

AN EXTENDED HARDY–HILBERT’S INEQUALITY WITH PARAMETERS AND APPLICATIONS

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Abstract. By using the weight coefficients, the Euler-Maclaurin summation formula and Abel’s summation by parts formula, an extended Hardy-Hilbert’s inequality with the power function as the interval variables and a new Hilbert-type inequality with the partial sums are given. As applications, the equivalent conditions of the best possible constant factor in a particular inequality related to a few parameters and some particular cases are considered.

1. Introduction

Assuming that $p > 1, \frac{1}{p} + \frac{1}{q} = 1, a_m, b_n \geq 0, 0 < \sum_{m=1}^{\infty} a_m^p < \infty$ and $0 < \sum_{n=1}^{\infty} b_n^q < \infty$, we have the following well known Hardy-Hilbert’s inequality with the best possible constant factor $\frac{\pi}{\sin(\pi/p)}$ (cf. [1], Theorem 315):

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)} \left(\sum_{m=1}^{\infty} a_m^p \right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} b_n^q \right)^{\frac{1}{q}}. \quad (1)$$

By introducing parameters $\lambda_i \in (0, 2] (i = 1, 2), \lambda_1 + \lambda_2 = \lambda \in (0, 4]$, an extension of (1) was provided by [2] in 2006 as follows:

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{(m+n)^{\lambda}} < B(\lambda_1, \lambda_2) \left[\sum_{m=1}^{\infty} m^{p(1-\lambda_1)-1} a_m^p \right]^{\frac{1}{p}} \left[\sum_{n=1}^{\infty} n^{q(1-\lambda_2)-1} b_n^q \right]^{\frac{1}{q}}, \quad (2)$$

where, the constant factor $B(\lambda_1, \lambda_2)$ is the best possible, and

$$B(u, v) := \int_0^{\infty} \frac{t^{u-1}}{(1+t)^{u+v}} dt \quad (u, v > 0)$$

is the beta function. For $\lambda = 1, \lambda_1 = \frac{1}{q}, \lambda_2 = \frac{1}{p}$, inequality (2) reduces to (1); for $p = q = 2, \lambda_1 = \lambda_2 = \frac{\lambda}{2}$, (2) reduces to Yang’s inequality in [3].

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Recently, applying (2), Adiyasuren et. al. [4] gave a new inequality with the kernel as (2) involving partial sums.

If $f(x), g(y) \geq 0, 0 < \int_0^\infty f^p(x)dx < \infty$ and $0 < \int_0^\infty g^q(y)dy < \infty$, then we still have the following Hardy-Hilbert's integral inequality (cf. [1], Theorem 316):

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dx dy < \frac{\pi}{\sin(\frac{\pi}{p})} \left(\int_0^\infty f^p(x)dx \right)^{\frac{1}{p}} \left(\int_0^\infty g^q(y)dy \right)^{\frac{1}{q}}, \tag{3}$$

where, the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible. Inequalities (1) and (3) with their extensions are playing an important role in analysis and its applications (cf. [5]–[15]).

In 1934, a half-discrete Hilbert-type inequality was given as follows (cf. [1], Theorem 351): If $K(t)$ ($t > 0$) is a decreasing function, $p > 1, \frac{1}{p} + \frac{1}{q} = 1, 0 < \phi(s) = \int_0^\infty K(t)t^{s-1}dt < \infty, a_n \geq 0, 0 < \sum_{n=1}^\infty a_n^p < \infty$, then

$$\int_0^\infty x^{p-2} \left(\sum_{n=1}^\infty K(nx)a_n \right)^p dx < \phi^p\left(\frac{1}{q}\right) \sum_{n=1}^\infty a_n^p. \tag{4}$$

Some new extensions of (4) were provided by [16]–[20].

In 2016, by means of the technique of real analysis, Hong et al. [21] considered some equivalent statements of the extensions of (1) with the best possible constant factor related to a few parameters. The other similar works about the extension of (3) were given by [22]–[39].

In this paper, following the way of [2, 4, 21], by the use of the weight coefficients, the idea of introduced parameters, the Euler-Maclaurin summation formula and Abel's summation by parts formula, an extended Hardy-Hilbert's inequality with the power function as the internal variables and a new Hilbert-type inequality with the partial sums are given in Lemma 3 and Theorem 1. As application, the equivalent conditions of the best possible constant factor in a particular inequality related to a few parameters and some particular cases are considered in Theorem 2 and Remark 3. The lemmas and theorems provide an extensive account of this type of inequalities.

2. Some Lemmas

In what follows, we suppose that $p > 1, \frac{1}{p} + \frac{1}{q} = 1, \lambda \in (0, 4], \alpha, \beta \in (0, 1], \lambda_1 \in (0, \frac{2}{\alpha} - 1] \cap (0, \lambda + 1), \lambda_2 \in (0, \frac{2}{\beta} - 1] \cap (0, \lambda + 1),$

$$k_\lambda(\lambda_i) := B(\lambda_i, \lambda - \lambda_i) \ (i = 1, 2).$$

For $a_m, b_n \geq 0$, we define the partial sums $A_m := \sum_{i=1}^m a_i$ and $B_n := \sum_{k=1}^n b_k$ ($m, n \in \mathbf{N} = \{1, 2, \dots\}$), such that $A_m = o(e^{tm^\alpha}), B_n = o(e^{tn^\beta})$ ($t > 0$), with

$$\begin{aligned} 0 < \sum_{m=1}^\infty m^{p[1-\alpha(1+\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1} A_m^p < \infty, \text{ and} \\ 0 < \sum_{n=1}^\infty n^{q[1-\beta(1+\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})]-1} B_n^q < \infty \end{aligned} \tag{5}$$

LEMMA 1. (cf. [5], (2.2.3)) (i) If $(-1)^i \frac{d^i}{dt^i} g(t) > 0, t \in [m, \infty)$ ($m \in \mathbf{N}$) with $g^{(i)}(\infty) = 0$ ($i = 0, 1, 2, 3$), $P_i(t), B_i$ ($i \in \mathbf{N}$) are the Bernoulli function and the Bernoulli number of i -order; then

$$\int_m^\infty P_{2q-1}(t)g(t)dt = -\varepsilon_q \frac{B_{2q}}{2q} g(m) \quad (0 < \varepsilon_q < 1; q = 1, 2, \dots) \tag{6}$$

In particular, for $q = 1$, in view of $B_2 = \frac{1}{6}$, we have

$$-\frac{1}{12}g(m) < \int_m^\infty P_1(t)g(t)dt < 0; \tag{7}$$

for $q = 2$, in view of $B_4 = -\frac{1}{30}$, we have

$$0 < \int_m^\infty P_3(t)g(t)dt < \frac{1}{120}g(m). \tag{8}$$

(ii) (cf. [5], (2.3.2)) If $f(t) (> 0) \in C^3[m, \infty)$, $f^{(i)}(\infty) = 0$ ($i = 0, 1, 2, 3$), then we have the following Euler-Maclaurin summation formula:

$$\sum_{k=m}^\infty f(k) = \int_m^\infty f(t)dt + \frac{1}{2}f(m) + \int_m^\infty P_1(t)f'(t)dt, \tag{9}$$

$$\int_m^\infty P_1(t)f'(t)dt = -\frac{1}{12}f'(m) + \frac{1}{6} \int_m^\infty P_3(t)f'''(t)dt. \tag{10}$$

LEMMA 2. For $s \in (0, 6]$, $s_2 \in (0, \frac{2}{\beta}] \cap (0, s)$, $k_s(s_2) = B(s_2, s - s_2)$, define the following weight coefficient:

$$\bar{\omega}_\alpha(s_2, m) := m^{\alpha(s-s_2)} \sum_{n=1}^\infty \frac{\beta n^{\beta s_2 - 1}}{(m^\alpha + n^\beta)^s} \quad (m \in \mathbf{N}). \tag{11}$$

We have the following inequalities:

$$0 < k_s(s_2) \left(1 - O\left(\frac{1}{m^{\alpha s_2}}\right) \right) < \bar{\omega}_\alpha(s_2, m) < k_s(s_2) \quad (m \in \mathbf{N}), \tag{12}$$

where, $O\left(\frac{1}{m^{\alpha s_2}}\right) := \frac{1}{k_s(s_2)} \int_0^{m^\alpha} \frac{u^{s_2-1}}{(1+u)^s} du > 0$.

Proof. For fixed $m \in \mathbf{N}$, we define the function $g(m, t)$ as follows:

$$g(m, t) := \frac{\beta t^{\beta s_2 - 1}}{(m^\alpha + t^\beta)^s} \quad (t > 0).$$

In view of (9), for

$$h(m) := \int_0^1 g(m, t) - \frac{1}{2}g(m, 1) - \int_1^\infty P_1(t)g'(m, t)dt,$$

we have

$$\begin{aligned} \sum_{n=1}^{\infty} g(m, n) &= \int_1^{\infty} g(m, t) dt + \frac{1}{2}g(m, 1) + \int_1^{\infty} P_1(t)g'(m, t) dt \\ &= \int_0^{\infty} g(m, t) dt - h(m). \end{aligned} \tag{13}$$

In the following, we show that $h(m) > 0$.

We obtain $-\frac{1}{2}g(m, 1) = -\frac{\beta}{2(m^\alpha+1)^s}$. Integration by parts, we find

$$\begin{aligned} \int_0^1 g(m, t) dt &= \beta \int_0^1 \frac{t^{\beta s_2 - 1}}{(m^\alpha + t^\beta)^s} dt \stackrel{u=t^\beta}{=} \int_0^1 \frac{u^{s_2 - 1}}{(m^\alpha + u)^s} du \\ &= \frac{1}{s_2} \int_0^1 \frac{du^{s_2}}{(m^\alpha + u)^s} = \frac{u^{s_2}}{s_2(m^\alpha + u)^s} \Big|_0^1 + \frac{s}{s_2} \int_0^1 \frac{u^{s_2}}{(m^\alpha + u)^{s+1}} du \\ &= \frac{1}{s_2} \frac{1}{(m^\alpha + 1)^s} + \frac{s}{s_2(s_2 + 1)} \int_0^1 \frac{du^{s_2+1}}{(m^\alpha + u)^{s+1}} \\ &> \frac{1}{s_2(m^\alpha + 1)^s} + \frac{s}{s_2(s_2 + 1)} \left[\frac{u^{s_2+1}}{(m^\alpha + u)^{s+1}} \right]_0^1 \\ &\quad + \frac{s(s+1)}{s_2(s_2 + 1)(m^\alpha + 1)^{s+2}} \int_0^1 u^{s_2+1} du \\ &= \frac{1}{s_2(m^\alpha + 1)^s} + \frac{s}{s_2(s_2 + 1)(m^\alpha + 1)^{s+1}} \\ &\quad + \frac{s(s+1)}{s_2(s_2 + 1)(s_2 + 2)(m^\alpha + 1)^{s+2}}. \end{aligned}$$

We obtain

$$\begin{aligned} -g'(m, t) &= -\frac{\beta(\beta s_2 - 1)t^{\beta s_2 - 2}}{(m^\alpha + t^\beta)^s} + \frac{\beta^2 s t^{\beta + \beta s_2 - 2}}{(m^\alpha + t^\beta)^{s+1}} \\ &= -\frac{\beta(\beta s_2 - 1)t^{\beta s_2 - 2}}{(m^\alpha + t^\beta)^s} + \frac{\beta^2 s(m^\alpha + t^\beta - m^\alpha)t^{\beta s_2 - 2}}{(m^\alpha + t^\beta)^{s+1}} \\ &= \frac{\beta(\beta s - \beta s_2 + 1)t^{\beta s_2 - 2}}{(m^\alpha + t^\beta)^s} - \frac{\beta^2 m^\alpha s t^{\beta s_2 - 2}}{(m^\alpha + t^\beta)^{s+1}}, \end{aligned}$$

and for $0 < s_2 \leq \frac{2}{\beta}, 0 < \beta \leq 1, s_2 < s \leq 6$, it follows that for $i = 0, 1, 2, 3$,

$$(-1)^i \frac{d^i}{dt^i} \left[\frac{t^{\beta s_2 - 1}}{(m^\alpha + t^\beta)^s} \right] > 0, \quad (-1)^i \frac{d^i}{dt^i} \left[\frac{t^{\beta s_2 - 1}}{(m^\alpha + t^\beta)^{s+1}} \right] > 0.$$

By (7), (8), (9) and (10), we obtain

$$\beta(\beta s - \beta s_2 + 1) \int_1^{\infty} P_1(t) \frac{t^{\beta s_2 - 2} dt}{(m^\alpha + t^\beta)^s} > -\frac{\beta(\beta s - \beta s_2 + 1)}{12(m^\alpha + 1)^s},$$

$$\begin{aligned}
 & -\beta^2 m^\alpha s \int_1^\infty P_1(t) \frac{t^{\beta s_2 - 2}}{(m^\alpha + t^\beta)^{s+1}} dt \\
 &= \frac{\beta^2 m^\alpha s}{12(m^\alpha + 1)^{s+1}} - \frac{\beta^2 m^\alpha s}{6} \int_1^\infty P_3(t) \left[\frac{t^{\beta s_2 - 2}}{(m^\alpha + t^\beta)^{s+1}} \right]'' dt \\
 &> \frac{\beta^2 m^\alpha s}{12(m^\alpha + 1)^{s+1}} - \frac{\beta^2 m^\alpha s}{720} \left[\frac{t^{\beta s_2 - 2}}{(m^\alpha + t^\beta)^{s+1}} \right]''_{t=1} \\
 &> \frac{\beta^2(m^\alpha + 1 - 1)s}{12(m^\alpha + 1)^{s+1}} - \frac{\beta^2(m^\alpha + 1)s}{720} \left[\frac{(s+1)(s+2)\beta^2}{(m^\alpha + 1)^{s+3}} \right. \\
 &\quad \left. + \frac{\beta(s+1)(5 - \beta - 2\beta s_2)}{(m^\alpha + 1)^{s+2}} + \frac{(2 - \beta s_2)(3 - \beta s_2)}{(m^\alpha + 1)^{s+1}} \right] \\
 &= \frac{\beta^2 s}{12(m^\alpha + 1)^s} - \frac{\beta^2 s}{12(m^\alpha + 1)^{s+1}} - \frac{\beta^2 s}{720} \left[\frac{(s+1)(s+2)\beta^2}{(m^\alpha + 1)^{s+2}} \right. \\
 &\quad \left. + \frac{\beta(s+1)(5 - \beta - 2\beta s_2)}{(m^\alpha + 1)^{s+1}} + \frac{(2 - \beta s_2)(3 - \beta s_2)}{(m^\alpha + 1)^s} \right],
 \end{aligned}$$

and then we have

$$h(m) > \frac{h_1}{(m^\alpha + 1)^s} + \frac{h_2}{(m^\alpha + 1)^{s+1}} + \frac{s(s+1)h_3}{(m^\alpha + 1)^{s+2}},$$

where,

$$\begin{aligned}
 h_1 &:= \frac{1}{s_2} - \frac{\beta}{2} - \frac{\beta - \beta^2 s_2}{12} - \frac{\beta^2 s(2 - \beta s_2)(3 - \beta s_2)}{720}, \\
 h_2 &:= \frac{1}{s_2(s_2 + 1)} - \frac{\beta^2}{12} - \frac{\beta^3(s+1)(5 - \beta - 2\beta s_2)}{720},
 \end{aligned}$$

and $h_3 := \frac{1}{s_2(s_2+1)(s_2+2)} - \frac{\beta^4(s+2)}{720}$.

Setting a function $g(y)(y \in (0, \frac{2}{\beta}])$ as follows:

$$g(y) := 720 - (420\beta + 6s\beta^2)y + (60\beta^2 + 5s\beta^3)y^2 - s\beta^4y^3.$$

We obtain for $\beta \in (0, 1], s \in (0, 6]$ that

$$\begin{aligned}
 g'(y) &= -(420\beta + 6s\beta^2) + 2(60\beta^2 + 5s\beta^3)y - 3s\beta^4y^2 \\
 &< -420\beta - 6s\beta^2 + (60\beta^2 + 5s\beta^3) \frac{4}{\beta} = 14s\beta^2 - 180\beta < 0,
 \end{aligned}$$

and then it follows that

$$h_1 = \frac{g(s_2)}{720s_2} \geq \frac{g(2/\beta^2)}{720s_2} = \frac{1}{6s_2} > 0.$$

We also find that for $s_2 \in (0, \frac{2}{\beta}]$ ($0 < \beta \leq 1, 0 < s \leq 6$),

$$h_2 > \frac{\beta^2}{6} - \frac{\beta^2}{12} - \frac{5(s+1)\beta^2}{720} = \left(\frac{1}{12} - \frac{7}{140}\right)\beta^2 > 0,$$

and $h_3 \geq [\frac{1}{24} - \frac{\beta(s+2)}{720}]\beta^3 > 0$.

Hence, we show that $h(m) > 0$ ($m \in \mathbf{N}$).

On the other-hand, by (9), for

$$H(m) := \frac{1}{2}g(m, 1) + \int_1^\infty P_1(t)g'(m, t)dt,$$

we have

$$\begin{aligned} \sum_{n=1}^\infty g(m, n) &= \int_1^\infty g(m, t)dt + \frac{1}{2}g(m, 1) + \int_1^\infty P_1(t)g'(m, t)dt \\ &= \int_1^\infty g(m, t)dt + H(m). \end{aligned} \tag{14}$$

In the following we show that $H(m) > 0$ ($m \in \mathbf{N}$).

In the same way, we obtain

$$g'(m, t) = -\frac{\beta(\beta s - \beta s_2 + 1)t^{\beta s_2 - 2}}{(m^\alpha + t^\beta)^s} + \frac{\beta^2 m^\alpha s t^{\beta s_2 - 2}}{(m^\alpha + t^\beta)^{s+1}},$$

$$-\beta(\beta s - \beta s_2 + 1) \int_1^\infty P_1(t) \frac{t^{\beta s_2 - 2}}{(m^\alpha + t^\beta)^s} dt > 0,$$

$$\beta^2 m^\alpha s \int_1^\infty P_1(t) \frac{t^{\beta s_2 - 2} dt}{(m^\alpha + t^\beta)^{s+1}} > \frac{-\beta^2 m^\alpha s}{12(m^\alpha + 1)^{s+1}} > \frac{-\beta^2 s}{12(m^\alpha + 1)^s}.$$

Hence, we have

$$\begin{aligned} H(m) &> \frac{\beta}{2(m^\alpha + 1)^s} - \frac{\beta^2 s}{12(m^\alpha + 1)^s} \\ &\geq \frac{\beta}{2(m^\alpha + 1)^s} - \frac{6\beta}{12(m^\alpha + 1)^s} = 0. \end{aligned}$$

Base on $h(m), H(m) > 0$ ($m \in \mathbf{N}$), by (11), (13) and (14), setting $t = m^{\alpha/\beta} u^{1/\beta}$, we obtain

$$\begin{aligned} \bar{\omega}_\alpha(s_2, m) &= m^{\alpha(s-s_2)} \sum_{m=1}^\infty g(m, n) < m^{\alpha(s-s_2)} \int_0^\infty g(m, t)dt \\ &= \beta m^{\alpha(s-s_2)} \int_0^\infty \frac{t^{\beta s_2 - 1}}{(m^\alpha + t^\beta)^s} dt = \int_0^\infty \frac{u^{s_2 - 1}}{(1 + u)^s} du \\ &= B(s_2, s - s_2) = k_s(s_2), \end{aligned}$$

$$\begin{aligned} \bar{\omega}_\alpha(s_2, m) &= m^{\alpha(s- s_2)} \sum_{m=1}^\infty g(m, n) > m^{\alpha(s- s_2)} \int_1^\infty g(m, t) dt \\ &= m^{\alpha(s- s_2)} \left(\int_0^\infty g(m, t) dt - \int_0^1 g(m, t) dt \right) \\ &= k_s(s_2) \left(1 - O\left(\frac{1}{m^{\alpha s_2}}\right) \right) > 0, \end{aligned}$$

where, $O\left(\frac{1}{m^{\alpha s_2}}\right) = \frac{1}{k_s(s_2)} \int_0^{\frac{1}{m^{\alpha s_2}}} \frac{u^{s_2-1}}{(1+u)^s} du > 0$, satisfying

$$0 < \int_0^{\frac{1}{m^{\alpha s_2}}} \frac{u^{s_2-1}}{(1+u)^s} du < \int_0^{\frac{1}{m^{\alpha s_2}}} u^{s_2-1} du = \frac{1}{s_2 m^{\alpha s_2}}.$$

Therefore, inequalities (12) follow.

The lemma is proved. \square

LEMMA 3. For $s \in (0, 6]$, $s_1 \in (0, \frac{2}{\alpha}] \cap (0, s)$, $s_2 \in (0, \frac{2}{\beta}] \cap (0, s)$, we have the following extended Hardy-Hilbert's inequality with the internal variables:

$$\begin{aligned} I &:= \sum_{n=1}^\infty \sum_{m=1}^\infty \frac{a_m b_n}{(m^\alpha + n^\beta)^s} \\ &\leq \left(\frac{1}{\beta} k_s(s_2)\right)^{\frac{1}{p}} \left(\frac{1}{\alpha} k_s(s_1)\right)^{\frac{1}{q}} \\ &\quad \times \left\{ \sum_{m=1}^\infty m^{p[1-\alpha(\frac{s-s_2}{p} + \frac{s_1}{q})]-1} a_m^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^\infty n^{q[1-\beta(\frac{s-s_1}{q} + \frac{s_2}{p})]-1} b_n^q \right\}^{\frac{1}{q}}. \end{aligned} \tag{15}$$

Proof. In the same way, we obtain the following inequalities for the next weight coefficient:

$$\begin{aligned} 0 < k_s(s_1) \left(1 - O\left(\frac{1}{n^{\beta s_1}}\right) \right) \\ < \omega_\beta(s_1, n) := n^{\beta(s- s_1)} \sum_{m=1}^\infty \frac{\alpha m^{\alpha s_1 - 1}}{(m^\alpha + n^\beta)^s} < k_s(s_1) \quad (n \in \mathbf{N}), \end{aligned} \tag{16}$$

where, $O\left(\frac{1}{n^{\beta s_1}}\right) := \frac{1}{k_s(s_1)} \int_0^{\frac{1}{n^{\beta s_1}}} \frac{u^{s_1-1}}{(1+u)^s} du > 0$.

By Holder's inequality (cf. [40]), we obtain

$$\begin{aligned} I &= \sum_{n=1}^\infty \sum_{m=1}^\infty \frac{1}{(m^\alpha + n^\beta)^s} \left[\frac{m^{\alpha(1- s_1)/q} (\beta n^{\beta-1})^{1/p} a_m}{n^{\beta(1- s_2)/p} (\alpha m^{\alpha-1})^{1/q}} \right] \\ &\quad \times \left[\frac{n^{\beta(1- s_2)/p} (\alpha m^{\alpha-1})^{1/q} b_n}{m^{\alpha(1- s_1)/q} (\beta n^{\beta-1})^{1/p}} \right] \end{aligned}$$

$$\begin{aligned} &\leq \left[\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\beta}{(m^\alpha + n^\beta)^s} \frac{m^{\alpha(1-s_1)(p-1)} n^{\beta-1}}{n^{\beta(1-s_2)} (\alpha m^{\alpha-1})^{p-1}} a_m^p \right]^{\frac{1}{p}} \\ &\quad \times \left[\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\alpha}{(m^\alpha + n^\beta)^s} \frac{n^{\beta(1-s_2)(q-1)} m^{\alpha-1}}{m^{\alpha(1-s_1)} (\beta n^{\beta-1})^{q-1}} b_n^q \right]^{\frac{1}{q}} \\ &= \left\{ \frac{1}{\beta} \sum_{m=1}^{\infty} \varpi_\alpha(s_2, m) m^{p[1-\alpha(\frac{s-s_2}{p} + \frac{s_1}{q})]-1} a_m^p \right\}^{\frac{1}{p}} \\ &\quad \times \left\{ \frac{1}{\alpha} \sum_{n=1}^{\infty} \omega_\beta(s_1, n) n^{q[1-\beta(\frac{s-s_1}{q} + \frac{s_2}{p})]-1} b_n^q \right\}^{\frac{1}{q}}. \end{aligned}$$

Then by (12) and (16), we have (15).

The lemma is proved. \square

REMARK 1. In particular, for $s = \lambda + 2 \in (2, 6], \lambda \in (0, 4]$,

$$\begin{aligned} s_1 &= \lambda_1 + 1 \in \left(0, \frac{2}{\alpha}\right], \lambda_1 \in \left(0, \frac{2}{\alpha} - 1\right] \cap (0, \lambda + 1), \\ s_2 &= \lambda_2 + 1 \in \left(0, \frac{2}{\beta}\right], \lambda_2 \in \left(0, \frac{2}{\beta} - 1\right] \cap (0, \lambda + 1) \end{aligned}$$

in (15), replacing a_m (resp. b_n) by A_m (resp. B_n), in view of (5), we still have

$$\begin{aligned} I_1 &:= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{A_m B_n}{(m^\alpha + n^\beta)^{\lambda+2}} < \left(\frac{1}{\beta} k_{\lambda+2}(\lambda_2 + 1)\right)^{\frac{1}{p}} \left(\frac{1}{\alpha} k_{\lambda+2}(\lambda_1 + 1)\right)^{\frac{1}{q}} \\ &\quad \times \left\{ \sum_{m=1}^{\infty} m^{p[1-\alpha(1+\frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q})]-1} A_m^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} n^{q[1-\beta(1+\frac{\lambda-\lambda_1}{q} + \frac{\lambda_2}{p})]-1} B_n^q \right\}^{\frac{1}{q}}. \end{aligned} \tag{17}$$

LEMMA 4. For $t > 0$, we have the following inequalities:

$$\sum_{m=1}^{\infty} e^{-tm^\alpha} a_m \leq t \sum_{m=1}^{\infty} e^{-tm^\alpha} A_m, \tag{18}$$

$$\sum_{n=1}^{\infty} e^{-tn^\beta} b_n \leq t \sum_{n=1}^{\infty} e^{-tn^\beta} B_n. \tag{19}$$

Proof. In view of $e^{-tm^\alpha} A_m = o(1)$ ($m \rightarrow \infty$), by Abel’s summation by parts formula, we find

$$\begin{aligned} \sum_{m=1}^{\infty} e^{-tm^\alpha} a_m &= \lim_{m \rightarrow \infty} e^{-tm^\alpha} A_m + \sum_{m=1}^{\infty} [e^{-tm^\alpha} - e^{-t(m+1)^\alpha}] A_m \\ &= \sum_{m=1}^{\infty} [e^{-tm^\alpha} - e^{-t(m+1)^\alpha}] A_m. \end{aligned}$$

Since $1 - e^{-t} < t$ and for $\alpha \in (0, 1]$,

$$e^{-t(m+1)^\alpha} \geq e^{-t(m^\alpha+1)} \iff e^{t[(m+1)^\alpha - m^\alpha - 1]} \leq 1 \iff (m+1)^\alpha - m^\alpha - 1 = \alpha(m + \theta_m)^{\alpha-1} - 1 \leq 0 \ (\theta_m \in (0, 1)).$$

we have

$$\begin{aligned} \sum_{m=1}^\infty e^{-tm^\alpha} a_m &\leq \sum_{m=1}^\infty [e^{-tm^\alpha} - e^{-t(m^\alpha+1)}] A_m \\ &= (1 - e^{-t}) \sum_{m=1}^\infty e^{-tm^\alpha} A_m \leq t \sum_{m=1}^\infty e^{-tm^\alpha} A_m, \end{aligned}$$

namely, (18) follows. In the same way, we obtain (19).

The lemma is proved. \square

3. Main results and applications

THEOREM 1. *We have the following inequality:*

$$\begin{aligned} H &:= \sum_{n=1}^\infty \sum_{m=1}^\infty \frac{a_m b_n}{(m^\alpha + n^\beta)^\lambda} \\ &< \frac{\Gamma(\lambda + 2)}{\Gamma(\lambda)} \left(\frac{1}{\beta} k_{\lambda+2}(\lambda_2 + 1) \right)^{\frac{1}{p}} \left(\frac{1}{\alpha} k_{\lambda+2}(\lambda_1 + 1) \right)^{\frac{1}{q}} \\ &\quad \times \left\{ \sum_{m=1}^\infty m^{p[1-\alpha(1+\frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q})-1]} A_m^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^\infty n^{q[1-\beta(1+\frac{\lambda-\lambda_1}{q} + \frac{\lambda_2}{p})-1]} B_n^q \right\}^{\frac{1}{q}}. \end{aligned} \tag{20}$$

In particular, for $\lambda_1 + \lambda_2 = \lambda$, we have

$$0 < \sum_{m=1}^\infty m^{p[1-\alpha(1+\lambda_1)]-1} A_m^p < \infty, \text{ and } 0 < \sum_{n=1}^\infty n^{q[1-\beta(1+\lambda_2)]-1} B_n^q < \infty,$$

and the following inequality:

$$\begin{aligned} H &= \sum_{n=1}^\infty \sum_{m=1}^\infty \frac{a_m b_n}{(m^\alpha + n^\beta)^\lambda} < \frac{\lambda_1 \lambda_2}{\beta^{1/p} \alpha^{1/q}} B(\lambda_1, \lambda_2) \\ &\quad \times \left\{ \sum_{m=1}^\infty m^{p[1-\alpha(1+\lambda_1)]-1} A_m^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^\infty n^{q[1-\beta(1+\lambda_2)]-1} B_n^q \right\}^{\frac{1}{q}}. \end{aligned} \tag{21}$$

Proof. In view of the fact that

$$\frac{1}{(m^\alpha + n^\beta)^\lambda} = \frac{1}{\Gamma(\lambda)} \int_0^\infty t^{\lambda-1} e^{-(m^\alpha + n^\beta)t} dt,$$

by (18) and (19), it follows that

$$\begin{aligned}
 H &= \frac{1}{\Gamma(\lambda)} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_m b_n \int_0^{\infty} t^{\lambda-1} e^{-(m^\alpha+n^\beta)t} dt \\
 &= \frac{1}{\Gamma(\lambda)} \int_0^{\infty} t^{\lambda-1} \sum_{m=1}^{\infty} e^{-m^\alpha t} a_m \sum_{n=1}^{\infty} e^{-n^\beta t} b_n dt \\
 &\leq \frac{1}{\Gamma(\lambda)} \int_0^{\infty} t^{\lambda+1} \sum_{m=1}^{\infty} e^{-m^\alpha t} A_m \sum_{n=1}^{\infty} e^{-n^\beta t} B_n dt \\
 &= \frac{1}{\Gamma(\lambda)} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_m B_n \int_0^{\infty} t^{\lambda+1} e^{-(m^\alpha+n^\beta)t} dt = \frac{\Gamma(\lambda+2)}{\Gamma(\lambda)} I_1.
 \end{aligned}$$

Then by (17), we have (20).

The theorem is proved. \square

REMARK 2. For $\alpha = \beta = 1$, $\lambda_i \in (0, 1] \cap (0, \lambda + 1)$ ($i = 1, 2$) in (20), we have

$$\begin{aligned}
 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{(m+n)^\lambda} &< \frac{\Gamma(\lambda+2)}{\Gamma(\lambda)} (k_{\lambda+2}(\lambda_2+1))^{\frac{1}{p}} (k_{\lambda+2}(\lambda_1+1))^{\frac{1}{q}} \\
 &\quad \times \left[\sum_{m=1}^{\infty} m^{-p(\frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q})-1} A_m^p \right]^{\frac{1}{p}} \left[\sum_{n=1}^{\infty} n^{-q(\frac{\lambda-\lambda_1}{q} + \frac{\lambda_2}{p})-1} B_n^q \right]^{\frac{1}{q}}. \tag{22}
 \end{aligned}$$

THEOREM 2. For $\lambda_i \in (0, 1] \cap (0, \lambda + 1)$ ($i = 1, 2$), the constant factor

$$\frac{\Gamma(\lambda+2)}{\Gamma(\lambda)} (k_{\lambda+2}(\lambda_2+1))^{\frac{1}{p}} (k_{\lambda+2}(\lambda_1+1))^{\frac{1}{q}}$$

in (20) is the best possible if and only if $\lambda_1 + \lambda_2 = \lambda \in (0, 2]$.

Proof. “ \Leftarrow ” (cf. [4]). If $\lambda_1 + \lambda_2 = \lambda \in (0, 2]$, then we find

$$\begin{aligned}
 k_{\lambda+2}(\lambda_2+1) &= k_{\lambda+2}(\lambda_1+1) = B(\lambda_1+1, \lambda_2+1) \\
 &= \frac{\Gamma(\lambda_1+1)\Gamma(\lambda_2+1)}{\Gamma(\lambda+2)} = \frac{\lambda_1 \lambda_2 \Gamma(\lambda_1)\Gamma(\lambda_2)}{\Gamma(\lambda+2)},
 \end{aligned}$$

and then (22) reduces to

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{(m+n)^\lambda} < \lambda_1 \lambda_2 B(\lambda_1, \lambda_2) \left(\sum_{m=1}^{\infty} \frac{A_m^p}{m^{p\lambda_1+1}} \right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} \frac{B_n^q}{n^{q\lambda_2+1}} \right)^{\frac{1}{q}}. \tag{23}$$

For any $0 < \varepsilon < \min\{p\lambda_1, q\lambda_2\}$, we set

$$\tilde{a}_m := m^{\lambda_1 - \frac{\varepsilon}{p} - 1}, \tilde{b}_n := n^{\lambda_2 - \frac{\varepsilon}{q} - 1} \quad (m, n \in \mathbf{N}).$$

Then it follows that for $m, n \in \mathbf{N}$,

$$\begin{aligned} \tilde{A}_m &:= \sum_{i=1}^m \tilde{a}_i = \sum_{i=1}^m i^{\lambda_1 - \frac{\varepsilon}{p} - 1} < \int_0^m t^{\lambda_1 - \frac{\varepsilon}{p} - 1} dt = \frac{m^{\lambda_1 - \frac{\varepsilon}{p}}}{\lambda_1 - \frac{\varepsilon}{p}}, \\ \tilde{B}_n &:= \sum_{k=1}^n \tilde{b}_k = \sum_{k=1}^n k^{\lambda_2 - \frac{\varepsilon}{q} - 1} < \int_0^n t^{\lambda_2 - \frac{\varepsilon}{q} - 1} dt = \frac{n^{\lambda_2 - \frac{\varepsilon}{q}}}{\lambda_2 - \frac{\varepsilon}{q}}. \end{aligned}$$

If there exists a positive constant $M \leq \lambda_1 \lambda_2 B(\lambda_1, \lambda_2)$, such that (23) is valid when replacing $\lambda_1 \lambda_2 B(\lambda_1, \lambda_2)$ by M , then in particular, by substitution of $a_m = \tilde{a}_m, b_n = \tilde{b}_n, A_m = \tilde{A}_m$ and $B_n = \tilde{B}_n$ in (23), we have

$$\tilde{I} := \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\tilde{a}_m \tilde{b}_n}{(m+n)^\lambda} < M \left(\sum_{m=1}^{\infty} m^{-p\lambda_1 - 1} \tilde{A}_m^p \right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} n^{-q\lambda_2 - 1} \tilde{B}_n^q \right)^{\frac{1}{q}}. \tag{24}$$

By (24) and the decreasingness property of series, we obtain

$$\begin{aligned} \tilde{I} &< \frac{M}{(\lambda_1 - \frac{\varepsilon}{p})(\lambda_2 - \frac{\varepsilon}{q})} \left(\sum_{m=1}^{\infty} m^{-p\lambda_1 - 1} m^{p\lambda_1 - \varepsilon} \right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} n^{-q\lambda_2 - 1} n^{q\lambda_2 - \varepsilon} \right)^{\frac{1}{q}} \\ &= \frac{M}{(\lambda_1 - \frac{\varepsilon}{p})(\lambda_2 - \frac{\varepsilon}{q})} \left(1 + \sum_{n=1}^{\infty} n^{-\varepsilon - 1} \right) \\ &< \frac{M}{(\lambda_1 - \frac{\varepsilon}{p})(\lambda_2 - \frac{\varepsilon}{q})} \left(1 + \int_0^{\infty} \frac{dy}{y^{\varepsilon + 1}} \right) = \frac{M(\varepsilon + 1)}{\varepsilon(\lambda_1 - \frac{\varepsilon}{p})(\lambda_2 - \frac{\varepsilon}{q})}. \end{aligned}$$

By (16) (for $\alpha = \beta = 1$), setting $\tilde{\lambda}_1 := \lambda_1 - \frac{\varepsilon}{p} \in (0, 1) \cap (0, \lambda)$ ($\tilde{\lambda}_2 := \lambda_2 + \frac{\varepsilon}{p} \in (0, \lambda)$), we find

$$\begin{aligned} \tilde{I} &= \sum_{n=1}^{\infty} \left[n^{\tilde{\lambda}_2} \sum_{m=1}^{\infty} \frac{m^{\tilde{\lambda}_1 - 1}}{(m+n)^\lambda} \right] n^{-\varepsilon - 1} = \sum_{n=1}^{\infty} \omega_1(\tilde{\lambda}_1, n) n^{-\varepsilon - 1} \\ &> B(\tilde{\lambda}_1, \tilde{\lambda}_2) \sum_{n=1}^{\infty} \left(1 - O\left(\frac{1}{n^{\tilde{\lambda}_1}}\right) \right) n^{-\varepsilon - 1} \\ &= B(\tilde{\lambda}_1, \tilde{\lambda}_2) \left(\sum_{n=1}^{\infty} n^{-\varepsilon - 1} - \sum_{n=1}^{\infty} O\left(\frac{1}{n^{\tilde{\lambda}_1 + \frac{\varepsilon}{q} + 1}}\right) \right) \\ &> B(\tilde{\lambda}_1, \tilde{\lambda}_2) \left(\int_1^{\infty} y^{-\varepsilon - 1} dy - O(1) \right) \\ &= \frac{1}{\varepsilon} B\left(\lambda_1 - \frac{\varepsilon}{p}, \lambda_2 + \frac{\varepsilon}{p}\right) (1 - \varepsilon O(1)). \end{aligned}$$

In view of the above results, we have

$$B\left(\lambda_1 - \frac{\varepsilon}{p}, \lambda_2 + \frac{\varepsilon}{p}\right) (1 - \varepsilon O(1)) < \varepsilon \tilde{I} < \frac{M(\varepsilon + 1)}{(\lambda_1 - \frac{\varepsilon}{p})(\lambda_2 - \frac{\varepsilon}{q})}.$$

For $\varepsilon \rightarrow 0^+$, in view of the continuity of beta function, we find $\lambda_1 \lambda_2 B(\lambda_1, \lambda_2) \leq M$. Hence, $M = \lambda_1 \lambda_2 B(\lambda_1, \lambda_2)$ is the best possible constant factor of (23).

“ \Rightarrow ”. We set $\widehat{\lambda}_1 := \frac{\lambda - \lambda_2}{p} + \frac{\lambda_1}{q}$, $\widehat{\lambda}_2 := \frac{\lambda - \lambda_1}{q} + \frac{\lambda_2}{p}$, and then reduce (22) to the following:

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{(m+n)^\lambda} < \frac{\Gamma(\lambda+2)}{\Gamma(\lambda)} (k_{\lambda+2}(\lambda_2+1))^{\frac{1}{p}} (k_{\lambda+2}(\lambda_1+1))^{\frac{1}{q}} \times \left(\sum_{m=1}^{\infty} m^{-p\widehat{\lambda}_1-1} A_m^p \right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} n^{-q\widehat{\lambda}_2-1} B_n^q \right)^{\frac{1}{q}}. \tag{25}$$

We find $\widehat{\lambda}_1 + \widehat{\lambda}_2 = \lambda$, $0 < \widehat{\lambda}_1, \widehat{\lambda}_2 < \lambda$, and $\widehat{\lambda}_1 \widehat{\lambda}_2 k_\lambda(\widehat{\lambda}_1) = \widehat{\lambda}_1 \widehat{\lambda}_2 B(\widehat{\lambda}_1, \widehat{\lambda}_2) \in \mathbf{R}_+ = (0, \infty)$. If the constant factor

$$\frac{\Gamma(\lambda+2)}{\Gamma(\lambda)} (k_{\lambda+2}(\lambda_2+1))^{\frac{1}{p}} (k_{\lambda+2}(\lambda_1+1))^{\frac{1}{q}}$$

in (23) (or (25)) is the best possible, then by (23) (for $\lambda_i = \widehat{\lambda}_i$ ($i = 1, 2$)), we have

$$\begin{aligned} & \frac{\Gamma(\lambda+2)}{\Gamma(\lambda)} (k_{\lambda+2}(\lambda_2+1))^{\frac{1}{p}} (k_{\lambda+2}(\lambda_1+1))^{\frac{1}{q}} \\ & \leq \widehat{\lambda}_1 \widehat{\lambda}_2 k_\lambda(\widehat{\lambda}_1) = \frac{\Gamma(\lambda+2)}{\Gamma(\lambda)} k_{\lambda+2}(\widehat{\lambda}_1+1), \end{aligned}$$

namely, $k_{\lambda+2}(\widehat{\lambda}_1+1) \geq (k_{\lambda+2}(\lambda_2+1))^{\frac{1}{p}} (k_{\lambda+2}(\lambda_1+1))^{\frac{1}{q}}$.

By Hölder’s inequality, we obtain

$$\begin{aligned} k_{\lambda+2}(\widehat{\lambda}_1+1) &= k_{\lambda+2} \left(\frac{\lambda - \lambda_2}{p} + \frac{\lambda_1}{q} + 1 \right) \\ &= \int_0^\infty \frac{1}{(1+u)^{\lambda+2}} u^{\frac{\lambda-\lambda_2}{p}} u^{\frac{\lambda_1}{q}} du \\ &\leq \left[\int_0^\infty \frac{u^{\lambda-\lambda_2}}{(1+u)^{\lambda+2}} du \right]^{\frac{1}{p}} \left[\int_0^\infty \frac{u^{\lambda_1}}{(1+u)^{\lambda+2}} du \right]^{\frac{1}{q}} \\ &= (k_{\lambda+2}(\lambda_2+1))^{\frac{1}{p}} (k_{\lambda+2}(\lambda_1+1))^{\frac{1}{q}}. \end{aligned} \tag{26}$$

Hence we have

$$k_{\lambda+2}(\widehat{\lambda}_1+1) = (k_{\lambda+2}(\lambda_2+1))^{\frac{1}{p}} (k_{\lambda+2}(\lambda_1+1))^{\frac{1}{q}}.$$

We observe that (26) keeps the form of equality if and only if there exist constants A and B , such that they are not both zero satisfying (cf. [40])

$$Au^{\lambda-\lambda_2} = Bu^{\lambda_1} \text{ a.e. in } \mathbf{R}_+.$$

Assuming that $A \neq 0$, we have

$$u^{\lambda-\lambda_1-\lambda_2} = B/A \text{ a.e. in } R_+,$$

and then $\lambda - \lambda_1 - \lambda_2 = 0$, namely, $\lambda_1 + \lambda_2 = \lambda (\in (0, 2])$.

The theorem is proved. \square

REMARK 3. (i) For $\lambda = 1, \lambda_1 = \frac{1}{r}, \lambda_2 = \frac{1}{s} (r > 1, \frac{1}{r} + \frac{1}{s} = 1, \text{ in (23)}, \text{ we have the following inequality with the best possible constant factor } \frac{\pi}{rs \sin(\pi/r)} :$

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{m+n} < \frac{\pi}{rs \sin(\pi/r)} \left(\sum_{m=1}^{\infty} \frac{A_m^p}{m^{\frac{p}{r}+1}} \right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} \frac{B_n^q}{n^{\frac{q}{s}+1}} \right)^{\frac{1}{q}}. \tag{27}$$

In particular, for $r = p, s = q$, we have

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{m+n} < \frac{\pi}{pq \sin(\pi/p)} \left(\sum_{m=1}^{\infty} \frac{A_m^p}{m^2} \right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} \frac{B_n^q}{n^2} \right)^{\frac{1}{q}}; \tag{28}$$

for $r = q, s = p$, we have the dual form of (28) as follows:

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{m+n} < \frac{\pi}{pq \sin(\pi/p)} \left[\sum_{m=1}^{\infty} \left(\frac{A_m}{m} \right)^p \right]^{\frac{1}{p}} \left[\sum_{n=1}^{\infty} \left(\frac{B_n}{n} \right)^q \right]^{\frac{1}{q}}; \tag{29}$$

for $p = q = 2$, both (28) and (29) reduce to

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{m+n} < \frac{\pi}{4} \left[\sum_{m=1}^{\infty} \left(\frac{A_m}{m} \right)^2 \sum_{n=1}^{\infty} \left(\frac{B_n}{n} \right)^2 \right]^{\frac{1}{2}}. \tag{30}$$

(ii) For $\lambda = 2, \lambda_1 = \lambda_2 = 1$ in (23), we have the following inequality with the best possible constant factor 1:

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{(m+n)^2} < \left(\sum_{m=1}^{\infty} \frac{A_m^p}{m^{p+1}} \right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} \frac{B_n^q}{n^{q+1}} \right)^{\frac{1}{q}}. \tag{31}$$

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