \sum_{s=0}^{m-1} \sum_{j=0}^{s-1} \tilde{f}_i(s, j),$$

$$\hat{E}_i(m) := \hat{W}_{i-1}^{-1} \left(\hat{W}_{i-1}(\hat{E}_{i-1}(m)) + \sum_{s=0}^{m-1} \tilde{g}_{i-1}(m, s) \right), \quad i = 2, 3, \dots, k,$$

and $M_1 \in \Lambda_1$ is arbitrarily given on the boundary of the lattice.

Proof. This follows immediately from Theorem 1. \square

COROLLARY 4. *Suppose that $(H_2 - H_5)$ hold and $f_i = 0$, $0 < p < 1$, $\psi(u(m)) = u(m)$ is a nonnegative function on Λ_1 satisfying*

$$u(m) \leq c(m) + \sum_{i=1}^k \sum_{s=0}^{m-1} \sum_{j=0}^{s-1} h_i(s, j)u^p(s)\varphi_i(u(j)).$$

Then

$$u(m) \leq \left[\bar{W}_k^{-1} \left(\bar{W}_k(\bar{E}_k(m)) + \sum_{s=0}^{m-1} \tilde{g}_k(m, s) \right) \right]^{\frac{1}{1-p}},$$

for $m \in \Lambda_{M_1}$, where

$$\bar{W}_i(u) := \int_0^u \frac{ds}{w_i(s^{\frac{1}{1-p}})}, \quad i = 1, 2, \dots, k, \quad u > 0,$$

$$\begin{aligned} \bar{E}_1(m) &:= \tilde{c}(m)^{\frac{1}{1-p}}, \\ \bar{E}_i(m) &:= \bar{W}_{i-1}^{-1} \left(\bar{W}_{i-1}(\bar{E}_{i-1}(m)) + \sum_{s=0}^{m-1} \tilde{g}_{i-1}(m,s) \right), i = 2, 3, \dots, k, \end{aligned}$$

and $M_1 \in \Lambda_1$ is arbitrarily given on the boundary of the lattice.

COROLLARY 5. *Suppose that $(H_2 - H_5)$ hold and $f_i = 0$, $\psi(u(m)) = u(m)$ is a nonnegative function on Λ_1 satisfying*

$$u(m) \leq c(m) + \sum_{i=1}^k \sum_{s=0}^{m-1} \sum_{j=0}^{s-1} h_i(s, j) \varphi_i(u(j)).$$

Then

$$u(m) \leq \bar{W}_k^{-1} \left(\bar{W}_k(\bar{E}_k(m)) + \sum_{s=0}^{m-1} \tilde{g}_k(m,s) \right),$$

for $m \in \Lambda_{M_1}$, where

$$\begin{aligned} \bar{W}_i(u) &:= \int_0^u \frac{ds}{w_i(s)}, i = 1, 2, \dots, k, u > 0, \\ \bar{E}_1(m) &:= \tilde{c}(m), \\ \bar{E}_i(m) &:= \bar{W}_{i-1}^{-1} \left(\bar{W}_{i-1}(\bar{E}_{i-1}(m)) + \sum_{s=0}^{m-1} \tilde{g}_{i-1}(m,s) \right), i = 2, 3, \dots, k, \end{aligned}$$

and $M_1 \in \Lambda_1$ is arbitrarily given on the boundary of the lattice.

COROLLARY 6. *Suppose that $(L_2 - L_5)$ hold and $p = 1$, $\psi(u(m, n, z)) = u(m, n, z)$ is a nonnegative function on Λ satisfying*

$$\begin{aligned} u(m, n, z) &\leq c(m, n, z) + \sum_{i=1}^k \left(\sum_{s=0}^{m-1} \sum_{t=0}^{n-1} \sum_{e=0}^{z-1} \sum_{j=0}^{s-1} \sum_{l=0}^{t-1} \sum_{r=0}^{e-1} f_i(s, t, e, j, l, r) u(s, t, e) \right. \\ &\quad \left. + \sum_{s=0}^{m-1} \sum_{t=0}^{n-1} \sum_{e=0}^{z-1} \sum_{j=0}^{s-1} \sum_{l=0}^{t-1} \sum_{r=0}^{e-1} h_i(s, t, e, j, l, r) u(s, t, e) \varphi_i(u(j, l, r)) \right). \end{aligned}$$

Then

$$u(m, n, z) \leq \exp \left[\hat{W}_k^{-1} \left(\hat{W}_k(\hat{E}_k(m, n, z)) + \sum_{s=0}^{m-1} \sum_{t=0}^{n-1} \sum_{e=0}^{z-1} \tilde{g}_k(m, n, z, s, t, e) \right) \right],$$

for $(m, n, z) \in \Lambda_{(M_1, N_1, Z_1)}$, where

$$\hat{W}_i(u) := \int_0^u \frac{ds}{w_i(\exp(s))}, i = 1, 2, \dots, k, u > 0,$$

$$\hat{E}_1(m, n, z) := \ln(\tilde{c}(m, n, z)) + \sum_{i=1}^k \sum_{s=0}^{m-1} \sum_{t=0}^{n-1} \sum_{e=0}^{z-1} \sum_{j=0}^{s-1} \sum_{l=0}^{t-1} \sum_{r=0}^{e-1} \tilde{f}_i(s, t, e, j, l, r),$$

$$\hat{E}_i(m, n, z) := \hat{W}_{i-1}^{-1} \left(\hat{W}_{i-1}(\hat{E}_{i-1}(m, n, z)) + \sum_{s=0}^{m-1} \sum_{t=0}^{n-1} \sum_{e=0}^{z-1} \tilde{g}_{i-1}(m, n, z, s, t, e) \right), i = 2, 3, \dots, k,$$

and $(M_1, N_1, Z_1) \in \Lambda_3$ is arbitrarily given on the boundary of the lattice.

COROLLARY 7. Suppose that $(L_2 - L_5)$ hold and $f_i = 0, 0 < p < 1, \psi(u(m, n, z)) = u(m, n, z)$ is a nonnegative function on Λ_3 satisfying

$$u(m, n, z) \leq c(m, n, z) + \sum_{i=1}^k \sum_{s=0}^{m-1} \sum_{t=0}^{n-1} \sum_{e=0}^{z-1} \sum_{j=0}^{s-1} \sum_{l=0}^{t-1} \sum_{r=0}^{e-1} h_i(s, t, e, j, l, r) u^p(s, t, e) \varphi_i(u(j, l, r)).$$

Then

$$u(m, n, z) \leq \left[\bar{W}_k^{-1} \left(\bar{W}_k(\bar{E}_k(m, n, z)) + \sum_{s=0}^{m-1} \sum_{t=0}^{n-1} \sum_{e=0}^{z-1} \tilde{g}_k(m, n, z, s, t, e) \right) \right]^{\frac{1}{1-p}},$$

for $(m, n, z) \in \Lambda_{(M_1, N_1, Z_1)}$, where

$$\bar{W}_i(u) := \int_0^u \frac{ds}{w_i(s^{\frac{1}{1-p}})}, i = 1, 2, \dots, k, u > 0,$$

$$\bar{E}_1(m, n, z) := \tilde{c}(m, n, z)^{\frac{1}{1-p}},$$

$$\bar{E}_i(m, n, z) := \bar{W}_{i-1}^{-1} \left(\bar{W}_{i-1}(\bar{E}_{i-1}(m, n, z)) + \sum_{s=0}^{m-1} \sum_{t=0}^{n-1} \sum_{e=0}^{z-1} \tilde{g}_{i-1}(m, n, z, s, t, e) \right), i = 2, 3, \dots, k,$$

and $(M_1, N_1, Z_1) \in \Lambda_3$ is arbitrarily given on the boundary of the lattice.

COROLLARY 8. Suppose that $(L_2 - L_5)$ hold and $f_i = 0, \psi(u(m, n, z)) = u(m, n, z)$ is a nonnegative function on Λ_3 satisfying

$$u(m, n, z) \leq c(m, n, z) + \sum_{i=1}^k \sum_{s=0}^{m-1} \sum_{t=0}^{n-1} \sum_{e=0}^{z-1} \sum_{j=0}^{s-1} \sum_{l=0}^{t-1} \sum_{r=0}^{e-1} h_i(s, t, e, j, l, r) \varphi_i(u(j, l, r)).$$

Then

$$u(m, n, z) \leq \bar{W}_k^{-1} \left(\bar{W}_k(\bar{E}_k(m, n, z)) + \sum_{s=0}^{m-1} \sum_{t=0}^{n-1} \sum_{e=0}^{z-1} \tilde{g}_k(m, n, z, s, t, e) \right),$$

for $(m, n, z) \in \Lambda_{(M_1, N_1, Z_1)}$, where

$$\bar{W}_i(u) := \int_0^u \frac{ds}{w_i(s)}, i = 1, 2, \dots, k, u > 0,$$

$$\bar{E}_1(m, n, z) := \tilde{c}(m, n, z),$$

$$\bar{E}_i(m, n, z) := \bar{W}_{i-1}^{-1} \left(\bar{W}_{i-1}(\bar{E}_{i-1}(m, n, z)) + \sum_{s=0}^{m-1} \sum_{t=0}^{n-1} \sum_{e=0}^{z-1} \tilde{g}_{i-1}(m, n, z, s, t, e) \right), i = 2, 3, \dots, k,$$

and $(M_1, N_1) \in \Lambda$ is arbitrarily given on the boundary of the lattice.

5. Analysis of the finite recursion

Consider the following finite recursion

$$u_i(t) = W_i^{-1} \left[W_i(u_{i-1}(t)) + \sum_{t=0}^{m-1} f_i(m, t) \right], \quad i = 1, 2, \dots, k, \tag{57}$$

for $t \in \mathbb{N}_0$, where f_i s are nonnegative continuous functions ($i = 1, 2, \dots, k$), $W_i \in C(\mathbb{R}_+, \mathbb{R})$ is strictly increasing such that $W_i(+\infty) = +\infty$, then the inverse W_i^{-1} is well defined on $[W_i(0), +\infty)$.

Define a mapping T_i on $(\mathbb{N}_0, \mathbb{R}_+)$

$$T_i u(t) = W_i(u(t)) + \sum_{t=0}^{m-1} f_i(m, t), \quad i = 1, 2, \dots, k. \tag{58}$$

For each $u : \mathbb{N}_0 \rightarrow \mathbb{R}_+$, we have $T_i u(t) \in [W_i(0), +\infty)$ for all $t \in \mathbb{N}_0$, then, $W_i^{-1}(T_i u_{i-1}(t))$ is well defined for all $t \in \mathbb{N}_0$.

The following Theorem gives the asymptoticity of u_n .

THEOREM 3. *Suppose that f_i, W_i are given as in (57), if $u_0 \in (\mathbb{N}_0, \mathbb{R}_+)$ satisfies $\lim_{t \rightarrow \infty} u_0(t) = 0$ and if (A₁)*

$$\lim_{t \rightarrow \infty} \sum_{s=0}^{\infty} f_i(t, s) = 0, \text{ for all } i,$$

then $\lim_{t \rightarrow \infty} u_n(t) = 0$.
if (A₂)

$$\lim_{u \rightarrow 0} W_i(u) = -\infty, \text{ for all } i,$$

and $\sum_{s=0}^{\infty} f_i(t, s)$ is bounded on \mathbb{N}_0 . Then $\lim_{t \rightarrow \infty} u_i(t) = 0, i = 1, 2, \dots, k$.

Proof. By definition (58) of T_i , the recursion (57) implies that

$$u_i(t) = (W_i^{-1}(T_i))u_{i-1}(t), \quad i = 1, 2, \dots, k. \tag{59}$$

In the case (A₁), due to W_i is continuous and strictly increasing, from (59), we have

$$\begin{aligned} \limsup_{t \rightarrow \infty} u_i(t) &= W_i^{-1}(\limsup_{t \rightarrow \infty} T_i u_{i-1}(t)) \\ &\leq W_i^{-1} \left(\limsup_{t \rightarrow \infty} W_i(u_{i-1}(t)) + \limsup_{t \rightarrow \infty} \sum_{s=0}^{\infty} f_i(t, s) \right) \\ &\leq W_i^{-1} \left(W_i(\limsup_{t \rightarrow \infty} u_{i-1}(t)) \right) \\ &= \limsup_{t \rightarrow \infty} u_{i-1}(t). \end{aligned}$$

By induction, we can prove that $0 \leq \limsup_{t \rightarrow \infty} u_n(t) \leq \limsup_{t \rightarrow \infty} u_0(t) = 0$.

In the case (A_2) . From $\lim_{t \rightarrow \infty} u_0(t) = 0$, we assume inductively that $\lim_{t \rightarrow \infty} u_{i-1}(t) = 0$, from (59), we have

$$\begin{aligned} \limsup_{t \rightarrow \infty} u_i(t) &= W_i^{-1}(\limsup_{t \rightarrow \infty} T_i u_{i-1}(t)) \\ &\leq W_i^{-1}(\limsup_{t \rightarrow \infty} W_i(u_{i-1}(t)) + \limsup_{t \rightarrow \infty} \sum_{s=0}^{\infty} f_i(t, s)) \\ &\leq W_i^{-1}(W_i(\limsup_{t \rightarrow \infty} u_{i-1}(t)) + \limsup_{t \rightarrow \infty} \sum_{s=0}^{\infty} f_i(t, s)) \\ &= \limsup_{u \rightarrow -\infty} W_i^{-1}(u) = 0, \end{aligned}$$

where $\sum_{s=0}^{\infty} f_i(t, s)$ is bounded guarantees that $\limsup_{u \rightarrow -\infty} \sum_{s=0}^{\infty} f_i(t, s)$ is finite, (A_2) guarantees that $\lim_{u \rightarrow 0^+} W_i(u) = -\infty$ and $\lim_{u \rightarrow -\infty} W_i^{-1}(u) = 0$. Then, by induction we also can prove that $\lim_{t \rightarrow \infty} u_n(t) = 0$. This completes the proof. \square

6. Applications

In this section, we apply our result to study the boundedness, uniqueness and continuous dependence of the solutions to the difference equations.

EXAMPLE 1. We consider the difference equation

$$v(m) = \frac{1}{m+3} + \sum_{s=0}^{m-1} 2^{-s} \sqrt{|v(s)|} + \sum_{s=0}^{m-1} s 3^{-s} v(s) + \sum_{s=0}^{m-1} \frac{s 2^{-s}}{200} e^{v(s)}, \tag{60}$$

for all $m \in \Lambda$, where Λ is defined as in the section 2 . From (60), we have

$$|v(m)| \leq \frac{1}{m+3} + \sum_{s=0}^{m-1} 2^{-s} \sqrt{|v(s)|} + \sum_{s=0}^{m-1} s 3^{-s} |v(s)| + \sum_{s=0}^{m-1} \frac{s 2^{-s}}{200} e^{|v(s)|}. \tag{61}$$

Let $|v(m)| = u(m)$, we obtain

$$u(m) \leq \frac{1}{m+3} + \sum_{s=0}^{m-1} 2^{-s} \sqrt{u(s)} + \sum_{s=0}^{m-1} s 3^{-s} u(s) + \sum_{s=0}^{m-1} \frac{s 2^{-s}}{200} e^{u(s)}, \tag{62}$$

where $c(m) = \frac{1}{m+3}$, $f_1(m, s) = 2^{-s}$, $w_1(u) = \sqrt{u}$, $f_2(m, s) = s 3^{-s}$, $w_2(u) = u$, $f_3(m, s) = \frac{s 2^{-s}}{200}$, $w_3(u) = e^u$. We can conclude that $\frac{w_3}{w_2} = \frac{e^u}{u}$ and $\frac{w_2}{w_1} = \frac{u}{\sqrt{u}}$ are nondecreasing for $u > 0$, then, we have

$$\begin{aligned} E_1(m) &= \tilde{c}(m) = \max_{2 \leq \tau \leq m} \frac{1}{\tau+3} = \frac{1}{5}, \\ \tilde{f}_i(m, s) &= f_i(m, s), \quad i = 1, 2, 3, \end{aligned}$$

$$\begin{aligned}
 W_1(u) &= \int_1^u \frac{ds}{\sqrt{s}} = 2(\sqrt{u} - 1), \quad W_1^{-1}(u) = \left(\frac{u}{2} + 1\right)^2, \\
 W_2(u) &= \int_1^u \frac{ds}{s} = \ln u, \quad W_2^{-1}(u) = e^u, \\
 W_3(u) &= \int_1^u \frac{ds}{e^s} = e^{-1} - e^{-u}, \quad W_3^{-1}(u) = \ln \frac{1}{e^{-1} - u},
 \end{aligned} \tag{63}$$

from (63), we have

$$\begin{aligned}
 E_2(m) &= W_1^{-1} \left[W_1(E_1(m)) + \sum_{s=0}^{m-1} 2^{-s} \right], \\
 &= W_1^{-1} \left[2(\sqrt{E_1(m)} - 1) + 2 - \left(\frac{1}{2}\right)^{m-1} \right], \\
 &= \left(\sqrt{E_1(m)} + 1 - \left(\frac{1}{2}\right)^m \right)^2,
 \end{aligned}$$

and

$$\begin{aligned}
 E_3(m) &= W_2^{-1} \left[W_2(E_2(m)) + \sum_{s=0}^{m-1} s3^{-s} \right], \\
 &= W_2^{-1} \left[\ln E_2(m) + \frac{3}{4} - \frac{5}{12} \left(\frac{1}{3}\right)^{m-2} - \frac{1}{2} \frac{m-2}{3^{m-1}} \right], \\
 &= E_2(m) \exp \left(\frac{3}{4} - \frac{5}{12} \left(\frac{1}{3}\right)^{m-2} - \frac{1}{2} \frac{m-2}{3^{m-1}} \right).
 \end{aligned}$$

Using Theorem 1, we obtain

$$\begin{aligned}
 &u(m) \\
 &\leq W_3^{-1} \left[W_3(E_3(m)) + \sum_{s=0}^{m-1} \frac{s2^{-s}}{200} \right], \\
 &= W_3^{-1} \left[e^{-1} - e^{-E_3(m)} + \frac{1}{200} \left(2 - \frac{3}{4} \frac{1}{2^{m-3}} - \frac{m-2}{2^{m-1}} \right) \right], \\
 &= \ln \frac{1}{\exp(-E_3(m)) - \frac{1}{200} \left(2 - \frac{3}{4} \frac{1}{2^{m-3}} - \frac{m-2}{2^{m-1}} \right)} \\
 &= \ln \frac{1}{\exp \left(-E_2(m) \exp \left(\frac{3}{4} - \frac{5}{12} \left(\frac{1}{3}\right)^{m-2} - \frac{1}{2} \frac{m-2}{3^{m-1}} \right) \right) - \frac{1}{200} \left(2 - \frac{3}{4} \frac{1}{2^{m-3}} - \frac{m-2}{2^{m-1}} \right)} \\
 &= \ln \frac{1}{\exp \left(- \left(\sqrt{\frac{1}{m+3}} + 1 - \left(\frac{1}{2}\right)^m \right)^2 \exp \left(\frac{3}{4} - \frac{5}{12} \left(\frac{1}{3}\right)^{m-2} - \frac{1}{2} \frac{m-2}{3^{m-1}} \right) \right) - \frac{1}{200} \left(2 - \frac{3}{4} \frac{1}{2^{m-3}} - \frac{m-2}{2^{m-1}} \right)}.
 \end{aligned}$$

The above function $\ln \frac{1}{s}$ always makes sense, since $\exp \left(- \left(\sqrt{\frac{1}{m+3}} + 1 - \left(\frac{1}{2}\right)^m \right)^2 \exp \left(\frac{3}{4} - \frac{5}{12} \left(\frac{1}{3}\right)^{m-2} - \frac{1}{2} \frac{m-2}{3^{m-1}} \right) \right)$ is a decreasing function, and $\frac{1}{200} \left(2 - \frac{3}{4} \frac{1}{2^{m-3}} - \frac{m-2}{2^{m-1}} \right)$ is a increas-

ing function. When $m = 2$, we have

$$\begin{aligned} & \exp\left(-\left(\sqrt{\frac{1}{2+3}}+1-\left(\frac{1}{2}\right)^2\right)^2 \exp\left(\frac{3}{4}-\frac{5}{12}\right)\right) \\ &= \exp\left(-\left(\sqrt{\frac{1}{5}}+\frac{3}{4}\right)^2 \exp\left(\frac{1}{3}\right)\right) \approx 0.1353, \\ & \frac{1}{200}\left(2-\frac{3}{4}\frac{1}{2^{2-3}}\right) = 0.0025. \end{aligned}$$

When $m \rightarrow \infty$, we have

$$\begin{aligned} & \lim_{m \rightarrow \infty} \exp\left(-\left(\sqrt{\frac{1}{2+3}}+1-\left(\frac{1}{2}\right)^m\right)^2 \exp\left(\frac{3}{4}-\frac{5}{12}\left(\frac{1}{3}\right)^{m-2}-\frac{1}{2}\frac{m-2}{3^{m-1}}\right)\right) \\ &= \exp\left(-\left(\frac{1}{5}+1\right)^2 \exp\left(\frac{3}{4}\right)\right) \approx 0.0119, \\ & \lim_{m \rightarrow \infty} \frac{1}{200}\left(2-\frac{3}{4}\frac{1}{2^{m-3}}-\frac{m-2}{2^{m-1}}\right) = 0.01. \end{aligned}$$

Therefore, in $\ln \frac{1}{s}$, $0 < s < 1$ always true. This implies that $u(m)$ is bounded for $m \in \mathbb{N}_0$.

EXAMPLE 2. We consider the partial difference equation with the initial boundary value conditions.

$$\begin{aligned} \Delta_3 \Delta_2 \Delta_1 \psi(z(m, n, c)) &= F(m, n, c, \varphi_1(z(m, n, c)), \dots, \varphi_k(z(m, n, c))), \tag{64} \\ \psi(z(m, 0, 0)) &= a_1(m), \psi(z(0, n, 0)) = a_2(n), \psi(z(0, 0, c)) = a_3(c), \tag{65} \end{aligned}$$

for all $(m, n, c) \in \Lambda_3$, where Λ_3 is defined as in the section 2, and $a_1(0) = a_2(0) = a_3(0) = 0$, $\psi(z) \in C(\mathbb{R}, \mathbb{R})$ is a strictly increasing odd function on \mathbb{R} , satisfying $\psi(0) = 0$ and $\psi(u) > 0$ for $u > 0$, $F : \Lambda_3 \times \mathbb{R}^k \rightarrow \mathbb{R}$, $a_1 : I_1 \rightarrow \mathbb{R}$ and $a_2 : I_2 \rightarrow \mathbb{R}$, $a_3 : I_3 \rightarrow \mathbb{R}$, $\varphi_i : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ are nondecreasing continuous functions and the ratio φ_{i+1}/φ_i are also nondecreasing, $\varphi_i(u) > 0$ for $u > 0$ $i = 1, 2, \dots, k$.

In the following, firstly, we apply our result to discuss boundedness on the solution of problem (64).

THEOREM 4. Assume that $F : \Lambda_3 \times \mathbb{R}^k \rightarrow \mathbb{R}$ is a continuous function satisfying

$$\begin{aligned} & |F(m, n, c, \varphi_1(u), \dots, \varphi_k(u))| \\ & \leq \sum_{i=1}^k [f_i(M, N, C, m, n, c)|u|^p + g_i(M, N, C, m, n, c)|u|^q \varphi_i(|u|)], \tag{66} \end{aligned}$$

$$|a_1(m) + a_2(n) + a_3(c)| \leq a(m, n, c), \tag{67}$$

for all $(m, n, c) \in \Lambda_3$, where $p > q > 0$ is a constant, $f_i(M, N, C, m, n, c)$, $g_i(M, N, C, m, n, c)$, $i = 1, 2, \dots, k$, are continuous nonnegative functions and nondecreasing in M, N, C for

each fixed m, n, c , $a(m, n, c) : \Lambda_3 \rightarrow \mathbb{R}_+$ is nondecreasing in each variable. If $z(m, n, c)$ is any solution of (64) with the condition (65), then, case one: if $\Psi^{-1}(\bar{z}(m, t, e)) > 1$,

$$|z(m, n, c)| \leq \Psi^{-1} \left\{ \Psi_p^{-1} \left[G_k^{-1} (G_k(H_k(m, n, c))) + \sum_{s=0}^{m-1} \sum_{t=0}^{n-1} \sum_{e=0}^{c-1} g_k(M, N, C, s, t, e) \right] \right\}, \tag{68}$$

for all $(m, n, c) \in \Lambda_{(M, N, C)}$, where $\Psi_p(u)$ is defined by (3), and

$$G_i(u) := \int_1^u \frac{ds}{\varphi_i(\Psi_p^{-1}(\Psi_p^{-1}(s)))}, \quad u > 0,$$

$$H_1(m, n, c) := \Psi_p(a(m, n, c)),$$

$$H_i(m, n, c) := G_{i-1}^{-1} [G_{i-1}(H_{i-1}(m, n, c)) + \sum_{s=0}^{m-1} \sum_{t=0}^{n-1} \sum_{e=0}^{c-1} g_{i-1}(M, N, C, s, t, e)],$$

Ψ_p^{-1} and G_k^{-1} denote the inverse function of Ψ_p and G .

Case two: if $\Psi^{-1}(\bar{z}(m, t, e)) < 1$,

$$|z(m, n, c)| \leq \Psi^{-1} \left\{ \Psi_q^{-1} \left[\tilde{G}_k^{-1} (\tilde{G}_k(\tilde{H}_k(m, n, c))) + \sum_{s=0}^{m-1} \sum_{t=0}^{n-1} \sum_{e=0}^{c-1} g_k(M, N, C, s, t, e) \right] \right\}, \tag{69}$$

for all $(m, n, c) \in \Lambda_{(M, N, C)}$, where $\Psi_q(u)$ is defined by (4), and

$$\tilde{G}_i(u) := \int_1^u \frac{ds}{\varphi_i(\Psi_q^{-1}(\Psi_q^{-1}(s)))}, \quad u > 0,$$

$$\tilde{H}_1(m, n, c) := \Psi_q(a(m, n, c)),$$

$$\tilde{H}_i(m, n, c) := \tilde{G}_{i-1}^{-1} [\tilde{G}_{i-1}(\tilde{H}_{i-1}(m, n, c)) + \sum_{s=0}^{m-1} \sum_{t=0}^{n-1} g_{i-1}(M, N, C, s, t, e)],$$

Ψ_q^{-1} and \tilde{G}_k^{-1} denote the inverse function of Ψ_q and \tilde{G} .

Proof. The solution $z(m, n, c)$ of (64) satisfies the following equivalent difference equation

$$\begin{aligned} \Psi(z(m, n, c)) &= a_1(m) + a_2(n) + a_3(c) \\ &\quad + \sum_{s=0}^{m-1} \sum_{t=0}^{n-1} \sum_{e=0}^{c-1} F(s, t, e, \varphi_1(z(s, t, e)), \dots, \varphi_k(z(s, t, e))). \end{aligned} \tag{70}$$

By (66), (67) and (70), we obtain

$$\begin{aligned} |\Psi(z(m, n, c))| &\leq a(m, n, c) + \sum_{s=0}^{m-1} \sum_{t=0}^{n-1} \sum_{e=0}^{c-1} |F(s, t, e, \varphi_1(z(s, t, e)), \dots, \varphi_k(z(s, t, e)))| \\ &\leq a(m, n, c) + \sum_{i=1}^k \sum_{s=0}^{m-1} \sum_{t=0}^{n-1} \sum_{e=0}^{c-1} f_i(M, N, C, s, t, e) |z(s, t, e)|^p \\ &\quad + \sum_{i=1}^k \sum_{s=0}^{m-1} \sum_{t=0}^{n-1} \sum_{e=0}^{c-1} |z(s, t, e)|^q g_i(M, N, C, s, t, e) \varphi_i(|z(s, t, e)|). \end{aligned} \tag{71}$$

Since $|\psi(z(m, n, c))| = \psi(|z(m, n, c)|)$, (71) has the same form of (52). Let $\tilde{z}(m, n, c)$ denote the function on the right-hand side of (71), then $|z(m, n, c)| \leq \psi^{-1}(\tilde{z}(m, n, c))$. Applying Theorem 2 to inequality (71), we obtain the estimation of $z(m, n, c)$ as given in (68) and (69).

If there exists a constant $M > 0$, such that

$$H_i(m, n, c) < M, \quad \sum_{s=0}^{m-1} \sum_{t=0}^{n-1} \sum_{e=0}^{c-1} g_i(M, N, C, s, t, e) < M, \quad i = 1, 2, \dots, k, \quad (72)$$

for all $(m, n, c) \in \Lambda_{(M, N, C)}$, then every solution $z(m, n, c)$ of (64) is bounded on $\Lambda_{(M, N, C)}$. □

Next, we discuss the uniqueness of the solutions of (64).

THEOREM 5. *Assume additionally that*

$$\begin{aligned} & |F(m, n, c, \varphi_1(u_1), \dots, \varphi_k(u_1)) - F(m, n, c, \varphi_1(u_2), \dots, \varphi_k(u_2))| \\ & \leq \sum_{i=1}^k h_i(M, N, C, m, n, c) |\psi(u_1) - \psi(u_2)|^q \varphi_i(|\psi(u_1) - \psi(u_2)|), \end{aligned} \quad (73)$$

for $u_1, u_2 \in \mathbb{R}$ and $(m, n, c) \in \Lambda_3$, where Λ_i is defined in the section 2, $h_i : \Lambda_6 \rightarrow \mathbb{R}_+$ are nonnegative functions, $i = 1, 2, \dots, k$, $\varphi_i : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ are continuous nondecreasing with the nondecreasing ratio φ_{i+1}/φ_i such that $\varphi_i(u) > 0$ for all $u > 0$, and $\int_0^1 \frac{ds}{\varphi_i(s)} = \infty$, for $i = 1, 2, \dots, k$, and $\psi \in C^1(\mathbb{R}, \mathbb{R})$ is a strictly increasing odd function satisfying $\psi(u) > 0$, for all $u > 0$. Then, (64) has at most one solution on Λ_3 .

Proof. Let $z(m, n, c)$ and $\tilde{z}(m, n, c)$ are two solutions of (64). From (70) and (73), we have

$$\begin{aligned} & |\psi(z(m, n, c)) - \psi(\tilde{z}(m, n, c))| \\ & \leq \sum_{i=1}^k \sum_{s=0}^{m-1} \sum_{t=0}^{n-1} \sum_{e=0}^{c-1} h_i(M, N, C, s, t, e) |\psi(z(s, t, e)) - \psi(\tilde{z}(s, t, e))|^q \\ & \quad \times \varphi_i(|\psi(z(s, t, e)) - \psi(\tilde{z}(s, t, e))|) \end{aligned} \quad (74)$$

for all $(m, n, c) \in \Lambda_3$, (74) is the special form of (52), where $f_i = 0$, $i = 1, 2, \dots, k$, $a(m, n, c) = 0$, $h_i(M, N, C, s, t, e)$, $i = 1, 2, \dots, k$, are continuous nonnegative functions and nondecreasing in M, N, C for each fixed s, t, e . Applying Theorem 2, we obtain an estimation of the difference $|\psi(z(m, n)) - \psi(\tilde{z}(m, n))|$ in the form (68), where $H_1(m, n, c) = 0$, because $\Psi_p(0) = 0$. Furthermore, by the definition of G_i , we conclude that

$$\lim_{u \rightarrow 0} G_i(u) = -\infty, \quad \lim_{u \rightarrow -\infty} G_i^{-1}(u) = 0, \quad i = 1, 2, \dots, k. \quad (75)$$

It follows that

$$G_i(H_i(m, n, c)) + \sum_{s=0}^{m-1} \sum_{t=0}^{n-1} \sum_{e=0}^{c-1} h_i(M, N, C, s, t, e) = -\infty,$$

$$G_i^{-1}[G_i(\tilde{H}_i(m,n,c)) + \sum_{s=0}^{m-1} \sum_{t=0}^{n-1} \sum_{e=0}^{c-1} h_i(M,N,C,s,t,e)] = 0, \quad i = 1, 2, \dots, k.$$

From (68), we deduce that $|\psi(z(m,n,c)) - \psi(\bar{z}(m,n,c))| \leq 0$, implying that $z(m,n,c) = \bar{z}(m,n,c)$, for all $(m,n,c) \in \Lambda_3$.

Finally, we consider the continuous dependence of solutions of (64) on the given functions F, a_1, a_2, a_3 . For that, we consider a perturbation of (64),

$$\Delta_3 \Delta_2 \Delta_1 \psi(z(m,n,c)) = \bar{F}(m,n,c, \varphi_1(z(m,n,c)), \dots, \varphi_k(z(m,n,c))), \tag{76}$$

$$\psi(z(m,0,0)) = \bar{a}_1(m), \psi(z(0,n,0)) = \bar{a}_2(n), \psi(z(0,0,c)) = \bar{a}_3(c), \tag{77}$$

for all $(m,n,c) \in \Lambda_3$, and $\bar{a}_1(0) = \bar{a}_2(0) = \bar{a}_3(0) = 0$, $\psi(z) \in C(\mathbb{R}, \mathbb{R})$ is a strictly increasing odd function on \mathbb{R} , satisfying $\psi(0) = 0$ and $\psi(u) > 0$ for $u > 0$, $F : \Lambda_3 \times \mathbb{R}^k \rightarrow \mathbb{R}$, $a_1 : I_1 \rightarrow \mathbb{R}$ and $a_2 : I_2 \rightarrow \mathbb{R}$, $a_3 : I_3 \rightarrow \mathbb{R}$, $\varphi_i : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ are nondecreasing continuous functions and the ratio φ_{i+1}/φ_i are also nondecreasing, $\varphi_i(u) > 0$ for $u > 0$ $i = 1, 2, \dots, k$. \square

THEOREM 6. *Suppose that*

$$\begin{aligned} \max |\bar{a}_1 - a_1| < \varepsilon, \max |\bar{a}_2 - a_2| < \varepsilon, \\ \max |\bar{a}_3 - a_3| < \varepsilon, \max |\bar{F} - F| < \varepsilon. \end{aligned} \tag{78}$$

where $\varepsilon > 0$ is an arbitrary small number. Then the solution $\bar{z}(m,n,c)$ of (76) is sufficiently close to the solution $z(m,n,c)$ of (64).

Proof. Let $z(m,n,c)$ and $\bar{z}(m,n,c)$ be the solutions of (76) and (64), respectively. Then, \bar{z} satisfies the equivalent difference equation

$$\begin{aligned} \psi(\bar{z}(m,n,c)) &= \bar{a}_1(m) + \bar{a}_2(n) + \bar{a}_3(c) \\ &+ \sum_{s=0}^{m-1} \sum_{t=0}^{n-1} \sum_{e=0}^{c-1} \bar{F}(s,t,e, \varphi_1(\bar{z}(s,t,e)), \dots, \varphi_k(\bar{z}(s,t,e))). \end{aligned} \tag{79}$$

From (70), (73), (78), (79), we have

$$\begin{aligned} &|\psi(z(m,n,c)) - \psi(\bar{z}(m,n,c))| \\ &\leq |a_1(m) - \bar{a}_1(m)| + |a_2(n) - \bar{a}_2(n)| + |a_3(c) - \bar{a}_3(c)| \\ &+ \sum_{s=0}^{m-1} \sum_{t=0}^{n-1} \sum_{e=0}^{c-1} |F(s,t,e, \varphi_1(z(s,t,e)), \dots, \varphi_k(z(s,t,e))) \\ &- \bar{F}(s,t,e, \varphi_1(\bar{z}(s,t,e)), \dots, \varphi_k(\bar{z}(s,t,e)))| \\ &\leq 3\varepsilon + \sum_{s=0}^{m-1} \sum_{t=0}^{n-1} \sum_{e=0}^{c-1} |F(s,t,e, \varphi_1(z(s,t,e)), \dots, \varphi_k(z(s,t,e))) \\ &- F(s,t,e, \varphi_1(\bar{z}(s,t,e)), \dots, \varphi_k(\bar{z}(s,t,e)))| \\ &+ \sum_{s=0}^{m-1} \sum_{t=0}^{n-1} \sum_{e=0}^{c-1} |F(s,t,e, \varphi_1(\bar{z}(s,t,e)), \dots, \varphi_k(\bar{z}(s,t,e))) \end{aligned}$$

$$\begin{aligned}
 & -\bar{F}(s, t, e, \varphi_1(\bar{z}(s, t, e)), \dots, \varphi_k(\bar{z}(s, t, e)))| \\
 \leq & (3+M_1N_1C_1)\varepsilon + \sum_{i=1}^k \sum_{s=0}^{m-1} \sum_{t=0}^{n-1} \sum_{e=0}^{c-1} h_i(M, N, C, m, n, c) |\psi(z) - \psi(\bar{z})|^q \varphi_i(|\psi(z) - \psi(\bar{z})|).
 \end{aligned} \tag{80}$$

(80) has the same form of Corollary 7, using Corollary 7 to (80), we obtain

$$|\psi(z(m, n, c)) - \psi(\bar{z}(m, n, c))| \leq [W_k^{-1}(W_k(E_k(m, n, z)) + \sum_{s=0}^{m-1} \sum_{t=0}^{n-1} \sum_{e=0}^{z-1} \tilde{g}_k(m, n, z, s, t, e))]^{\frac{1}{1-p}},$$

for $(m, n, z) \in \Lambda_{(M_1, N_1, Z_1)}$, where $E_1 = (3 + M_1N_1C_1)\varepsilon$, then $E_1 \rightarrow 0$ as $\varepsilon \rightarrow 0$. Since

$$E_k = W_k^{-1}(W_k(E_k(m, n, z)) + \sum_{s=0}^{m-1} \sum_{t=0}^{n-1} \sum_{e=0}^{z-1} \tilde{g}_k(m, n, z, s, t, e)),$$

from (75), we have $\lim_{\varepsilon \rightarrow 0} E_k = 0$, and that

$$\lim_{\varepsilon \rightarrow 0} |\psi(z(m, n, c)) - \psi(\bar{z}(m, n, c))| = 0.$$

Thus, $\psi(z)$ depends continuously on F, a_1, a_2, a_3 . \square

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