

\mathcal{L}_p -CONVERGENCE FOR WEIGHTED SUMS OF ARRAYS OF ROWWISE EXTENDED NEGATIVELY DEPENDENT RANDOM VARIABLES

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Abstract. In this paper, \mathcal{L}_p -convergence for weighted sums of arrays of rowwise extended negatively dependent (rowwise END) random variables are investigated and some sufficient conditions for convergence are established. Additionally, the relationships among the convergence rates, weights of the sums and the dominating sequence of the rowwise END arrays are in a sense revealed. The results obtained in this paper generalise some corresponding ones for independent and some dependent random variables.

1. Introduction

In this paper, we consider an array of real-valued random variables $\{X_{ni}, i \in \mathbb{I}_n, n \geq 1\}$ defined on some probability space (Ω, \mathcal{F}, P) , where $\mathbb{I}_n \subset \mathbb{Z}$ is an arbitrary subset and no matter how many (finite or infinite) elements are in it.

The concept of rowwise END array was introduced by Lita da Silva (2016) [7] as follows.

DEFINITION 1.1. An array $\{X_{ni}, i \in \mathbb{I}_n, n \geq 1\}$ of random variables is said to be rowwise upper extended negatively dependent (rowwise UEND) if for each $n \geq 1$, there exists an $M_n > 0$ such that

$$P\left(\bigcap_{i \in \mathbb{J}} \{X_{ni} > x_i\}\right) \leq M_n \prod_{i \in \mathbb{J}} P\{X_{ni} > x_i\}$$

holds for any finite subset $\mathbb{J} \subset \mathbb{I}_n$ and all $x_i \in \mathbb{R} (i \in \mathbb{J})$. An array $\{X_{ni}, i \in \mathbb{I}_n, n \geq 1\}$ of random variables is said to be rowwise lower extended negatively dependent (rowwise LEND) if for each $n \geq 1$, there exists an $M_n > 0$ such that

$$P\left(\bigcap_{i \in \mathbb{J}} \{X_{ni} \leq x_i\}\right) \leq M_n \prod_{i \in \mathbb{J}} P\{X_{ni} \leq x_i\}$$

holds for any finite subset $\mathbb{J} \subset \mathbb{I}_n$ and all $x_i \in \mathbb{R} (i \in \mathbb{J})$. An array $\{X_{ni}, i \in \mathbb{I}_n, n \geq 1\}$ of random variables is said to be rowwise extended negatively dependent (rowwise END) if it is both rowwise UEND and rowwise LEND.

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We point out that the above concept covers the concept of widely orthant dependent (WOD) sequence (see [11]) and END sequence (see [12]). Of course the concept of rowwise END arrays covers all negative dependence structures and more interestingly, it covers certain positive dependence structures and some others. To make it clear, one can refer to the following examples.

EXAMPLE 1.1. Let $\{X_{ni}, 1 \leq i \leq n, n \geq 1\}$ be a lower triangular array and the joint density of n -th row $\{X_{n1}, \dots, X_{nm}\}$ be

$$P\{X_{n1} = \dots = X_{nm} = j\} = \frac{1}{n}, \quad j = -m + 1, \dots, m - 1$$

for $n = 2m - 1$ and

$$P\{X_{n1} = \dots = X_{nm} = j\} = \frac{1}{n}, \quad j = -m, \dots, -1, 1, \dots, m$$

for $n = 2m$, where $m \in \mathbb{N}_+$. Then for any given $n \geq 1$, one can easily verify that $\text{Corr}(X_i, X_j) = 1$ for $1 \leq i, j \leq n$ and $i \neq j$. However, $\{X_{ni}, 1 \leq i \leq n, n \geq 1\}$ is a rowwise END array with every dominating sequence $\{M_n, n \geq 1\}$ satisfying $M_n \geq n^{n-1}$.

EXAMPLE 1.2. Let $\{X_{ni}, 1 \leq i \leq n, n \geq 1\}$ be a lower triangular array and the joint density of n -th row $\{X_{n1}, \dots, X_{nm}\}$ be

$$P\{X_{n1} = \dots = X_{nm} = j\} = \frac{1}{2}, \quad j = -1, 1.$$

Then for any given $n \geq 1$, one can easily verify that $\text{Corr}(X_i, X_j) = 1$ for $1 \leq i, j \leq n$ and $i \neq j$. However, $\{X_{ni}, 1 \leq i \leq n, n \geq 1\}$ is a rowwise END array with every dominating sequence $\{M_n, n \geq 1\}$ and $M_n \geq 2^{n-1}$.

After the introduction of the definition for a rowwise END array, several works have been done. Lita da Sliva (2016) [7] studied the limiting behaviour for rowwise UEND arrays, which extended the corresponding results obtained in Lita da Silva (2015) [8]. Lita da Sliva (2016) [6] established \mathcal{L}_p -convergence for rowwise END arrays. Lita da Silva (2020) [9] obtained strong laws of large numbers for rowwise END arrays, and so on.

Xu et al. (2016) [1] established the following results on \mathcal{L}_p -convergence.

THEOREM 1.1. *Let $p \in (1, 2)$ and $\{X_i, i \in \mathbb{Z}\}$ be a sequence of END random variables with $E[X_i] = 0, i \in \mathbb{Z}$. Let $\{a_{ni}, i \in \mathbb{Z}, n \geq 1\}$ be an array of constants such that for any $\tau \geq 1, \sum_{i \in \mathbb{Z}} |a_{ni}|^\tau = O(n)$. Denote $T_n = \sum_{i \in \mathbb{Z}} a_{ni} X_i$.*

(1) *If $x^p \sup_{i \in \mathbb{Z}} P\{|X_i| > x\} \rightarrow 0$ as $x \rightarrow \infty$, then $n^{-\frac{1}{p}} T_n \xrightarrow{P} 0$.*

(2) If $x^p \sup_{i \in \mathbb{Z}} P\{|X_i| > x\} \rightarrow 0$ as $x \rightarrow \infty$ and $\sup_{i \in \mathbb{Z}} E[|X_i|^p] < \infty$, then for any $p' \in (0, p)$, $n^{-\frac{1}{p}} T_n \xrightarrow{\mathcal{L}_{p'}} 0$.

(3) If $\sup_{i \in \mathbb{Z}} E[|X_i|^p I(|X_i| > x)] \rightarrow 0$ as $x \rightarrow \infty$, then $n^{-\frac{1}{p}} T_n \xrightarrow{\mathcal{L}_p} 0$.

Lita da Silva (2020) [10] pointed out that the assertion (1) of Theorem 1.1 does not in general work for an array of random variables by the following counterexample.

EXAMPLE 1.3. For each $n \geq 1$ and suppose $\{X_{ni}, i \in \mathbb{Z}\}$ given by $X_{ni} = 0$, a.s. for all $i \neq n$, and X_{nn} having probability density

$$f_n(t) = \begin{cases} \frac{n^2}{|t|^3}, & |t| \geq n, \\ 0, & \text{elsewhere.} \end{cases}$$

Then $\{X_{ni}, i \in \mathbb{Z}\}$ is END with $E[X_{ni}] = 0$ for all i . And for any $p \in (1, 2)$ and $x > n$,

$$x^p \sup_{i \in \mathbb{Z}} P\{|X_{ni}| > x\} = x^p P\{|X_{nn}| > x\} = n^2 x^{p-2} \rightarrow 0, \quad x \rightarrow \infty.$$

Taking $a_{ni} = \begin{cases} 1, & i = n \\ 0, & i \neq n \end{cases}$. Then for any $\varepsilon > 0$, there exists some $N = N_{\varepsilon, p} > 0$ such that for all $n > N$,

$$P\left\{\left|\sum_{i \in \mathbb{Z}} a_{ni} X_{ni}\right| > \varepsilon n^{\frac{1}{p}}\right\} = P\{|X_{nn}| > \varepsilon n^{\frac{1}{p}}\} = 1.$$

A proper reformulation of Theorem 1.1 (1) for a rowwise END array with dominating sequence satisfying $M_n = O(1)(n \rightarrow \infty)$ was then made by Lita da Silva (2020) [10] as follows.

THEOREM 1.2. Let $p \in (1, 2)$ and $\{X_{ni}, i \in \mathbb{Z}, n \geq 1\}$ be an array of rowwise END random variables with means zero and dominating sequence satisfying $M_n = O(1)(n \rightarrow \infty)$. Let $\{a_{ni}, i \in \mathbb{Z}, n \geq 1\}$ be an array of constants such that for any $\tau \geq 1$, $\sum_{i \in \mathbb{Z}} |a_{ni}|^\tau = O(n)$. Denote $T_n = \sum_{i \in \mathbb{Z}} a_{ni} X_{ni}$. If

(a) $x^p \sup_{i \in \mathbb{Z}} \sup P\{|X_{ni}| > x\} \rightarrow 0$ as $x \rightarrow \infty$ for every $n \geq 1$,

(b) $\int_0^1 y n \sup_{i \in \mathbb{Z}} P\{|X_{ni}| > y n^{\frac{1}{p}}\} dy \rightarrow 0$ as $n \rightarrow \infty$,

(c) $\int_1^\infty n \sup_{i \in \mathbb{Z}} P\{|X_{ni}| > y n^{\frac{1}{p}}\} dy \rightarrow 0$ as $n \rightarrow \infty$,

then $n^{-\frac{1}{p}} T_n \xrightarrow{P} 0$.

Inspired by the above works, we will further study \mathcal{L}_p -convergence and the purpose of this article is threefold:

1. generalise the above results to a rowwise END array $\{X_{ni}, 1 \leq i \leq k_n, n \geq 1\}$ with dominating sequence $\{M_n, n \geq 1\}$;
2. weaken the conditions in a sense;
3. reveal the relationships between the rate of convergence, weights of the sums and the dominating sequence in a sense.

Before presenting the main result, we introduce some notations. The letter C denotes a positive constant, whose value is not important and may change in each appearance. Denote by $\mathcal{L}_p(p > 0)$ the collection of all random variables X satisfying $E[|X|^p] < \infty$. $X_n \xrightarrow{P} X$ means a sequence of random variables X_n converges to a random variable X in probability, that is for all $\varepsilon > 0, P\{|X_n - X| > \varepsilon\} \rightarrow 0$ as $n \rightarrow \infty$. $X_n \xrightarrow{\mathcal{L}_p} X$ means $E[|X_n - X|^p] \rightarrow 0$ as $n \rightarrow \infty$. Set two sequences of real numbers $\{a_n\}$ and $\{b_n\}$, then $a_n = O(b_n)$ stands for $a_n \leq Cb_n$ for all n and $a_n \asymp b_n$ means $0 < \liminf_{n \rightarrow \infty} \frac{a_n}{b_n} \leq \limsup_{n \rightarrow \infty} \frac{a_n}{b_n} < \infty$. $I(A)$ is the indicator function on the measurable set $A \in \mathcal{F}$.

Denote $x^+ = x \vee 0$ and $x^- = x^+ - x$. For the sake of convenience, we write ‘ $\stackrel{\Delta}{=}$ ’ to define the corresponding expressions.

2. Main results and Lemmas

THEOREM 2.1. *Let $\{X_{ni}, i \in \mathbb{I}_n, n \geq 1\}$ be an array of zero-mean rowwise END random variables with dominating sequence $\{M_n, n \geq 1\}$. Let $\{\beta_n^{(1)}, n \geq 1\}$, $\{\beta_n^{(2)}, n \geq 1\}$ and $\{b_n, n \geq 1\}$ be three sequences of positive constants, and $\{a_{ni}, i \in \mathbb{I}_n, n \geq 1\}$ be an array of constants satisfying*

$$\sum_{i \in \mathbb{I}_n} |a_{ni}| = O(\beta_n^{(1)}) \text{ and } \sum_{i \in \mathbb{I}_n} |a_{ni}|^2 = O(\beta_n^{(2)}). \tag{1}$$

Denote $T_n = \sum_{i \in \mathbb{I}_n} a_{ni}X_{ni}$. If the following two conditions hold:

$$(C1) \quad \begin{cases} \beta_n^{(1)} \sup_{i \in \mathbb{I}_n} P\{|X_{ni}| > b_n\} \rightarrow 0, \\ M_n \beta_n^{(2)} \sup_{i \in \mathbb{I}_n} P\{|X_{ni}| > b_n\} \rightarrow 0, \end{cases} \quad n \rightarrow \infty,$$

$$(C2) \quad \begin{cases} M_n \beta_n^{(2)} \int_0^1 y \sup_{i \in \mathbb{I}_n} P\{|X_{ni}| > b_n y\} dy \rightarrow 0, \\ \beta_n^{(1)} \int_1^\infty \sup_{i \in \mathbb{I}_n} P\{|X_{ni}| > b_n y\} dy \rightarrow 0, \end{cases} \quad n \rightarrow \infty,$$

then $\frac{T_n}{b_n} \xrightarrow{P} 0$ as $n \rightarrow \infty$.

THEOREM 2.2. *Let $p \in (1, 2)$ and $\{X_{ni}, i \in \mathbb{I}_n, n \geq 1\}$ be an array of zero-mean rowwise END random variables with dominating sequence $\{M_n, n \geq 1\}$. Let $\{\beta_n^{(1)}, n \geq 1\}$, $\{\beta_n^{(2)}, n \geq 1\}$ and $\{b_n, n \geq 1\}$ be three sequences of positive constants, and $\{a_{ni}, i \in \mathbb{I}_n, n \geq 1\}$ be an array of constants satisfying (1). Denote $T_n = \sum_{i \in \mathbb{I}_n} a_{ni} X_{ni}$. If the conditions*

(C1') For any $\varepsilon > 0$,

$$\begin{cases} \beta_n^{(1)} \sup_{i \in \mathbb{I}_n} P\{|X_{ni}| > b_n \varepsilon\} \rightarrow 0, \\ M_n \beta_n^{(2)} \sup_{i \in \mathbb{I}_n} P\{|X_{ni}| > b_n \varepsilon\} \rightarrow 0, \end{cases} \quad n \rightarrow \infty,$$

and

$$(C3) \quad \begin{cases} \sup_{n \geq 1} \beta_n^{(1)} b_n^{-p} \sup_{i \in \mathbb{I}_n} E[|X_{ni}|^p] < \infty, \\ \sup_{n \geq 1} M_n \beta_n^{(2)} b_n^{-p} \sup_{i \in \mathbb{I}_n} E[|X_{ni}|^p] < \infty, \end{cases}$$

hold, then $\sup_{n \geq 1} E[|\frac{T_n}{b_n}|^p] < \infty$.

REMARK 2.1. On the one hand, it is easy to see that (C1') is equivalent to the following weaker condition

(C1'') For any $\varepsilon \in (0, 1)$,

$$\begin{cases} \beta_n^{(1)} \sup_{i \in \mathbb{I}_n} P\{|X_{ni}| > b_n \varepsilon\} \rightarrow 0, \\ M_n \beta_n^{(2)} \sup_{i \in \mathbb{I}_n} P\{|X_{ni}| > b_n \varepsilon\} \rightarrow 0, \end{cases} \quad n \rightarrow \infty.$$

Either (C1') or (C1'') implies (C1).

On the other hand, we can find that (C1') and (C3) together imply (C2). In fact, note that

$$g_n^{(1)}(y) = M_n \beta_n^{(2)} y \sup_{i \in \mathbb{I}_n} P\{|X_{ni}| > b_n y\}, \quad y \in (0, 1)$$

and

$$g_n^{(2)}(y) = \beta_n^{(1)} \sup_{i \in \mathbb{I}_n} P\{|X_{ni}| > b_n y\}, \quad y \in (1, \infty)$$

both approach 0 pointwisely as $n \rightarrow \infty$ by (C1'). And, by Markov's inequality, we have

$$g_n^{(1)}(y) \leq y^{1-p} \sup_n M_n \beta_n^{(2)} b_n^{-p} \sup_{i \in \mathbb{I}_n} E[|X_{ni}|^p] = g^{(1)}(y)$$

and

$$g_n^{(2)}(y) \leq y^{-p} \sup_n \beta_n^{(1)} b_n^{-p} \sup_{i \in \mathbb{I}_n} E[|X_{ni}|^p] = g^{(2)}(y)$$

for all n . Since $g^{(1)}(y)$ and $g^{(2)}(y)$ are integrable in $(0, 1)$ and $(1, \infty)$ respectively due to (C3), applying Lebesgue's dominated convergence theorem, we conclude that (C1') and (C3) together imply (C2).

REMARK 2.2. Let $r > 0$ and $\{\xi_n, n \geq 1\}$ be a sequence of random variables with $\sup_n E[|\xi_n|^r] < \infty$ and ξ_n converges to some random variable ξ in probability, then for every $r' \in (0, r)$, $\xi_n \xrightarrow{\mathcal{L}^{r'}} \xi$ as $n \rightarrow \infty$. Therefore, taking into account the results of Theorems 2.1 and 2.2 as well as the fact mentioned above, we have $\frac{T_n}{b_n} \xrightarrow{\mathcal{L}^{p'}} 0$ as $n \rightarrow \infty$ for all $p' \in (0, p)$ under the conditions of Theorem 2.2.

THEOREM 2.3. Let $p \in (1, 2)$ and $\{X_{ni}, i \in \mathbb{I}_n, n \geq 1\}$ be an array of zero-mean rowwise END random variables with dominating sequence $\{M_n, n \geq 1\}$. Let $\{\beta_n^{(1)}, n \geq 1\}$, $\{\beta_n^{(2)}, n \geq 1\}$ and $\{b_n, n \geq 1\}$ be three sequences of positive constants, and $\{a_{ni}, i \in \mathbb{I}_n, n \geq 1\}$ be an array of constants satisfying (1). Denote $T_n = \sum_{i \in \mathbb{I}_n} a_{ni} X_{ni}$. If (C3) and (C4) for each $\varepsilon > 0$,

$$\begin{cases} \beta_n^{(1)} b_n^{-p} \sup_{i \in \mathbb{I}_n} E[|X_{ni}|^p I(|X_{ni}| > b_n \varepsilon)] \rightarrow 0 \\ M_n \beta_n^{(2)} b_n^{-p} \sup_{i \in \mathbb{I}_n} E[|X_{ni}|^p I(|X_{ni}| > b_n \varepsilon)] \rightarrow 0, \end{cases} \quad n \rightarrow \infty,$$

hold, then $\frac{T_n}{b_n} \xrightarrow{\mathcal{L}^p} 0$ as $n \rightarrow \infty$.

REMARK 2.3. Clearly, (C4) implies (C1') and together with (C3) yields (C2).

Taking $\mathbb{I}_n = \{1, \dots, n\}$ and $\beta_n = n$, we obtain the following Corollary 2.1 for weighted sums of (lower triangular) rowwise END arrays.

COROLLARY 2.1. Let $\{X_{ni}, 1 \leq i \leq n, n \geq 1\}$ be a zero-mean rowwise END array with dominating sequence $\{M_n, n \geq 1\}$. Let $\{b_n, n \geq 1\}$ be a sequence of positive constants, and $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of constants satisfying $\sum_{i=1}^n |a_{ni}|^2 = O(n)$. Set $p \in (1, 2)$ and denote $T_n = \sum_{i=1}^n a_{ni} X_{ni}$.

(i) If $nM_n \max_{1 \leq i \leq n} P\{|X_{ni}| > b_n\} \rightarrow 0, nM_n \max_{1 \leq i \leq n} \int_0^1 y P\{|X_{ni}| > b_n y\} dy \rightarrow 0$ and $n \max_{1 \leq i \leq n} \int_1^\infty P\{|X_{ni}| > b_n y\} dy \rightarrow 0$ as $n \rightarrow \infty$, then $\frac{T_n}{b_n} \xrightarrow{P} 0$.

(ii) If for any $\varepsilon > 0, nM_n \max_{1 \leq i \leq n} P\{|X_{ni}| > b_n \varepsilon\} \rightarrow 0$ as $n \rightarrow \infty$ and

$\sup_{n \geq 1} nM_n b_n^{-p} \max_{1 \leq i \leq n} E[|X_{ni}|^p] < \infty$, then $\frac{T_n}{b_n} \xrightarrow{\mathcal{L}^{p'}} 0$ for all $p' \in (0, p)$.

(iii) If $\sup_{n \geq 1} nM_n b_n^{-p} \max_{1 \leq i \leq n} E[|X_{ni}|^p] < \infty$, and for any $\varepsilon > 0$,

$nM_n b_n^{-p} \max_{1 \leq i \leq n} E[|X_{ni}|^p I(|X_{ni}| > b_n \varepsilon)] \rightarrow 0$ as $n \rightarrow \infty$, then $\frac{T_n}{b_n} \xrightarrow{\mathcal{L}^p} 0$.

REMARK 2.4. Since $\frac{1}{n} \sum_{i=1}^n |a_{ni}| \leq \left(\frac{1}{n} \sum_{i=1}^n |a_{ni}|^2 \right)^{\frac{1}{2}}$, one can see that $\sum_{i=1}^n |a_{ni}|^2 = O(n)$ implies $\sum_{i=1}^n |a_{ni}| = O(n)$. So the condition $\sum_{i=1}^n |a_{ni}|^2 = O(n)$ in Corollary 2.1 is weaker than the corresponding ones in Theorem 1.1 and 1.2. Particularly, taking $a_{ni} = 1, 1 \leq i \leq n, n \geq 1$ in Corollary 2.1 leads to the corresponding modes of convergence for partial sums of (lower triangular) rowwise END arrays.

We have the following corollary for END sequence.

COROLLARY 2.2. Let $\{X_i, i \in \mathbb{Z}\}$ be a zero-mean END sequence $\{b_n, n \geq 1\}$ and $\{\beta_n, n \geq 1\}$ be two sequences of positive constants. Suppose $\{a_{ni}, i \in \mathbb{Z}, n \geq 1\}$ be an array of constants satisfying $\sum_{i \in \mathbb{Z}} |a_{ni}|^\tau = O(\beta_n)$ for $\tau = 1$ and 2. Set $p \in (1, 2)$ and denote $T_n = \sum_{i \in \mathbb{Z}} a_{ni} X_i$.

(i) If $\beta_n \sup_i P\{|X_i| > b_n\} \rightarrow 0, \beta_n \int_0^1 y \sup_i P\{|X_i| > b_n y\} dy \rightarrow 0$ and $\beta_n \int_1^\infty \sup_i P\{|X_i| > b_n y\} dy \rightarrow 0$ as $n \rightarrow \infty$, then $\frac{T_n}{b_n} \xrightarrow{P} 0$.

(ii) If for any $\varepsilon > 0, \beta_n \sup_i P\{|X_i| > b_n \varepsilon\} \rightarrow 0$ as $n \rightarrow \infty$ and $\sup_{n \geq 1} \beta_n b_n^{-p} \sup_i E[|X_i|^p] < \infty$, then $\frac{T_n}{b_n} \xrightarrow{\mathcal{L}_{p'}} 0$ for all $p' \in (0, p)$.

(iii) If $\sup_{n \geq 1} \beta_n b_n^{-p} \sup_i E[|X_i|^p] < \infty$, and for any $\varepsilon > 0, \beta_n b_n^{-p} \sup_i E[|X_i|^p I(|X_i| > b_n \varepsilon)] \rightarrow 0$ as $n \rightarrow \infty$, then $\frac{T_n}{b_n} \xrightarrow{\mathcal{L}_p} 0$.

REMARK 2.5. Taking $\beta_n \asymp b_n^p$ in Corollary 2.2, since for any given $\delta > 0$, there is an $N > 0$ such that

$$\sup_i E[|X_i|^p] \leq \sup_i E[|X_i|^p I(|X_i| \leq b_N)] + \sup_i E[|X_i|^p I(|X_i| > b_N)] \leq b_N^p + \delta < \infty,$$

we conclude that Corollary 2.2 can serve as a generalisation of Theorem 1.1. Additionally setting that $\{X_i, i \in \mathbb{Z}\}$ is stochastically dominated by (or identically distributed with) some random variable X and $b_n \rightarrow \infty$ with $n \rightarrow \infty$, we conclude that three statements (a) $\sup_i E[|X_i|^p I(|X_i| > b_n \varepsilon)] \rightarrow 0$ for all $\varepsilon > 0$, (b) $\sup_i E[|X_i|^p] < \infty$ and (c) $X \in \mathcal{L}_p$ are equivalent.

The following lemmas are of significance in the proofs of our main results. The first lemma is about a basic property for rowwise END arrays due to Lita da Silva (2016) [7].

LEMMA 2.1. Let $\{X_{ni}, i \in \mathbb{I}_n, n \geq 1\}$ be an array of random variables and $\{f_{ni}, i \in \mathbb{I}_n, n \geq 1\}$ be an array of real-valued functions.

- (1) If $\{X_{ni}, i \in \mathbb{I}_n, n \geq 1\}$ is rowwise UEND, LEND, or END with dominating sequence $\{M_n, n \geq 1\}$ and $f_{ni}, i \in \mathbb{I}_n, n \geq 1$ are all nondecreasing, then $\{f_{ni}(X_{ni}), i \in \mathbb{I}_n, n \geq 1\}$ is still rowwise UEND, LEND, or END respectively.
- (2) If $\{X_{ni}, i \in \mathbb{I}_n, n \geq 1\}$ is rowwise UEND, LEND, or END with dominating sequence $\{M_n, n \geq 1\}$ and $f_{ni}, i \in \mathbb{I}_n, n \geq 1$ are all nonincreasing, then $\{f_{ni}(X_{ni}), i \in \mathbb{I}_n, n \geq 1\}$ is still rowwise LEND, UEND, or END respectively.

For each case, the dominating sequence $\{M_n, n \geq 1\}$ remains unchanged.

The second lemma is a Rosenthal type inequality for rowwise END arrays with dominating sequence $\{M_n, n \geq 1\}$, which was established by Lita da Silva (2016) [6].

LEMMA 2.2. Let $\theta \geq 2$ and $\{X_{ni}, i \in \mathbb{I}_n, n \geq 1\} \subset \mathcal{L}_\theta$ be an array of zero-mean rowwise END random variables with dominating sequence $\{M_n, n \geq 1\}$. Then there exists a constant $C > 0$ depending only on θ such that

$$E \left[\left| \sum_{i \in \mathbb{I}_n} X_{ni} \right|^\theta \right] \leq C \left(\sum_{i \in \mathbb{I}_n} E[|X_{ni}|^\theta] + M_n \left(\sum_{i \in \mathbb{I}_n} E[X_{ni}^2] \right)^{\frac{\theta}{2}} \right).$$

The next three lemmas play significant roles in the proofs of the main results.

LEMMA 2.3. Assume the conditions from Theorem 2.1 hold. Set $X'_{ni} = T(X_{ni}, b_n)$ and $X''_{ni} = X_{ni} - X'_{ni}$,

$$J_1 = P \left\{ \left| \sum_{i \in \mathbb{I}_n} a_{ni}(X'_{ni} - E[X'_{ni}]) \right| > \frac{\varepsilon b_n}{2} \right\}, \quad J_2 = P \left\{ \left| \sum_{i \in \mathbb{I}_n} a_{ni}(X''_{ni} - E[X''_{ni}]) \right| > \frac{\varepsilon b_n}{2} \right\},$$

where $T(x, t)$ is defined as

$$T(x, t) = -tI(x < -t) + xI(|x| \leq t) + tI(x > t), \quad -\infty < x < \infty, \quad t > 0.$$

Then for all $\varepsilon > 0$, when $n \rightarrow \infty$, $J_1 \rightarrow 0$, $J_2 \rightarrow 0$.

LEMMA 2.4. Assume the conditions from Theorem 2.2 hold. Set

$$Y'_{ni}(s) = T(X_{ni}, s^{\frac{1}{p}}) \quad \text{and} \quad Y''_{ni}(s) = X_{ni} - Y'_{ni}(s) \quad \text{for all } s > 0,$$

$$J_3 = \sup_{n \geq 1} b_n^{-p} \int_{b_n^p}^\infty P \left\{ \left| \sum_{i \in \mathbb{I}_n} a_{ni}(Y'_{ni}(s) - E[Y'_{ni}(s)]) \right| > \frac{s^{\frac{1}{p}}}{2} \right\} ds,$$

$$J_4 = \sup_{n \geq 1} b_n^{-p} \int_{b_n^p}^\infty P \left\{ \left| \sum_{i \in \mathbb{I}_n} a_{ni}(Y''_{ni}(s) - E[Y''_{ni}(s)]) \right| > \frac{s^{\frac{1}{p}}}{2} \right\} ds,$$

where $T(x, t) (-\infty < x < \infty, t > 0)$ is defined as the same as Lemma 2.3. Then $J_3 < \infty$, $J_4 < \infty$.

LEMMA 2.5. Assume the conditions from Theorem 2.3 hold. Let $Y'_{ni}(s)$ and $Y''_{ni}(s)$ be defined as the same as Lemma 2.4, and

$$J_5 = b_n^{-p} \int_{b_n^p \varepsilon}^{\infty} P \left\{ \left| \sum_{i \in \mathbb{I}_n} a_{ni} (Y'_{ni}(s) - E[Y'_{ni}(s)]) \right| > \frac{s^{\frac{1}{p}}}{2} \right\} ds,$$

$$J_6 = b_n^{-p} \int_{b_n^p \varepsilon}^{\infty} P \left\{ \left| \sum_{i \in \mathbb{I}_n} a_{ni} (Y''_{ni}(s) - E[Y''_{ni}(s)]) \right| > \frac{s^{\frac{1}{p}}}{2} \right\} ds.$$

Then for all $\varepsilon \in (0, 1)$, when $n \rightarrow \infty$, $J_5 \rightarrow 0$, $J_6 \rightarrow 0$.

3. Proofs of the Lemmas and the main results

Without loss of generality, we suppose in this section that $a_{ni} \geq 0$ for each $i \in \mathbb{I}_n, n \geq 1$ and $\sum_{i \in \mathbb{I}_n} a_{ni}^\tau \leq \beta_n$ for $\tau = 1, 2$ with respect to (1).

Proof of Lemma 2.3. By Markov’s inequality, Lemma 2.2 and the definition of X'_{ni} , we have

$$J_1 \leq C b_n^{-2} E \left[\left| \sum_{i \in \mathbb{I}_n} a_{ni} (X'_{ni} - E[X'_{ni}]) \right|^2 \right] \leq C M_n b_n^{-2} \sum_{i \in \mathbb{I}_n} a_{ni}^2 E[|X'_{ni}|^2]$$

$$\leq C M_n \sum_{i \in \mathbb{I}_n} a_{ni}^2 P\{|X_{ni}| > b_n\} + C M_n b_n^{-2} \sum_{i \in \mathbb{I}_n} a_{ni}^2 [E|X_{ni}|^2 I(|X_{ni}| \leq b_n)]$$

$$\triangleq C(J_{11} + J_{12}).$$

It follows directly from (1) and (C1) that

$$J_{11} \leq C M_n \beta_n^{(2)} \sup_{i \in \mathbb{I}_n} P\{|X_{ni}| > b_n\} \rightarrow 0, \quad n \rightarrow \infty.$$

By (1) and (C2), we have

$$J_{12} \leq C M_n b_n^{-2} \sum_{i \in \mathbb{I}_n} a_{ni}^2 \int_0^{b_n} x P\{|X_{ni}| > x\} dx$$

$$= C M_n \sum_{i \in \mathbb{I}_n} a_{ni}^2 \int_0^1 y P\{|X_{ni}| > y b_n\} dy$$

$$\leq C M_n \beta_n^{(2)} \sup_{i \in \mathbb{I}_n} \int_0^1 y P\{|X_{ni}| > y b_n\} dy \rightarrow 0, \quad n \rightarrow \infty.$$

It can be checked that

$$|X''_{ni}| \leq |X_{ni}| I(|X_{ni}| > b_n) = (|X_{ni}| - b_n)_+ + b_n I(|X_{ni}| > b_n).$$

Applying Markov's inequality, (C1) and (C2) leads to

$$\begin{aligned}
 J_2 &\leq Cb_n^{-1}E \left[\left| \sum_{i \in \mathbb{I}_n} a_{ni}(X''_{ni} - EX''_{ni}) \right| \right] \leq Cb_n^{-1} \sum_{i \in \mathbb{I}_n} a_{ni}E[|X''_{ni}|] \\
 &\leq Cb_n^{-1} \sum_{i \in \mathbb{I}_n} a_{ni} \int_{b_n}^{\infty} P\{|X_{ni}| > x\} dx + C \sum_{i \in \mathbb{I}_n} a_{ni}P\{|X_{ni}| > b_n\} \\
 &= C \sum_{i \in \mathbb{I}_n} a_{ni} \int_1^{\infty} P\{|X_{ni}| > b_n y\} dy + C \sum_{i \in \mathbb{I}_n} a_{ni}P\{|X_{ni}| > b_n\} \\
 &\leq C\beta_n^{(1)} \sup_{i \in \mathbb{I}_n} \int_1^{\infty} P\{|X_{ni}| > b_n y\} dy + C\beta_n^{(1)} \sup_{i \in \mathbb{I}_n} P\{|X_{ni}| > b_n\} \rightarrow 0, \quad n \rightarrow \infty.
 \end{aligned}$$

The proof is completed. \square

Proof of Lemma 2.4. By Remark 2.1, conditions (C1) and (C2) can be used in this proof. By Markov's inequality, Lemma 2.2 and the definition of $Y'_{ni}(s)$, we have

$$\begin{aligned}
 J_3 &\leq C \sup_{n \geq 1} b_n^{-p} \int_{b_n^p}^{\infty} s^{-\frac{2}{p}} E \left[\left| \sum_{i \in \mathbb{I}_n} a_{ni}(Y'_{ni}(s) - E[Y'_{ni}(s)]) \right|^2 \right] ds \\
 &\leq C \sup_{n \geq 1} M_n b_n^{-p} \int_{b_n^p}^{\infty} s^{-\frac{2}{p}} \sum_{i \in \mathbb{I}_n} a_{ni}^2 E[|Y'_{ni}(s)|^2] ds \\
 &\leq C \sup_{n \geq 1} M_n b_n^{-p} \int_{b_n^p}^{\infty} \sum_{i \in \mathbb{I}_n} a_{ni}^2 P\{|X_{ni}| > s^{\frac{1}{p}}\} ds \\
 &\quad + C \sup_{n \geq 1} M_n b_n^{-p} \int_{b_n^p}^{\infty} s^{-\frac{2}{p}} \sum_{i \in \mathbb{I}_n} a_{ni}^2 E[|X_{ni}|^2 I(|X_{ni}| \leq s^{\frac{1}{p}})] ds \\
 &\triangleq C(J_{31} + J_{32}).
 \end{aligned}$$

It follows directly from (1) and (C3) that

$$J_{31} \leq CM_n \beta_n^{(2)} b_n^{-p} \sup_{i \in \mathbb{I}_n} E[|X_{ni}|^p] < \infty.$$

Note that $|X_{ni}|^2 I(|X_{ni}| \leq s^{\frac{1}{p}}) \leq |X_{ni}|^2 \wedge s^{\frac{2}{p}}$, we have

$$\begin{aligned}
 &\int_{b_n^p}^{\infty} s^{-\frac{2}{p}} E[|X_{ni}|^2 I(|X_{ni}| \leq s^{\frac{1}{p}})] ds \\
 &\leq C \int_{b_n^p}^{\infty} s^{-\frac{2}{p}} \left(\int_0^{s^{\frac{1}{p}}} t P\{|X_{ni}| > t\} dt \right) ds \\
 &= C \int_0^{\infty} t P\{|X_{ni}| > t\} \left(\int_{b_n^p \vee t^p}^{\infty} s^{-\frac{2}{p}} ds \right) dt \\
 &= C \int_0^{b_n} b_n^{p-2} t P\{|X_{ni}| > t\} dt + C \int_{b_n}^{\infty} t^{p-1} P\{|X_{ni}| > t\} dt
 \end{aligned}$$

for each $i \in \mathbb{I}_n, n \geq 1$, thus applying (1), (C2) and (C3) yields

$$\begin{aligned} J_{32} &\leq C \sup_{n \geq 1} M_n \beta_n^{(2)} b_n^{-2} \sup_{i \in \mathbb{I}_n} \int_0^{b_n} t P\{|X_{ni}| > t\} dt \\ &\quad + C \sup_{n \geq 1} M_n \beta_n^{(2)} b_n^{-p} \sup_{i \in \mathbb{I}_n} \int_{b_n}^\infty t^{p-1} P\{|X_{ni}| > t\} dt \\ &\leq C \sup_{n \geq 1} M_n \beta_n^{(2)} \sup_{i \in \mathbb{I}_n} \int_0^1 y P\{|X_{ni}| > b_n y\} dy \\ &\quad + C \sup_{n \geq 1} M_n \beta_n^{(2)} b_n^{-p} \sup_{i \in \mathbb{I}_n} E[|X_{ni}|^p] < \infty. \end{aligned}$$

It can be checked that

$$|Y''_{ni}(s)| \leq |X_{ni}| I(|X_{ni}| > s^{\frac{1}{p}}) = (|X_{ni}| - s^{\frac{1}{p}})_+ + s^{\frac{1}{p}} I(|X_{ni}| > s^{\frac{1}{p}}).$$

Applying Markov's inequality,

$$\begin{aligned} J_4 &\leq C \sup_{n \geq 1} b_n^{-p} \int_{b_n^p}^\infty s^{-\frac{1}{p}} E \left[\left| \sum_{i \in \mathbb{I}_n} a_{ni} (Y''_{ni}(s) - E[Y''_{ni}(s)]) \right| \right] ds \\ &\leq C \sup_{n \geq 1} b_n^{-p} \int_{b_n^p}^\infty s^{-\frac{1}{p}} \sum_{i \in \mathbb{I}_n} a_{ni} E[|Y''_{ni}(s)|] ds \\ &\leq C \sup_{n \geq 1} b_n^{-p} \int_{b_n^p}^\infty s^{-\frac{1}{p}} \sum_{i \in \mathbb{I}_n} a_{ni} \left(\int_{s^{\frac{1}{p}}}^\infty P\{|X_{ni}| > t\} dt \right) ds \\ &\quad + C \sup_{n \geq 1} b_n^{-p} \int_{b_n^p}^\infty \sum_{i \in \mathbb{I}_n} a_{ni} P\{|X_{ni}| > s^{\frac{1}{p}}\} ds \\ &\stackrel{\Delta}{=} C(J_{41} + J_{42}). \end{aligned}$$

Note that

$$\begin{aligned} \int_{b_n^p}^\infty s^{-\frac{1}{p}} \left(\int_{s^{\frac{1}{p}}}^\infty P\{|X_{ni}| > t\} dt \right) ds &= \int_{b_n}^\infty P\{|X_{ni}| > t\} \left(\int_{b_n^p}^{t^p} s^{-\frac{1}{p}} ds \right) dt \\ &\leq \int_{b_n}^\infty t^{p-1} P\{|X_{ni}| > t\} dt \end{aligned}$$

for each $i \in \mathbb{I}_n, n \geq 1$, taking into account (1) and (C3), we obtain

$$\begin{aligned} J_{41} &\leq C \sup_{n \geq 1} \beta_n^{(1)} b_n^{-p} \sup_{i \in \mathbb{I}_n} \int_{b_n^p}^\infty s^{-\frac{1}{p}} \left(\int_{s^{\frac{1}{p}}}^\infty P\{|X_{ni}| > t\} dt \right) ds \\ &\leq C \sup_{n \geq 1} \beta_n^{(1)} b_n^{-p} \sup_{i \in \mathbb{I}_n} \int_{b_n}^\infty t^{p-1} P\{|X_{ni}| > t\} dt \\ &\leq C \sup_{n \geq 1} \beta_n^{(1)} b_n^{-p} \sup_{i \in \mathbb{I}_n} E[|X_{ni}|^p] < \infty, \end{aligned}$$

and

$$\begin{aligned} J_{42} &\leq C \sup_{n \geq 1} \beta_n^{(1)} b_n^{-p} \sup_{i \in \mathbb{I}_n} \int_{b_n^p \varepsilon}^{\infty} P\{|X_{ni}| > s^{\frac{1}{p}}\} ds \\ &\leq C \sup_{n \geq 1} \beta_n^{(1)} b_n^{-p} \sup_{i \in \mathbb{I}_n} E[|X_{ni}|^p] < \infty. \end{aligned}$$

The proof is completed. \square

Proof of Lemma 2.5. By remark 2.3, conditions (C1), (C1') and (C2) can be used in this proof. By Markov's inequality, Lemma 2.2 and the definition of $Y'_{ni}(s)$, we get

$$\begin{aligned} J_5 &\leq C b_n^{-p} \int_{b_n^p \varepsilon}^{\infty} s^{-\frac{2}{p}} E \left[\left| \sum_{i \in \mathbb{I}_n} a_{ni} (Y'_{ni}(s) - E[Y'_{ni}(s)]) \right|^2 \right] ds \\ &\leq C M_n b_n^{-p} \int_{b_n^p \varepsilon}^{\infty} s^{-\frac{2}{p}} \sum_{i \in \mathbb{I}_n} a_{ni}^2 E[|Y'_{ni}(s)|^2] ds \\ &\leq C M_n b_n^{-p} \int_{b_n^p \varepsilon}^{\infty} \sum_{i \in \mathbb{I}_n} a_{ni}^2 P\{|X_{ni}| > s^{\frac{1}{p}}\} ds \\ &\quad + C M_n b_n^{-p} \int_{b_n^p \varepsilon}^{\infty} s^{-\frac{2}{p}} \sum_{i \in \mathbb{I}_n} a_{ni}^2 E[|X_{ni}|^2 I(|X_{ni}| \leq s^{\frac{1}{p}})] ds \\ &\triangleq C(J_{51} + J_{52}). \end{aligned}$$

It follows from (1) and (C4) that

$$\begin{aligned} J_{51} &\leq C M_n \beta_n^{(2)} b_n^{-p} \sup_{i \in \mathbb{I}_n} \int_{b_n^p \varepsilon}^{\infty} P\{|X_{ni}| > s^{\frac{1}{p}}\} ds \\ &\leq C M_n \beta_n^{(2)} b_n^{-p} \sup_{i \in \mathbb{I}_n} E[|X_{ni}|^p I(|X_{ni}| > b_n \varepsilon^{\frac{1}{p}})] \rightarrow 0, \quad n \rightarrow \infty. \end{aligned}$$

Since

$$\begin{aligned} &\int_{b_n^p \varepsilon}^{\infty} s^{-\frac{2}{p}} E[|X_{ni}|^2 I(|X_{ni}| \leq s^{\frac{1}{p}})] ds \\ &\leq C \int_{b_n^p \varepsilon}^{\infty} s^{-\frac{2}{p}} \left(\int_0^{s^{\frac{1}{p}}} t P\{|X_{ni}| > t\} dt \right) ds \\ &= C \int_0^{\infty} t P\{|X_{ni}| > t\} \left(\int_{(b_n^p \varepsilon) \vee t^p}^{\infty} s^{-\frac{2}{p}} ds \right) dt \\ &= C \int_0^{b_n \varepsilon^{\frac{1}{p}}} b_n^{p-2} t P\{|X_{ni}| > t\} dt + C \int_{b_n \varepsilon^{\frac{1}{p}}}^{\infty} t^{p-1} P\{|X_{ni}| > t\} dt \end{aligned}$$

for each $i \in \mathbb{I}_n, n \geq 1$, taking into account (1), (C2) and (C4), we obtain

$$\begin{aligned}
 J_{52} &\leq CM_n \beta_n^{(2)} b_n^{-p} \sup_{i \in \mathbb{I}_n} \int_{b_n^p \varepsilon}^{\infty} s^{-\frac{2}{p}} E[|X_{ni}|^2 I(|X_{ni}| \leq s^{\frac{1}{p}})] ds \\
 &\leq CM_n \beta_n^{(2)} b_n^{-2} \sup_{i \in \mathbb{I}_n} \int_0^{b_n \varepsilon^{\frac{1}{p}}} t P\{|X_{ni}| > t\} dt \\
 &\quad + CM_n \beta_n^{(2)} b_n^{-p} \sup_{i \in \mathbb{I}_n} \int_{b_n \varepsilon^{\frac{1}{p}}}^{\infty} t^{p-1} P\{|X_{ni}| > t\} dt \\
 &\leq CM_n \beta_n^{(2)} \sup_{i \in \mathbb{I}_n} \int_0^{\varepsilon^{\frac{1}{p}}} y P\{|X_{ni}| > b_n y\} dy \\
 &\quad + CM_n \beta_n^{(2)} b_n^{-p} \sup_{i \in \mathbb{I}_n} E[|X_{ni}|^p I(|X_{ni}| > b_n \varepsilon^{\frac{1}{p}})] \\
 &\leq CM_n \beta_n^{(2)} \sup_{i \in \mathbb{I}_n} \int_0^1 y P\{|X_{ni}| > b_n y\} dy \\
 &\quad + CM_n \beta_n^{(2)} b_n^{-p} \sup_{i \in \mathbb{I}_n} E[|X_{ni}|^p I(|X_{ni}| > b_n \varepsilon^{\frac{1}{p}})] \rightarrow 0, \quad n \rightarrow \infty.
 \end{aligned}$$

Similarly, by Markov's inequality, the definition of $Y_{ni}''(s)$, (1) and (C4), we have

$$\begin{aligned}
 J_6 &\leq C b_n^{-p} \int_{b_n^p \varepsilon}^{\infty} s^{-\frac{1}{p}} E \left[\left| \sum_{i \in \mathbb{I}_n} a_{ni} (Y_{ni}''(s) - E[Y_{ni}''(s)]) \right| \right] ds \\
 &\leq C b_n^{-p} \int_{b_n^p \varepsilon}^{\infty} s^{-\frac{1}{p}} \sum_{i \in \mathbb{I}_n} a_{ni} E[|Y_{ni}''(s)|] ds \\
 &\leq C b_n^{-p} \int_{b_n^p \varepsilon}^{\infty} s^{-\frac{1}{p}} \sum_{i \in \mathbb{I}_n} a_{ni} E[|X_{ni}| I(|X_{ni}| > s^{\frac{1}{p}})] ds \\
 &\leq C b_n^{-p} \int_{b_n^p \varepsilon}^{\infty} s^{-\frac{1}{p}} \sum_{i \in \mathbb{I}_n} a_{ni} [E[(|X_{ni}| - s^{\frac{1}{p}})_+] + s^{\frac{1}{p}} P\{|X_{ni}| > s^{\frac{1}{p}}\}] ds \\
 &\leq C \beta_n^{(1)} b_n^{-p} \sup_{i \in \mathbb{I}_n} \int_{b_n^p \varepsilon}^{\infty} s^{-\frac{1}{p}} \left(\int_{s^{\frac{1}{p}}}^{\infty} P\{|X_{ni}| > t\} dt \right) ds \\
 &\quad + C \beta_n^{(1)} b_n^{-p} \sup_{i \in \mathbb{I}_n} \int_{b_n^p \varepsilon}^{\infty} P\{|X_{ni}| > s^{\frac{1}{p}}\} ds \\
 &= C \beta_n^{(1)} b_n^{-p} \sup_{i \in \mathbb{I}_n} \int_{b_n \varepsilon^{\frac{1}{p}}}^{\infty} P\{|X_{ni}| > t\} \left(\int_{b_n^p}^{t^p} s^{-\frac{1}{p}} ds \right) dt \\
 &\quad + C \beta_n^{(1)} b_n^{-p} \sup_{i \in \mathbb{I}_n} E[(|X_{ni}|^p - b_n^p \varepsilon)_+] \\
 &\leq C \beta_n^{(1)} b_n^{-p} \sup_{i \in \mathbb{I}_n} \int_{b_n \varepsilon^{\frac{1}{p}}}^{\infty} t^{p-1} P\{|X_{ni}| > t\} dt + C \beta_n^{(1)} b_n^{-p} \sup_{i \in \mathbb{I}_n} E[(|X_{ni}|^p - b_n^p \varepsilon)_+] \\
 &\leq C \beta_n^{(1)} b_n^{-p} \sup_{i \in \mathbb{I}_n} E[|X_{ni}|^p I(|X_{ni}| > b_n \varepsilon^{\frac{1}{p}})] \rightarrow 0, \quad n \rightarrow \infty.
 \end{aligned}$$

The proof is completed. \square

Proof of Theorem 2.1. Let X'_{ni} and X''_{ni} be defined as the same as Lemma 2.3. Note that $E[X_{ni} = 0]$, taking into account Lemma 2.3, we have for any $\varepsilon > 0$,

$$P\{|T_n| > \varepsilon b_n\} = P\left\{\left|\sum_{i \in \mathbb{I}_n} a_{ni}(X'_{ni} - E[X'_{ni}]) + \sum_{i \in \mathbb{I}_n} a_{ni}(X''_{ni} - E[X''_{ni}])\right| > \varepsilon b_n\right\} \leq J_1 + J_2 \rightarrow 0$$

as $n \rightarrow \infty$, which completes the proof. \square

Proof of Theorem 2.2. Let $Y'_{ni}(s)$ and $Y''_{ni}(s)$ be defined as the same as Lemma 2.4. Note that $E[X_{ni}] = 0$, taking into account Lemma 2.4, we have

$$\begin{aligned} \sup_n b_n^{-p} E[|T_n|^p] &= \sup_n b_n^{-p} \left(\int_0^{b_n^p} + \int_{b_n^p}^\infty \right) P\{|T_n| > s^{\frac{1}{p}}\} ds \\ &\leq 1 + \sup_n b_n^{-p} \int_{b_n^p}^\infty P\left\{\left|\sum_{i \in \mathbb{I}_n} a_{ni}(Y'_{ni}(s) - E[Y'_{ni}(s)]) + \sum_{i \in \mathbb{I}_n} a_{ni}(Y''_{ni}(s) - E[Y''_{ni}(s)])\right| > s^{\frac{1}{p}}\right\} ds \\ &\leq 1 + J_3 + J_4 < \infty. \end{aligned}$$

Then the proof is completed. \square

Proof of Theorem 2.3. Let $Y'_{ni}(s)$ and $Y''_{ni}(s)$ be defined as the same as Lemma 2.5. We know by Lemma 2.5 that for any $\varepsilon \in (0, 1)$, there exists an $N = N_\varepsilon > 0$ such that when $n \geq N$, then $J_5 < \varepsilon$ and $J_6 < \varepsilon$.

Note that $E[X_{ni}] = 0$, taking into account Lemma 2.5, we have for any given $\varepsilon \in (0, 1)$ and N mentioned above, when $n \geq N$, then

$$\begin{aligned} b_n^{-p} E[|T_n|^p] &= b_n^{-p} \left(\int_0^{b_n^p \varepsilon} + \int_{b_n^p \varepsilon}^\infty \right) P\{|T_n| > s^{\frac{1}{p}}\} ds \\ &\leq \varepsilon + b_n^{-p} \int_{b_n^p \varepsilon}^\infty P\left\{\left|\sum_{i \in \mathbb{I}_n} a_{ni}(Y'_{ni}(s) - E[Y'_{ni}(s)]) + \sum_{i \in \mathbb{I}_n} a_{ni}(Y''_{ni}(s) - E[Y''_{ni}(s)])\right| > s^{\frac{1}{p}}\right\} ds \\ &\leq \varepsilon + J_5 + J_6 < 3\varepsilon, \end{aligned}$$

and the proof is hence completed due to the arbitrariness of ε . \square

REFERENCES

- [1] C. XU, M. M. XI, X. J. WANG, AND H. XIA, *L^r Convergence for Weighted sums of Extended Negatively Dependent Random Variables*, J. Math. Inequal., **10** (4) (2016), pp. 1157–1167.
- [2] J. G. YAN, *Strong stability of a type of Jamison weighted sums for END random variables*, J. Korean Math. Soc., **54** (3) (2017), 897–907.
- [3] J. G. YAN, *Almost sure convergence for weighted sums of WNOD random variables and its applications in nonparametric regression models*, Commun. Statist. Theory Methods, **47** (16) (2018), 3893–3909.
- [4] J. G. YAN, *Complete Convergence and Complete Moment Convergence for Maximal Weighted Sums of Extended Negatively Dependent Random Variables*, Acta Math Sinica, English Series, **34** (10) (2018), 1501–1516.
- [5] J. G. YAN, *Complete Convergence in Marcinkiewicz-Zygmund Type SLLN for END Random Variables and Its Applications*, Commun. Statist. Theory Methods, **48** (20) (2019), 5074–5098.
- [6] J. LITA DA SILVA, *Convergence in p -Mean for Arrays of Row-wise Extended Negatively Dependent Random Variables*, Acta Math. Hungar., **150** (2) (2016), 346–362.
- [7] J. LITA DA SILVA, *Limiting Behaviour for Arrays of Row-wise Upper Extended Negatively Dependent Random Variables*, Acta Math. Hungar., **148** (2) (2016), 481–492.
- [8] J. LITA DA SILVA, *Limiting Behaviour for Arrays of Upper Extended Negatively Dependent Random Variables*, Bull. Aust. Math. Soc., **92** (2015), 159–167.
- [9] J. LITA DA SILVA, *Strong laws of large numbers for arrays of row-wise extended negatively dependent random variables with applications*, J. Nonparametr. Stat., **32** (1) (2020), 20–41.
- [10] J. LITA DA SILVA, *Some Comments on Chen Xu, Mengmei Xi, Xuejun Wang and Hao Xia's Paper " L^r Convergence for Weighted Sums of Extended Negatively Dependent Random Variables"*, Math. Inequal. Appl., **23** (1) (2020), 75–60.
- [11] K. Y. WANG, Y. B. WANG, AND Q. W. GAO, *Uniform asymptotics for the finite-time ruin probability of a new dependent risk model with a constant interest rate*, Methodol. Comput. Appl. Probab., **15** (1) (2013), 109–124.
- [12] L. LIU, *Precise large deviations for dependent random variables with heavy tails*, Statist. Probab. Lett., **79** (9) (2009), 1290–1298.
- [13] X. XU, J. G. YAN, *Complete moment convergence for randomly weighted sums of END sequences and its applications*, Commun. Statist. Theory Methods, **50** (12) (2021), 2877–2899.

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