

## STABLE FUNCTIONS OF JANOWSKI TYPE

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(Communicated by J. Pečarić)

*Abstract.* A function  $f \in \mathcal{A}_1$  is said to be stable with respect to  $g \in \mathcal{A}_1$  if

$$\frac{s_n(f(z))}{f(z)} \prec \frac{1}{g(z)}, \quad z \in \mathbb{D},$$

holds for all  $n \in \mathbb{N}$  where  $\mathcal{A}_1$  denote the class of analytic functions  $f$  in the unit disk  $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$  normalized by  $f(0) = 1$ . Here  $s_n(f(z))$ , the  $n^{\text{th}}$  partial sum of  $f(z) = \sum_{k=0}^{\infty} a_k z^k$  is given by  $s_n(f(z)) = \sum_{k=0}^n a_k z^k$ ,  $n \in \mathbb{N} \cup \{0\}$ . In this work, we consider the following function

$$v_\lambda(A, B, z) = \left( \frac{1 + Az}{1 + Bz} \right)^\lambda$$

for  $-1 \leq B < A \leq 1$  and  $0 \leq \lambda \leq 1$  for our investigation. The main purpose of this paper is to prove that  $v_\lambda(A, B, z)$  is stable with respect to  $v_\lambda(0, B, z) = \frac{1}{(1 + Bz)^\lambda}$  for  $0 < \lambda \leq 1$  and  $-1 \leq B < A \leq 0$ . Further, we prove that  $v_\lambda(A, B, z)$  is not stable with respect to itself, when  $0 < \lambda \leq 1$  and  $-1 \leq B < A < 0$ .

### Introduction and Main Results

Let  $\mathcal{A}$  denote the family of functions  $f$  that are analytic in the unit disk  $\mathbb{D} := \{z : |z| < 1\}$ . Let  $\mathcal{A}_1$  is the subset of  $\mathcal{A}$  with the normalization  $f(0) = 1$ . A single valued function  $f \in \mathcal{A}_1$  is said to be univalent in a domain  $\Delta \subseteq \mathbb{C}$  if  $f$  is one-to-one in  $\Delta$ . The class of all univalent functions with the normalization  $f(0) = 0 = f'(0) - 1$  is denoted by  $\mathcal{S}$ . Let  $\Omega$  be the family of functions  $\omega$ , regular in  $\mathbb{D}$  and satisfying the conditions  $\omega(0) = 0$  and  $|\omega(z)| < 1$  for all  $z \in \mathbb{D}$ . For  $f, g \in \mathcal{A}$ , the function  $f$  is said to be subordinate to  $g$ , denoted by  $f \prec g$  if and only if there exists an analytic function  $\omega \in \Omega$  such that  $f = g \circ \omega$ . In particular, if  $g$  is univalent in  $\mathbb{D}$  then  $f(0) = g(0)$  and  $f(\mathbb{D}) \subseteq g(\mathbb{D})$  hold.

The function  $zf(z) \in \mathcal{A}_1$  is starlike of order  $\lambda$  if  $\operatorname{Re} \left( \frac{zf'(z)}{f(z)} \right) > \lambda$  for all  $z \in \mathbb{D}$  and  $0 \leq \lambda < 1$ . The class of all starlike functions, denoted by  $\mathcal{S}^*(\lambda)$  is a subclass of

*Mathematics subject classification* (2020): 30C45.

*Keywords and phrases:* Analytic functions, starlike functions, subordination and stable functions.

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$\mathcal{S}$ . The  $n^{\text{th}}$  partial sum  $s_n(f(z))$  of  $f(z) = \sum_{k=0}^{\infty} a_k z^k$  is given by  $s_n(f(z)) = \sum_{k=0}^n a_k z^k$ ,  $n = 0, 1, 2, \dots$ . For more details about the univalent functions, its subclasses and subordination properties, we refer [2, 3, 5].

The concept of stable functions was first introduced by Ruscheweyh and Salinas [6], while discussing the class of starlike functions of order  $\lambda$ , where  $1/2 \leq \lambda < 1$ . However, the class of starlike functions of order  $\lambda \in [1/2, 1)$  is comparatively a much narrow class but it has many interesting properties too. Ruscheweyh and Salinas [6] proved the following result.

**THEOREM 1.** [6] *Let  $\lambda \in [1/2, 1)$  and  $zf \in \mathcal{S}^*(\lambda)$ , then*

$$\frac{s_n(f, z)}{f(z)} \prec (1-z)^\lambda, \quad n \in \mathbb{N}, z \in \mathbb{D}.$$

Theorem 1 has several applications in Gegenbauer polynomial sums and motivated by Theorem 1, Ruscheweyh and Salinas [6] introduced the concept of Stable functions which is stated as follows. For some  $n \in \mathbb{N}$ , a function  $F$  is said to be  $n$ -stable function with respect to  $G$  if

$$\frac{s_n(F(z))}{F(z)} \prec \frac{1}{G(z)}, \quad \text{for } F, G \in \mathcal{A}_1 \text{ and } z \in \mathbb{D}.$$

Moreover, the function  $F$  is said to be stable with respect to  $G$ , if  $F$  is  $n$ -stable with respect to  $G$  for every  $n \in \mathbb{N}$ . Particularly, if the function  $F$  is  $n$ -stable with respect to itself. Then for every  $n \in \mathbb{N}$ ,  $F$  is stable. In the present context, for  $-1 \leq B < A \leq 1$ , we define a function

$$v_\lambda(A, B, z) := \left( \frac{1 + Az}{1 + Bz} \right)^\lambda \quad \text{for } z \in \mathbb{D} \text{ and } \lambda \in (0, 1].$$

For  $\lambda = 1/2$ , Ruscheweyh and Salinas [7] proved that  $v_{1/2}(1, -1, z)$  is stable function with respect to itself. The stability of  $v_{1/2}(1, -1, z)$  is equivalent to the simultaneous non-negativity of general class of sine and cosine sums given by Vietoris [11], the most celebrated theorem of positivity of trigonometric sums. Ruscheweyh and Salinas [7] conjectured that  $v_\lambda(1, -1, z)$  is stable for  $0 < \lambda < 1/2$ . Using computer algebra, for  $\lambda = 1/4$  it was shown in [7] that  $v_{1/4}(1, -1, z)$  is  $n$ -stable for  $n = 1, 2, 3, \dots, 5000$ . In the limiting case, the validation of stability of  $v_\lambda(1, -1, z)$  for  $0 < \lambda < 1/2$  interpreted in terms of positivity of trigonometric polynomials.

Further extensions of Vietoris Theorem and stable functions to Cesàro stable functions and Generalized Cesàro stable functions have been studied in [4] and [9] respectively. In this direction, conjectures are also proposed in [9] that linked Generalized Cesàro stable functions with the positivity of trigonometric sums. Chakraborty and Vasudevarao [1] considered  $A = 1 - 2\alpha$ ,  $B = -1$  and proved the following result.

THEOREM 2. [1] For  $0 < \lambda \leq 1$  and  $1/2 \leq \alpha < 1$ ,

$$v_\lambda(1 - 2\alpha, -1, z) = \left( \frac{1 + (1 - 2\alpha)z}{1 - z} \right)^\lambda$$

is stable with respect to  $v_\lambda(0, -1, z) = \frac{1}{(1 - z)^\lambda}$ .

Chakraborty and Vasudevarao [1] also proved that  $v_\lambda(1 - 2\alpha, -1, z)$  is not stable with respect to itself when  $1/2 < \alpha < 1$  and  $0 < \lambda \leq 1$ . For  $\lambda = 1$ , the function  $v_1(A, B, z) = \frac{1 + Az}{1 + Bz}$  have been studied widely by many researchers. The analytic functions of  $\mathcal{A}_1$  subordinate to  $\frac{1 + Az}{1 + Bz}$  have been studied by Janowski [3] and the class of such functions is denoted by  $\mathcal{P}(A, B)$ . The functions of  $\mathcal{P}(A, B)$  are called Janowski functions. Moreover, the set of functions  $zf \in \mathcal{A}_1$ , for which  $\frac{zf'(z)}{f(z)} \prec \frac{1 + Az}{1 + Bz}$  holds, called Janowski starlike functions and the class of such functions is denoted by  $\mathcal{S}^*(A, B)$ . It can be easily seen that  $\mathcal{S}^*(1, -1) \equiv \mathcal{S}^*$ .

In this paper, we show that  $v_\lambda(A, B, z)$  is stable with respect to  $v_\lambda(0, B, z) = 1/(1 + Bz)^\lambda$  for  $0 < \lambda \leq 1$  and  $-1 \leq B < A \leq 0$ . Further,  $v_\lambda(A, B, z)$  is not stable with respect to itself, when  $0 < \lambda \leq 1$  and  $-1 \leq B < A < 0$ . We can write  $v_\lambda(A, B, z)$  as,

$$\begin{aligned} v_\lambda(A, B, z) &= \left( \frac{1 + Az}{1 + Bz} \right)^\lambda \\ &= (1 + Az)^\lambda (1 + Bz)^{-\lambda} \\ &= \left( 1 + \sum_{k=1}^{\infty} \frac{[\lambda]_k}{k!} A^k z^k \right) \left( 1 + \sum_{k=1}^{\infty} \frac{(\lambda)_k}{k!} (-B)^k z^k \right) \\ &= 1 + \sum_{n=1}^{\infty} \left( \sum_{k=0}^n \frac{[\lambda]_k}{k!} \frac{(\lambda)_{n-k}}{(n-k)!} A^k (-B)^{n-k} \right) z^n, \end{aligned} \tag{1}$$

where  $[\lambda]_k$  and  $(\lambda)_k$  denote the factorial polynomials given as

$$\begin{cases} [\lambda]_k = \lambda(\lambda - 1)(\lambda - 2)(\lambda - 3) \cdots (\lambda - k + 1), & \text{and} \\ (\lambda)_k = \frac{\Gamma(\lambda + k)}{\Gamma(\lambda)} = \lambda(\lambda + 1) \cdots (\lambda + k - 1) \end{cases}$$

for  $k = 1, 2, \dots$  respectively with  $[\lambda]_0 = 1 = (\lambda)_0$  and  $\Gamma$  is well-known gamma function. So  $v_\lambda(A, B, z)$  can be written as

$$v_\lambda(A, B, z) = 1 + \sum_{n=1}^{\infty} a_n(A, B, \lambda) z^n,$$

where

$$a_n := a_n(A, B, \lambda) = \sum_{k=0}^n \frac{[\lambda]_k}{k!} \frac{(\lambda)_{n-k}}{(n-k)!} A^k (-B)^{n-k}.$$

Now, we state two lemmas which will help to prove our main results.

LEMMA 1. For  $0 < \lambda \leq 1$  and  $-1 \leq B < A \leq 0$ , we have

$$\sum_{k=0}^n \frac{[\lambda]_k}{k!} \frac{(\lambda)_{n-k}}{(n-k)!} A^k (-B)^{n-k} > 0.$$

LEMMA 2. Let  $v_\lambda(A, B, z)$  be defined by (1). Then for  $\lambda \in (0, 1]$  and  $-1 \leq B < A \leq 0$ ,

$$(m+1)(n+1) \left( \sum_{k=0}^{n+1} \frac{[\lambda]_k}{k!} \frac{(\lambda)_{n+1-k}}{(n+1-k)!} A^k B^{n+1-k} \right) - mn \left( \sum_{k=0}^n \frac{[\lambda]_k}{k!} \frac{(\lambda)_{n-k}}{(n-k)!} A^k B^{n-k} \right) \geq 0 \tag{2}$$

holds for all  $m, n \in \mathbb{N}$ .

Now, we state main results of this paper which are about the stability of  $v_\lambda(A, B, z)$  with respect to  $v_\lambda(0, B, z)$  and  $v_\lambda(A, B, z)$  itself.

THEOREM 3. For  $\lambda \in (0, 1]$  and  $-1 \leq B < A \leq 0$ ,  $v_\lambda(A, B, z)$  given in (1) is stable with respect to  $v_\lambda(0, B, z) = \frac{1}{(1+Bz)^\lambda}$ .

If we substitute  $A = 0$  in Theorem 3, we get the following corollary which is also a generalization of the result given by Ruscheweyh and Salinas [6].

COROLLARY 1. For  $\lambda \in (0, 1]$  and  $-1 \leq B < 0$ ,  $v_\lambda(0, B, z) = \frac{1}{(1+Bz)^\lambda}$  is stable function.

Now for  $0 < \mu \leq \lambda \leq 1$ , we have the following corollary of Theorem 3.

COROLLARY 2. For  $0 < \mu \leq \lambda \leq 1$  and for  $-1 \leq B < 0$  we have

$$\frac{s_n(v_\mu(0, B, z))}{v_\lambda(0, B, z)} \prec \frac{1}{v_\lambda(0, B, z)}, \quad \text{for } z \in \mathbb{D}.$$

Theorem 3 also generalizes result of Chakraborty and Vasudevarao [1] as if we substitute  $A = 1 - 2\alpha$  and  $B = -1$  in Theorem 3, reduces to Theorem 2. In other words, Theorem 2 is a particular case of Theorem 3.

THEOREM 4. For  $\lambda \in (0, 1]$  and  $-1 \leq B < A < 0$ ,  $v_\lambda(A, B, z) = \left( \frac{1+Az}{1+Bz} \right)^\lambda$  is not stable with respect to itself.

**Proof of Main Results**

*Proof of Lemma 1.* Consider,

$$\begin{aligned}
 1 &= (1-z)^\lambda (1-z)^{-\lambda} \\
 &= 1 + \sum_{n=1}^{\infty} \left( \sum_{k=0}^n \frac{[\lambda]_k}{k!} \frac{(\lambda)_{n-k}}{(n-k)!} (-1)^k \right) z^n
 \end{aligned}$$

Comparing the coefficients of  $z^n$  on both the sides we have

$$\sum_{k=0}^n \frac{[\lambda]_k}{k!} \frac{(\lambda)_{n-k}}{(n-k)!} (-1)^k = 0,$$

which can be expanded as

$$\begin{aligned}
 &\frac{(\lambda)(\lambda+1)\cdots(\lambda+n-1)}{(n)!} + \frac{(\lambda)(\lambda+1)\cdots(\lambda+n-2)}{(n-1)!} \left(\frac{\lambda}{1!}\right) (-1) \\
 &+ \frac{(\lambda)(\lambda+1)\cdots(\lambda+n-3)}{(n-2)!} \frac{\lambda(\lambda-1)}{2!} (-1)^2 + \cdots \\
 &+ \frac{\lambda}{1!} \frac{(\lambda)(\lambda-1)\cdots(\lambda-n+2)}{(n-1)!} (-1)^{n-1} + \frac{(\lambda)(\lambda-1)\cdots(\lambda-n+1)}{n!} (-1)^n = 0.
 \end{aligned}$$

Since  $0 \leq \lambda < 1$ , so only first term in the above equation is positive. By multiplying  $2^{nd}, 3^{rd}, \dots, (n+1)^{th}$  terms by  $\frac{\alpha}{\beta}, \frac{\alpha^2}{\beta^2}, \dots, \frac{\alpha^n}{\beta^n}$  respectively, we obtain for  $0 \leq \alpha < \beta$ ,

$$\begin{aligned}
 &\frac{(\lambda)(\lambda+1)\cdots(\lambda+n-1)}{(n)!} + \frac{(\lambda)(\lambda+1)\cdots(\lambda+n-2)}{(n-1)!} \left(\frac{\lambda}{1!}\right) (-1) \frac{\alpha}{\beta} \\
 &+ \frac{(\lambda)(\lambda+1)\cdots(\lambda+n-3)}{(n-2)!} \frac{\lambda(\lambda-1)}{2!} \left(\frac{-\alpha}{\beta}\right)^2 + \cdots \\
 &+ \frac{\lambda}{1!} \frac{(\lambda)(\lambda-1)\cdots(\lambda-n+2)}{(n-1)!} \left(\frac{-\alpha}{\beta}\right)^{n-1} + \frac{(\lambda)(\lambda-1)\cdots(\lambda-n+1)}{n!} \left(\frac{-\alpha}{\beta}\right)^n \geq 0.
 \end{aligned}$$

After multiplying by  $\beta^n$  we obtain

$$\beta^n \sum_{k=0}^n \frac{[\lambda]_k}{k!} \frac{(\lambda)_{n-k}}{(n-k)!} (-1)^k \left(\frac{\alpha}{\beta}\right)^k \geq 0 \tag{3}$$

By substituting  $\alpha = -A, \beta = -B$  in (3) so that for  $-1 \leq B < A \leq 0$ , the lemma is proved.  $\square$

*Proof of Lemma 2.* Let  $v_\lambda(A, B, z)$  be defined by (1). Then,

$$\begin{aligned}
 v_\lambda(A, B, z) &= \left(\frac{1 + Az}{1 + Bz}\right)^\lambda = 1 + a_1z + a_2z^2 + a_3z^3 + \dots \\
 v'_\lambda(A, B, z) &= \lambda \left(\frac{1 + Az}{1 + Bz}\right)^{\lambda-1} \left(\frac{(1 + Bz)A - (1 + Az)B}{(1 + Bz)^2}\right) \\
 &= \lambda(A - B) \frac{(1 + Az)^{\lambda-1}}{(1 + Bz)^{\lambda+1}} \\
 (1 + Bz)v'_\lambda(A, B, z) &= \lambda(A - B)(1 + Az)^{\lambda-1}(1 + Bz)^{-\lambda}
 \end{aligned} \tag{4}$$

Since  $0 > A > B$ ,  $0 < \lambda \leq 1$ ,  $(1 + Az)^{\lambda-1} = 1 + (\lambda - 1)Az + \frac{(\lambda-1)(\lambda-2)}{2!}A^2z^2 + \dots$  and  $(1 + Bz)^{-\lambda} = 1 - \lambda Bz + \frac{\lambda(\lambda+1)}{2!}B^2z^2 + \dots$  have positive Taylor series coefficients. A simple computation yields that

$$\begin{aligned}
 (1 + Bz)v'_\lambda(A, B, z) &= (a_1 + 2a_2z + 3a_3z^2 + \dots)(1 + Bz) \\
 &= a_1 + \sum_{n=1}^\infty ((n + 1)a_{n+1} + Bna_n)z^n.
 \end{aligned} \tag{5}$$

Since right hand side of (4) has positive Taylor coefficients, from (4) and (5) we conclude that

$$(n + 1)a_{n+1} + Bna_n > 0, \quad n \in \mathbb{N}. \tag{6}$$

The left hand side of the expression given in (2) can be rewritten as

$$(m + 1)(n + 1)a_{n+1} + mnBa_n. \tag{7}$$

Equivalently, (7) can be written as

$$m((n + 1)a_{n+1} + Bna_n) + (n + 1)a_{n+1}.$$

Using (6) and the fact that  $a_n \geq 0$  for  $m, n \in \mathbb{N}$ , the lemma is proved for  $\lambda \in (0, 1]$  and  $-1 \leq B < A \leq 0$ .  $\square$

Before going to proceed further for the proof of Theorem 3, it is easy to verify the following relations.

$$\begin{aligned}
 s'_n(v_\lambda(A, B, z), z) &= s_{n-1}(v'_\lambda(A, B, z), z), \\
 z s'_n(v_\lambda(A, B, z), z) &= s_n(zv'_\lambda(A, B, z), z), \\
 z^2 s'_n(v_\lambda(A, B, z), z) &= s_n(z^2v'_\lambda(A, B, z), z).
 \end{aligned} \tag{8}$$

Now, we are ready to give the proof of Theorem 3.

*Proof of Theorem 3.* To show that  $v_\lambda(A, B, z)$  is stable with respect to  $v_\lambda(0, B, z)$ , it is enough to show that

$$\frac{s_n(v_\lambda(A, B, z), z)}{v_\lambda(A, B, z), z} \prec \frac{1}{v_\lambda(0, B, z)}, \quad z \in \mathbb{D}$$

for all  $n \in \mathbb{N}$ , i.e., to prove that

$$\frac{(1+Bz)^\lambda s_n(v_\lambda(A,B,z),z)}{(1+Az)^\lambda} \prec (1+Bz)^\lambda, \quad z \in \mathbb{D},$$

which can be equivalently written as

$$\frac{(1+Bz)s_n(v_\lambda(A,B,z),z)^{\frac{1}{\lambda}}}{(1+Az)} \prec (1+Bz).$$

To show that, it is enough to prove that

$$\left| \frac{(1+Bz)s_n(v_\lambda(A,B,z),z)^{\frac{1}{\lambda}}}{(1+Az)} - 1 \right| \leq |B| \leq 1, \quad z \in \mathbb{D}.$$

For fixed  $n$  and  $\lambda$ , we consider the following function

$$h(z) = 1 - \frac{(1+Bz)s_n(v_\lambda(A,B,z),z)^{\frac{1}{\lambda}}}{(1+Az)}, \quad z \in \mathbb{D}.$$

It is easy to see that

$$v'_\lambda(A,B,z) = \lambda(A-B) \frac{(1+Az)^{\lambda-1}}{(1+Bz)^{\lambda+1}} = \lambda(A-B) \frac{v_\lambda(A,B,z)}{(1+Bz)(1+Az)},$$

which can be rewritten in the following form

$$v_\lambda(A,B,z) - \frac{(1+(A+B)z+ABz^2)}{\lambda(A-B)} v'_\lambda(A,B,z) = 0 \quad \text{for } z \in \mathbb{D}. \tag{9}$$

A simple calculations gives that

$$\begin{aligned} h'(z) &= \frac{A-B}{(1+Az)^2} s_n(v_\lambda(A,B,z),z)^{1/\lambda} \\ &\quad - \frac{(1+Bz)}{(1+Az)\lambda} s_n(v_\lambda(A,B,z),z)^{\frac{1}{\lambda}-1} s'_n(v_\lambda(A,B,z),z) \\ &= \frac{(A-B)s_n(v_\lambda(A,B,z),z)^{\frac{1}{\lambda}}}{(1+Az)^2} \left( s_n(v_\lambda(A,B,z),z) - \frac{(1+Az)(1+Bz)}{(A-B)\lambda} s'_n(v_\lambda(A,B,z),z) \right) \end{aligned} \tag{10}$$

Using relations (8) in (10), we get

$$\begin{aligned} h'(z) &= \frac{(A-B)s_n(v_\lambda(A,B,z),z)^{\frac{1}{\lambda}-1}}{(1+Az)^2} \left[ s_n \left( v_\lambda(A,B,z) - \frac{(1+Az)(1+Bz)}{(A-B)\lambda} v'_\lambda(A,B,z), z \right) \right. \\ &\quad + \frac{(n+1)}{\lambda(A-B)} \sum_{k=0}^{n+1} \frac{[\lambda]_k}{k!} \frac{(\lambda)_{n-k+1}}{(n-k+1)!} A^k (-B)^{n-k+1} z^n \\ &\quad \left. - \frac{nAB}{\lambda(A-B)} \sum_{k=0}^n \frac{[\lambda]_k}{k!} \frac{(\lambda)_{n-k}}{(n-k)!} A^k (-B)^{n-k} z^{n+1} \right]. \end{aligned} \tag{11}$$

Substituting (9) in (11) and using definition of  $a_n$ , the following form of  $h'(z)$  can be obtained.

$$\begin{aligned} h'(z) &= \frac{z^n s_n(v_\lambda(A, B, z), z)^{\frac{1}{\lambda}-1}}{\lambda(1 + Az)^2} ((n + 1)a_{n+1} - ABzn a_n) \\ &= \frac{z^n s_n(v_\lambda(A, B, z), z)^{\frac{1}{\lambda}-1}}{\lambda} ((n + 1)a_{n+1} - ABzn a_n) (1 + Az)^{-2} \\ &= \frac{z^n s_n(v_\lambda(A, B, z), z)^{\frac{1}{\lambda}-1}}{\lambda} ((n + 1)a_{n+1} - ABzn a_n) (1 - 2Az + 3A^2z^2 - 4A^3z^3 + \dots) \\ &= \frac{z^n s_n(v_\lambda(A, B, z), z)^{\frac{1}{\lambda}-1}}{\lambda} \left( (n + 1)a_{n+1} + \sum_{m=1}^\infty (m + 1)(n + 1)a_{n+1} + mnBa_n(-A)^m z^m \right) \end{aligned}$$

Since  $A \in (-1, 0]$ , we have  $-A \geq 0$ . Therefore in view of Lemma 1, we obtain  $a_n > 0$  for all  $n \in \mathbb{N}$ . Further, from Lemma 2, we obtain  $(m + 1)(n + 1)a_{n+1} + Bmna_n > 0$  for all  $n, m \in \mathbb{N}$ . Thus

$$(n + 1)a_{n+1} + \sum_{m=0}^\infty [(m + 1)(n + 1)a_{n+1} + Bmna_n](-A)^m z^m$$

represents a series of positive Taylor’s coefficients. Since  $a_n > 0$  for all  $n \in \mathbb{N}$ , the function  $v_\lambda(A, B, z)$  has a series representation with positive Taylor coefficients. Hence,

$$|s_n(v_\lambda(A, B, z), z)| \leq s_n(v_\lambda(A, B, z), |z|)$$

holds and consequently

$$|h'(z)| \leq h'(|z|), \quad \text{for all } z \in \mathbb{D} \text{ holds.} \tag{12}$$

Since  $h(0) = 0$  and  $h(-B) = 1$ , using (12) we obtain

$$|h(z)| = \left| \int_0^z h'(t) dt \right| \leq \int_0^{-B} h' \left( \frac{-tz}{B} \right) dt \leq \int_0^{-B} h'(t) dt = 1, \quad z \in \mathbb{D}.$$

Therefore,

$$\left| \frac{(1 + Bz)(s_n(v_\lambda(A, B, z), z))^{\frac{1}{\lambda}}}{(1 + Az)} - 1 \right| < 1, \quad z \in \mathbb{D}.$$

which implies that

$$\frac{s_n(v_\lambda(A, B, z), z)}{v_\lambda(A, B, z)} \prec \frac{1}{v_\lambda(0, B, z)}.$$

Therefore,  $v_\lambda(A, B, z)$  is  $n$ -stable with respect to  $v_\lambda(0, B, z)$  for all  $n \in \mathbb{N}$ . Hence  $v_\lambda(A, B, z)$  is stable with respect to  $v_\lambda(0, B, z)$  for all  $0 < \lambda \leq 1$  and  $-1 \leq B < A \leq 0$ .  $\square$

For the proof of Corollary 2, we need the following proposition which follows the same procedure as given in [8].

PROPOSITION 1. Let  $\alpha, \beta > 0$  and  $B \in [-1, 0)$ . If  $F \prec (1 + Bz)^\alpha$  and  $G \prec (1 + Bz)^\beta$  then  $FG \prec (1 + Bz)^{\alpha+\beta}$  for  $z \in \mathbb{D}$ .

*Proof.* The function  $\log(1 + Bz)$  is convex univalent for  $z \in \mathbb{D}$  and  $B \in [-1, 0)$ . Our claim follows from

$$\begin{aligned} \frac{1}{\alpha + \beta} \log(F(z)G(z)) &= \frac{\alpha}{\alpha + \beta} \log(1 + Bu(z)) + \frac{\beta}{\alpha + \beta} \log(1 + Bv(z)) \\ &\prec \log(1 + Bz), \end{aligned}$$

where  $u, v$  are analytic functions such that  $|u(z)| \leq |z|$  and  $|v(z)| \leq |z|$  for  $z \in \mathbb{D}$ .  $\square$

Now we are ready to give proof of Corollary 2.

*Proof of Corollary 2.* For  $0 < \mu \leq \lambda \leq 1$  we have,

$$\begin{aligned} &\frac{1}{\lambda} \log \left( (1 + Bz)^\lambda s_n(v_\mu(0, B, z), z) \right) \\ &= \frac{1}{\lambda} \log \left[ (1 + Bz)^{\lambda-\mu} (1 + Bz)^\mu s_n(v_\mu(0, B, z), z) \right] \\ &= \frac{1}{\lambda} \log(1 + Bz)^{\lambda-\mu} + \frac{1}{\lambda} \log \left[ (1 + Bz)^\mu s_n(v_\mu(0, B, z), z) \right] \\ &= \frac{1}{\lambda} \log(1 + Bu(z))^{\lambda-\mu} + \frac{1}{\lambda} \log(1 + Bw(z))^\mu \\ &\prec (1 + Bz)^\lambda \end{aligned}$$

for  $|u(z)| \leq |z|$  and  $|w(z)| \leq |z|$ . Therefore,  $(1 + Bz)^\lambda s_n(v_\mu(0, B, z), z) \prec (1 + Bz)^\lambda$  holds for all  $z \in \mathbb{D}$  and  $0 < \mu \leq \lambda \leq 1$ .  $\square$

Now we prove that  $v_\lambda(A, B, z)$  is not stable with respect to itself for  $\lambda \in (0, 1]$  and  $-1 \leq B < A \leq 0$ .

*Proof of Theorem 4.* For  $-1 \leq B < A \leq 0$ , to prove that  $v_\lambda(A, B, z)$  is stable with respect to itself, we need to show that

$$\frac{s_n(v_\lambda(A, B, z), z)}{v_\lambda(A, B, z)} \prec \frac{1}{v_\lambda(A, B, z)}, \quad z \in \mathbb{D}. \tag{13}$$

Equivalently  $G(z) \prec H(z)$  where

$$G(z) := \frac{(1 + Bz)s_n(v_\lambda(A, B, z), z)^{\frac{1}{\lambda}}}{1 + Az} \quad \text{and} \quad H(z) := \frac{1 + Bz}{1 + Az}$$

Since  $G(z)$  and  $H(z)$  are analytic in  $\mathbb{D}$  for  $-1 \leq B < A \leq 0$  and  $H(z)$  is univalent in  $\mathbb{D}$ . In the point of view of the subordination, we have  $G(z) \prec H(z)$  if and only if  $G(0) = H(0)$  and  $G(\mathbb{D}) \subseteq H(\mathbb{D})$  and  $G = H \circ \omega_1$ , where  $\omega_1 \in \Omega$  analytic in  $\mathbb{D}$  satisfying  $\omega_1(0) = 0$  and  $|\omega_1(z)| < 1$ .

In view of the Schwartz Lemma, we have  $|\omega_1(z)| \leq |z|$  for  $z \in \mathbb{D}$  and  $|\omega'_1(0)| \leq 1$ . If  $G \prec H$ , it follows that  $|G'(0)| \leq |H'(0)|$  and  $G(|z| \leq r) \subseteq H(|z| \leq r)$ ,  $0 \leq r < 1$ .

Let  $\omega = H(z) = \frac{1+Bz}{1+Az}$ , then  $z = \frac{1+B\omega}{1+A\omega}$ . Therefore, the image of  $|z| \leq r$  under  $H(z)$  is  $\left| \frac{1+B\omega}{1+A\omega} \right| \leq r$  which after simplification is equivalent to  $|w - C(r,A,B)| \leq R(r,A,B)$  where

$$C(r,A,B) := \frac{r^2A - B}{B^2 - r^2A^2}$$

and

$$R(r,A,B) := \frac{r(A - B)}{B^2 - r^2A^2}.$$

To show that  $G \not\prec H$ , it is enough to show that  $G(|z| \leq r) \not\subseteq H(|z| \leq r)$ . To prove that  $G(|z| \leq r) \not\subseteq H(|z| \leq r)$ , it is enough to choose a point  $z_0$  with  $|z_0| \leq r_0$  such that  $G(z_0)$  does not lie in the disk  $|w - C(r,A,B)| \leq R(r,A,B)$  for some  $-1 \leq B < A \leq 0$ .

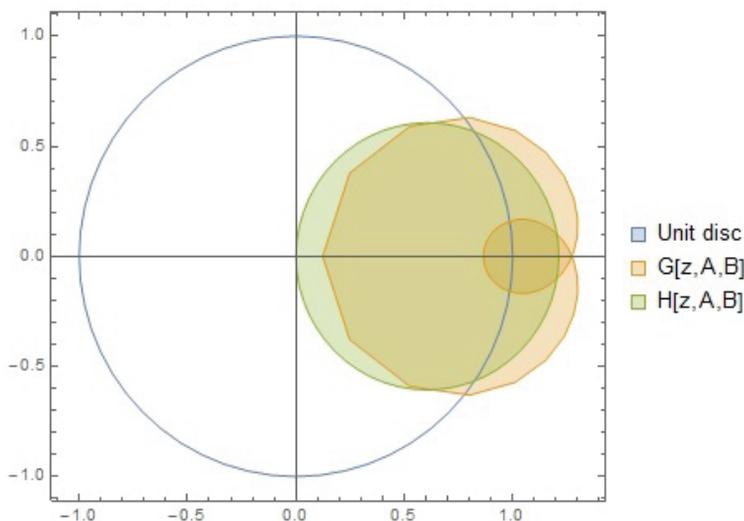


Figure 1:  $G(z_0, A, B) \not\prec H(z_0, A, B)$  for  $z_0 = 0.915282 - 0.357037i$ ,  $A = -0.679$ ,  $B = -0.97$ , and  $\lambda = 0.3$ .

Choose  $z_0 = 0.915282 - 0.357037i$ ,  $r_0 = 0.98$ ,  $A = -0.679$ ,  $B = -0.97$ ,  $\lambda = 0.3$  and  $n = 1$ . Then  $G(z_0) = 0.8697 + 0.5845i$ ,  $C(r_0, A, B) = 0.634444$  and  $R(r_0, A, B) = 0.576521$ . Clearly  $G(z_0)$  does not lie in the disk  $|w - C(r_0, A, B)| \leq R(r_0, A, B)$ . Therefore  $G \not\prec H$  i.e., (13) does not hold. The graphical illustration of these values is also given here in Figure . Hence  $v_\lambda(A, B, z)$  is not stable with respect to itself.  $\square$

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(Received June 1, 2020)

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