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SOME MULTIDIMENSIONAL OPIAL TYPE INEQUALITIES

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Abstract. In this note, we establish some multidimensional Opial type inequalities, which are generalized from the one-dimensional case. Based on calculus and some fundamental inequalities, we first present an elementary proof in Euclidean spaces. Then by using the property of the Minkowski gradient and the so called adapted frame field, we further extend the multidimensional Opial type inequalities to a Minkowski space.

1. Introduction

Due to considerable applications in ordinary, partial and difference equations and some important theoretical value, Opial-type inequalities and their variant extensions have been studied by many people for about sixty years. The classical Opial inequality [4] shows that if a function $g \geqslant 0$ is continuously differentiable on a closed interval $[0,h]$ with $g(0) = g(h) = 0$, then

$$\int_0^h |g(t)g'(t)|dt \leqslant \frac{h}{4}\int_0^h |g'(t)|^2 dt,$$

where $\frac{h}{4}$ is the best possible constant. Immediately after Opial's work, Olech [3] observed that positivity of $g(t)$ is not necessary and if $g(t)$ is absolutely continuous in $[0,h]$ and satisfies $g(0) = 0$, then it holds that

$$\int_0^h |g(t)g'(t)|dt \leqslant \frac{h}{2}\int_0^h |g'(t)|^2 dt,$$

with $\frac{h}{2}$ being the best possible constant. Among a large number of papers, Yang [7] gave a generalization in the following.

THEOREM 1.1. [7] *Let $f(x)$ be an absolutely continuous function on $[a,b]$, and $f(a) = 0$. Then for $p \geqslant 0, q \geqslant 1$,*

$$\int_a^b |f(x)|^p |f'(x)|^q dx \leqslant \frac{q}{p+q}(b-a)^p \int_a^b |f'(x)|^{p+q} dx.$$

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By using calculus and some fundamental inequalities, He [2] offered a simple proof of Theorem 1.1. On the other hand, Pachpatte [5] discussed the multidimensional case and obtained the following result.

THEOREM 1.2. [5] *Let* $Q=\{(x_1,\cdots,x_n)\in\mathbb{R}^n|a_k\leqslant x_k\leqslant b_k, 1\leqslant k\leqslant n\}$. *Suppose* f *is a* C^1 *function on* Q *and satisfies* $f|_{\partial Q}=0$. *Then for any* $p,q\geqslant 1$,

$$\int_Q |f|^p|\nabla f|^q dx\leqslant M\int_Q |\nabla f|^{p+q}dx,$$

where $M=\frac{1}{n2^p}\left[\sum_{k=1}^n (b_k-a_k)^\alpha\right]^\beta, \alpha=\frac{p(p+q)}{q}, \beta=\frac{q}{p+q}$, ∇f *is the gradient of* f, *and* $dx=dx_1\cdots dx_n$.

REMARK 1.3. Recently, Opial's type inequalities involving higher order partial derivatives, which generalizes Pachpatte's type inequality, were obtained in [8].

2. Multidimensional Opial type inequality in Euclidean spaces

By borrowing the skill from He [2], we first generalize Theorem 1.1 into the multidimensional case in Euclidean spaces as follows.

THEOREM 2.1. *Let* $Q=\{(x_1,\cdots,x_n)\in\mathbb{R}^n|a_k\leqslant x_k\leqslant b_k, 1\leqslant k\leqslant n\}$. *Suppose* f *is an absolutely continuous function on* Q *and satisfies* $f|_{x_k=a_k}=0, \forall k$. *Then for any* $p\geqslant 0, q\geqslant 1$,

$$\int_Q |f|^p|\nabla f|^q dx\leqslant M\int_Q |\nabla f|^{p+q}dx, \tag{2.1}$$

where $M=\frac{q}{p+q}\min\{(b_k-a_k)^p, 1\leqslant k\leqslant n\}$, ∇f *is the gradient of* f, *and* $dx=dx_1\cdots dx_n$.

REMARK 2.2. Theorem 2.1 is different from Theorem 1.2 since f vanishes on the boundary $\{x_k=a_k\}\cap\partial Q$ for $1\leqslant k\leqslant n$ and $p\geqslant 0$, while in Theorem 1.2 f vanishes on the whole boundary and $p\geqslant 1$, and thus our condition is a bit weakly than Pachpatte's. In Theorem 2.1 we give a new control constant M for Opial's type inequalities. In addition, by similar arguments as in [8], we can also derive the corresponding result for higher order partial derivatives.

Proof. To prove Theorem 2.1, we follow the arguments in [2] with some necessary modifications. Since Inequality (2.1) obviously holds if $p=0$, we might as well consider $p>0$ in the following proof. With no loss of generality, we can assume $a_k=0$ for $1\leqslant k\leqslant n$. Let

$$\overline{Q}=\{(x_2,\cdots,x_n)\in\mathbb{R}^{n-1}|0\leqslant x_k\leqslant b_k, 2\leqslant k\leqslant n\}.$$

Set

$$F(t)=\frac{qt^p}{p+q}\int_0^t ds\int_{\overline{Q}}|\nabla f(s,\bar{x})|^{p+q}d\bar{x}-\int_0^t ds\int_{\overline{Q}}|f(s,\bar{x})|^p|\nabla f(s,\bar{x})|^q d\bar{x},$$

where $d\bar{x}=dx_2\cdots dx_n$. Then for almost every $t\geqslant 0$, we have

$$\begin{aligned}F'(t)=&\frac{pqt^{p-1}}{p+q}\int_0^t ds\int_{\overline{Q}}|\nabla f(s,\bar{x})|^{p+q}d\bar{x}+\frac{qt^p}{p+q}\int_{\overline{Q}}|\nabla f(t,\bar{x})|^{p+q}d\bar{x}\\&-\int_{\overline{Q}}|f(t,\bar{x})|^p|\nabla f(t,\bar{x})|^q d\bar{x}.\end{aligned}\tag{2.2}$$

Using Hölder inequality and noting $f(0,\bar{x})=0$, we obtain

$$\begin{aligned}|f(t,\bar{x})|&=\left|\int_0^t\frac{\partial}{\partial s}f(s,\bar{x})ds\right|\\&\leqslant\int_0^t\left|\frac{\partial}{\partial s}f(s,\bar{x})\right|ds\\&\leqslant\int_0^t|\nabla f(s,\bar{x})|ds\\&\leqslant\left(\int_0^t ds\right)^{\frac{1}{r}}\left(\int_0^t|\nabla f(s,\bar{x})|^{p+q}ds\right)^{\frac{1}{p+q}},\end{aligned}$$

where $\frac{1}{r}+\frac{1}{p+q}=1$. This gives

$$\frac{|f(t,\bar{x})|^{p+q}}{t^{p+q-1}}\leqslant\int_0^t|\nabla f(s,\bar{x})|^{p+q}ds.$$

Substituting it into (2.2) yields

$$\begin{aligned}F'(t)\geqslant&\frac{pq}{(p+q)t^q}\int_{\overline{Q}}|f(t,\bar{x})|^{p+q}d\bar{x}+\frac{qt^p}{p+q}\int_{\overline{Q}}|\nabla f(t,\bar{x})|^{p+q}d\bar{x}\\&-\int_{\overline{Q}}|f(t,\bar{x})|^p|\nabla f(t,\bar{x})|^q d\bar{x}\\\geqslant&\frac{1}{(p+q)t^q}\left[pq\int_{\overline{Q}}|f(t,\bar{x})|^{p+q}d\bar{x}+qt^{p+q}\int_{\overline{Q}}|\nabla f(t,\bar{x})|^{p+q}d\bar{x}\right.\\&\left.-(p+q)t^q\int_{\overline{Q}}|f(t,\bar{x})|^p|\nabla f(t,\bar{x})|^q d\bar{x}\right]\\\geqslant&\frac{1}{(p+q)t^q}\left[p\int_{\overline{Q}}|f(t,\bar{x})|^{p+q}d\bar{x}+qt^{p+q}\int_{\overline{Q}}|\nabla f(t,\bar{x})|^{p+q}d\bar{x}\right.\\&\left.-(p+q)t^q\int_{\overline{Q}}|f(t,\bar{x})|^p|\nabla f(t,\bar{x})|^q d\bar{x}\right]\\=&\frac{1}{(p+q)t^q}\int_{\overline{Q}}\left[p|f(t,\bar{x})|^{p+q}+qt^{p+q}|\nabla f(t,\bar{x})|^{p+q}\right.\\&\left.-(p+q)t^q|f(t,\bar{x})|^p|\nabla f(t,\bar{x})|^q\right]d\bar{x}.\end{aligned}\tag{2.3}$$

Write

$$A=|f(t,\bar{x})|^{p+q},\quad B=(t|\nabla f(t,\bar{x})|)^{p+q},\quad \alpha=\frac{p}{p+q},\quad \beta=\frac{q}{p+q}.$$

Then by using Young inequality we derive

$$A^{\alpha}B^{\beta}\leqslant\frac{(A^{\alpha})^{\frac{1}{\alpha}}}{\frac{1}{\alpha}}+\frac{(B^{\beta})^{\frac{1}{\beta}}}{\frac{1}{\beta}}=\alpha A+\beta B,$$

which implies

$$t^{q}|f(t,\bar{x})|^{p}|\nabla f(t,\bar{x})|^{q}\leqslant\frac{p}{p+q}|f(t,\bar{x})|^{p+q}+\frac{q}{p+q}t^{p+q}|\nabla f(t,\bar{x})|^{p+q}.$$

Therefore, it follows from (2.3) that $F'(t)\geqslant 0$. Thus we have

$$\int_{Q}|f|^{p}|\nabla f|^{q}dx\leqslant\frac{q}{p+q}(b_1-a_1)^{p}\int_{Q}|\nabla f|^{p+q}dx. \tag{2.4}$$

By similar arguments as above, we can deduce (2.4) for any $2\leqslant k\leqslant n$:

$$\int_{Q}|f|^{p}|\nabla f|^{q}dx\leqslant\frac{q}{p+q}(b_k-a_k)^{p}\int_{Q}|\nabla f|^{p+q}dx.$$

This ends the proof. □

Let f_i $(i=1,\cdots,m)$ be the functions as in Theorem 2.1. Then using the elementary inequality, we have

$$\begin{aligned}
&\int_{Q}\prod_{i=1}^{m}|f_i|^{p_i}|\nabla f_i|^{q_i}dx\\
=&\int_{Q}\left[\left\{\prod_{i=1}^{m}|f_i|^{p_i}|\nabla f_i|^{q_i}\right\}^{\frac{1}{m}}\right]^{m}dx\\
\leqslant&\int_{Q}\left[\frac{1}{m}\sum_{i=1}^{m}|f_i|^{p_i}|\nabla f_i|^{q_i}\right]^{m}dx\\
\leqslant&\left(\frac{1}{m}\right)^{m}\int_{Q}m^{m-1}\left[\sum_{i=1}^{m}|f_i|^{mp_i}|\nabla f_i|^{mq_i}\right]dx\\
\leqslant&\frac{1}{m}\sum_{i=1}^{m}M_i\int_{Q}|\nabla f_i|^{m(p_i+q_i)}dx.
\end{aligned}$$

Therefore, we obtain the following.

THEOREM 2.3. *Let $Q=\{(x_1,\cdots,x_n)\in\mathbb{R}^n|a_k\leqslant x_k\leqslant b_k,1\leqslant k\leqslant n\}$. Suppose f_i $(i=1,\cdots,m)$ are the absolutely continuous functions on Q and satisfies $f_i|_{x_k=a_k}=0,\forall i,k$. Then for any $p_i\geqslant 0,q_i\geqslant 1$,*

$$\int_{Q}\prod_{i=1}^{m}|f_i|^{p_i}|\nabla f_i|^{q_i}dx\leqslant\frac{1}{m}\sum_{i=1}^{m}M_i\int_{Q}|\nabla f_i|^{m(p_i+q_i)}dx,$$

where $M_i = \frac{q_i}{p_i+q_i}\min\{(b_k-a_k)^{mp_i}, 1 \leqslant k \leqslant n\}$, ∇f_i *is the gradient of* f_i, *and* $dx = dx_1 \cdots dx_n$.

3. Multidimensional Opial type inequality in Minkowski spaces

In this section, we will further extend the multidimensional Opial type inequalities into Minkowski spaces, which are more general than Euclidean spaces.

In what follows, we demonstrate the so called Minkowski spaces. For more details, we refer to [1]. Recall that in Euclidean space $(\mathbb{R}^n, \|\cdot\|)$, the norm of a vector $y = (y^1, \cdots, y^n)$ is defined as

$$\|y\| := \sqrt{\sum_{i=1}^{n}(y^i)^2}.$$

Thus, by computing the Hessian of $\|y\|^2$, the metric on Euclidean space is given by

$$g := \sum_{i=1}^{n} \delta_{ij} dx^i dx^j,$$

where $\delta_{ii} = 1$ for $1 \leqslant i \leqslant n$ and $\delta_{ij} = 0$ if $i \neq j$.

Now we equip with a norm $F(y)$ on $\mathbb{R}^n$ such that the metric is

$$g := \sum_{i=1}^{n} g_{ij}(y) dx^i dx^j, \quad g_{ij} := \frac{1}{2}\frac{\partial^2(F^2)}{\partial y^i \partial y^j}.$$

Here the functions $g_{ij}(y)$ are smooth in $\mathbb{R}^n \setminus 0$, and $g_{ij}(cy) = g_{ij}(y)$ for any $c > 0$. We also require the matrix (g_{ij}) is positive definite. Then $(\mathbb{R}^n, F)$ is called a Minkowski space. Obviously, a Euclidean space is a special Minkowski space.

In a Minkowski space, the gradient operator is a nonlinear operator which is defined by the Legendre transformation. It is more complicated than that in the Euclidean situation. For simplicity, we only consider it in a Randers-Minkowski space $(\mathbb{R}^n, F)$ with $F = \alpha + \beta$, where α is a Euclidean metric and β is a 1-form:

$$\alpha(y) = \sqrt{\sum_{i,j=1}^{n} a_{ij} y^i y^j}, \quad \beta(y) = \sum_{i=1}^{n} b_i y^i.$$

For a smooth function $f(x)$, define

$$|df| := \sqrt{\sum_{i,j=1}^{n} a^{ij}\frac{\partial f}{\partial x^i}\frac{\partial f}{\partial x^j}}, \quad \langle \beta, df\rangle := \sum_{i,j=1}^{n} a^{ij} b_i \frac{\partial f}{\partial x^j}, \quad \|\beta\| := \sqrt{\sum_{i,j=1}^{n} a^{ij} b_i b_j},$$

where $(a^{ij}) = (a_{ij})^{-1}$. Then the gradient of f is given by (see [6])

$$\begin{aligned}\nabla f = &\frac{\sqrt{(1-\|\beta\|^2)|df|^2 + \langle \beta, df\rangle^2} - \langle \beta, df\rangle}{(1-\|\beta\|^2)^2} \\ &\times \left\{ \frac{(1-\|\beta\|^2)a^{ij}\frac{\partial f}{\partial x^j} + \langle \beta, df\rangle a^{ij} b_j}{\sqrt{(1-\|\beta\|^2)|df|^2 + \langle \beta, df\rangle^2}} - a^{ij} b_j \right\} \frac{\partial}{\partial x^i}.\end{aligned}$$

One can find that if f is a smooth function, then the gradient ∇f is smooth in $\{x \in \mathbb{R}^n | df(x) \neq 0\}$ and only continuous at x where $df(x) = 0$. We remark that this property is true in any Minkowski space. In general, ∇f can be written as

$$\nabla f(x) = \begin{cases} g^{ij}(\nabla f)\frac{\partial f}{\partial x^j}\frac{\partial}{\partial x^i}, & df(x) \neq 0; \\ 0, & df(x) = 0, \end{cases}$$

but is not necessarily written explicitly, where $(g^{ij}) = (g_{ij})^{-1}$.

In a minkowski space $(\mathbb{R}^n, F)$, the distance from the point x to point y is $F(y-x)$. But in general, $F(y-x) \neq F(x-y)$ unless F is reversible. Notice that ∇f points into the direction in which f increases the most. That is to say,

$$F(\nabla f(x_0)) = \limsup_{x \to x_0} \frac{f(x) - f(x_0)}{F(x - x_0)}. \tag{3.1}$$

Now we choose a basis $\{e_1, \cdots, e_n\}$ in Minkowski space $\mathbb{R}^n$ such that $F(e_i) = 1$ for $1 \leqslant i \leqslant n$. Here we cannot define the angle between e_i and e_j by the Minkowski metric. We call $\{e_1, \cdots, e_n\}$ an *adapted frame field*. In a Minkowski space with an adapted frame field, coordinate component x^i is the distance function from origin along the direction e_i. Note that $e_i = \frac{\partial}{\partial x^i}$. Then formula (3.1) means that

$$\left| \frac{\partial f}{\partial x^i} \right| \leqslant F(\nabla f), \forall i.$$

With this inequality in hand, we can derive Opial's type inequality by following the discussions above step by step. Namely, we can establish Opial type inequality in the Minkowski case as follows.

THEOREM 3.1. *Let $(\mathbb{R}^n, F)$ be a Minkowski space with an adapted frame field, and $Q = \{(x_1, \cdots, x_n) \in \mathbb{R}^n | a_k \leqslant x_k \leqslant b_k, 1 \leqslant k \leqslant n\}$. Suppose f is an absolutely continuous function on Q and satisfies $f|_{x_k = a_k} = 0, \forall k$. Then for any $p \geqslant 0, q \geqslant 1$,*

$$\int_Q |f|^p F(\nabla f)^q dx \leqslant M \int_Q F(\nabla f)^{p+q} dx,$$

where $M = \frac{q}{p+q} \min\{(b_k - a_k)^p, 1 \leqslant k \leqslant n\}$, ∇f is the Minkowski gradient of f, and $dx = dx_1 \cdots dx_n$.

THEOREM 3.2. *Let $(\mathbb{R}^n, F)$ be a Minkowski space with an adapted frame field, and $Q = \{(x_1, \cdots, x_n) \in \mathbb{R}^n | a_k \leqslant x_k \leqslant b_k, 1 \leqslant k \leqslant n\}$. Suppose f_i $(i = 1, \cdots, m)$ are the absolutely continuous functions on Q and satisfies $f_i|_{x_k = a_k} = 0, \forall i, k$. Then for any $p_i \geqslant 0, q_i \geqslant 1$,*

$$\int_Q \prod_{i=1}^m |f_i|^{p_i} |\nabla f_i|^{q_i} dx \leqslant \frac{1}{m} \sum_{i=1}^m M_i \int_Q |\nabla f_i|^{m(p_i + q_i)} dx,$$

where $M_i = \frac{q_i}{p_i + q_i} \min\{(b_k - a_k)^{mp_i}, 1 \leqslant k \leqslant n\}$, ∇f_i is the Minkowski gradient of f_i, and $dx = dx_1 \cdots dx_n$.

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