

## SOME APPROXIMATION RESULTS ON A CLASS OF NEW TYPE $\lambda$ -BERNSTEIN POLYNOMIALS

REŞAT ASLAN AND MOHAMMAD MURSALEEN\*

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*Abstract.* The main concern of this article is to acquire some approximation properties of a new class of Bernstein polynomials based on Bézier basis functions with shape parameter  $\lambda \in [-1, 1]$ . We prove Korovkin type approximation theorem and estimate the degree of convergence in terms of the modulus of continuity, for the functions belong to Lipschitz type class and Peetre's  $K$ -functional, respectively. Additionally, with the help of Maple software, we present the comparison of the convergence of newly defined operators to the certain functions with some graphical illustrations and error estimation tables. Also, we conclude that the error estimation of our newly defined operators in some cases is better than classical Bernstein operators [3], Cai et al. [4] and Izgi [10].

### 1. Introduction

Bernstein [3] proposed the following polynomials for a simple way of proving the Weierstrass approximation theorem:

$$B_m(\mu; y) = \sum_{k=0}^m q_{m,k}(y) \mu\left(\frac{k}{m}\right), \quad y \in [0, 1], \quad (1.1)$$

where  $m \in \mathbb{N}$ ,  $\mu \in C[0, 1]$  and Bernstein basis functions  $q_{m,k}(y)$  are given as:

$$q_{m,k}(y) = \binom{m}{k} y^k (1-y)^{m-k}. \quad (1.2)$$

These polynomials still shed light on many studies in approximation theory. Due to the significance of Bernstein polynomials, extensive studies have been done by some authors on many modifications and generalizations of polynomials (1.1). Recently, one of these studies was presented by Izgi [10] as follows:

$$F_{m,a,b}(\mu; y) = \sum_{k=0}^m q_{m,k,a,b}(y) \mu\left(\frac{k(m+a)}{m(m+b)}\right), \quad y \in \left[0, \frac{m+a}{m+b}\right], \quad (1.3)$$

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\* Corresponding author.

where  $m \in \mathbb{N}$ ,  $0 \leq a \leq b$ ,  $q_{m,k,a,b}(y) = \left(\frac{m+b}{m+a}\right)^m \binom{m}{k} y^k \left(\frac{m+a}{m+b} - y\right)^{m-k}$ .

He derived the Korovkin type theorem and estimated in connection with the modulus of continuity the order of convergence of operators (1.3). Moreover, he concluded that in some special cases of variables  $a$  and  $b$ , the error estimation of operators (1.3) is better than the classical Bernstein operators (1.1).

In 1960's, a French engineer Bézier handled the Bernstein basis functions to develop the shape design of surface and curve of cars. These basis functions are known today as Bézier basis. Moreover, these magnificent polynomials of Bernstein led to many application areas of mathematics, such as computer graphics, computer-aided geometric design (CAGD), numerical solution of partial differential equations and so on. Some applications in CAGD, one can refer to ([11, 12, 17, 9, 23]).

Very recently, the Bézier basis with shape parameter  $\lambda \in [-1, 1]$ , which is presented by Ye et al. [26], has attracted attention by some authors. Firstly, Cai et al. [4] introduced  $\lambda$ -Bernstein polynomials as below:

$$B_m^\lambda(\mu; y) = \sum_{k=0}^m \tilde{q}_{m,k}(\lambda; y) \mu \left(\frac{k}{m}\right), \quad m \in \mathbb{N}, \tag{1.4}$$

where  $\tilde{q}_{m,k}(\lambda; y)$  are Bézier basis with shape parameter  $\lambda \in [-1, 1]$ .

For the operators defined by (1.4), they derived various approximation theorems, namely, Korovkin type convergence, local approximation, convergence of the element of Lipschitz continuous function and Voronovskaya-type asymptotic. Özger [18] achieved several statistical approximation results of univariate  $\lambda$ -Bernstein operators and proved a statistical Voronovskaya type asymptotic theorem. Also, he defined bivariate  $\lambda$ -Bernstein operators and reached several approximation results of these operators. Further, Cai et al. [6] obtained some statistical approximation properties of a new generalization of  $\lambda$ -Bernstein operators via  $q$ -calculus. Cai and Cheng [5] constructed a new kind of  $\lambda$ -Bernstein operators related on  $(p, q)$ -calculus (see [16]) and arrived Korovkin type convergence theorem, order of approximation for the Lipschitz type continuous function. Mursaleen et al. [15] investigated some approximation results of Chlodowsky type  $q$ -Bernstein-Stancu operators with the help of Bézier basis with  $\lambda \in [-1, 1]$ . Acu et al. [1] introduced the Kantorovich type  $\lambda$ -Bernstein operators and arrived several approximation features such as order of convergence, in connection with Ditzian-Totik modulus of smoothness the Voronovskaya and Grüss-Voronovskaya type theorems. Further, we refer the readers some other interesting papers established on the Bézier basis with shape parameter  $\lambda \in [-1, 1]$  (see: [22, 19, 20, 21, 24, 25, 14]).

Now, motivated by all above mentioned works, based on operators (1.3), we define a new class of Bernstein operators related on the Bézier basis with shape parameter  $\lambda \in [-1, 1]$  as below:

$$F_{m,a,b}^\lambda(\mu; y) = \sum_{k=0}^m \tilde{q}_{m,k,a,b}(\lambda; y) \mu \left(\frac{k(m+a)}{m(m+b)}\right), \quad y \in \left[0, \frac{m+a}{m+b}\right], \tag{1.5}$$

where  $\tilde{q}_{m,k,a,b}(\lambda; y)$  are Bézier basis with shape parameter  $\lambda \in [-1, 1]$  as:

$$\tilde{q}_{m,0,a,b}(\lambda; y) = q_{m,0,a,b}(y) - \frac{\lambda}{m+1} q_{m+1,1,a,b}(y),$$

$$\begin{aligned} \tilde{q}_{m,j,a,b}(\lambda; y) &= q_{m,j,a,b}(y) + \lambda \left( \frac{m-2j+1}{m^2-1} q_{m+1,j,a,b}(y) \right. \\ &\quad \left. - \frac{m-2j-1}{m^2-1} q_{m+1,j+1,a,b}(y) \right) \quad (j = 1, 2, \dots, m-1), \end{aligned} \tag{1.6}$$

$$\tilde{q}_{m,m,a,b}(\lambda; y) = q_{m,m,a,b}(y) - \frac{\lambda}{m+1} q_{m+1,m,a,b}(y),$$

and  $m \in \mathbb{N}$ ,  $0 \leq a \leq b$ ,  $q_{m,k,a,b}(y) = \left(\frac{m+b}{m+a}\right)^m \binom{m}{k} y^k \left(\frac{m+a}{m+b} - y\right)^{m-k}$ . When  $a = b$  and  $\lambda = 0$ , they reduce to (1.2).

The structure of this work is organized as follows: In section 2, some preliminaries results such as moments and central moments are computed. In section 3, Korovkin type convergence theorem is derived, the order of convergence with regard to the ordinary modulus of continuity, for the functions belong to Lipschitz type class and Peetre’s  $K$ -functional are discussed, respectively. In the final section, we compare the convergence behaviour of our newly defined operators (1.5) by different parameters of  $m, a, b$  and  $\lambda$  with the help of some graphical illustrations and error estimation tables. Further, we conclude that the error estimation of operators (1.5), in some cases, is better than operators (1.1), (1.3) and (1.4).

### 2. Preliminaries

LEMMA 1. [10] *Let the operators  $F_{m,a,b}(\mu; y)$  be defined by (1.3). Then, the following expressions verify:*

$$\begin{aligned} F_{m,a,b}(1; y) &= 1; \\ F_{m,a,b}(t; y) &= y; \\ F_{m,a,b}(t^2; y) &= y^2 + \frac{y[(m+a) - (m+b)y]}{m(m+b)}. \end{aligned}$$

LEMMA 2. *Let the operators  $F_{m,a,b}^\lambda(\mu; y)$  be defined by (1.5). Then, we have the following identities:*

$$F_{m,a,b}^\lambda(1; y) = 1; \tag{2.1}$$

$$F_{m,a,b}^\lambda(t; y) = y + \frac{\left(\frac{m+a}{m+b}\right) \left[1 + \left(\frac{m+b}{m+a}\right)^{m+1} - \left(1 - \frac{m+b}{m+a}\right)^{m+1} - 2\frac{m+b}{m+a}y\right]}{m(m-1)} \lambda; \tag{2.2}$$

$$\begin{aligned} F_{m,a,b}^\lambda(t^2; y) &= y^2 + \frac{y[(m+a) - (m+b)y]}{m(m+b)} + \left( \frac{2\frac{m+a}{m+b}y - 4y^2 + 2\left(\frac{m+a}{m+b}\right)^2 \left(\frac{m+b}{m+a}\right)^{m+1}}{m(m-1)} \right. \\ &\quad \left. + \frac{\left(\frac{m+a}{m+b}\right)^2 \left[ \left(1 - \frac{m+b}{m+a}\right)^{m+1} + \left(\frac{m+b}{m+a}\right)^{m+1} - 1 \right]}{m^2(m-1)} \right) \lambda. \end{aligned} \tag{2.3}$$

*Proof.* From the definition of operators (1.5) and  $\tilde{q}_{m,j,a,b}(\lambda; y)$  (1.6), it is easy to

see  $\sum_{k=0}^m \tilde{q}_{m,k,a,b}(\lambda; y) = 1$ , hence we get (2.1).

$$\begin{aligned}
 F_{m,a,b}^{\lambda}(t; y) &= \sum_{k=0}^m \frac{k(m+a)}{m(m+b)} \tilde{q}_{m,k,a,b}(\lambda; y) \\
 &= \sum_{k=0}^{m-1} \frac{k(m+a)}{m(m+b)} \left[ q_{m,k,a,b}(y) + \lambda \left( \frac{m-2k+1}{m^2-1} q_{m+1,k,a,b}(y) \right. \right. \\
 &\quad \left. \left. - \frac{m-2k-1}{m^2-1} q_{m+1,k+1,a,b}(y) \right) \right] + q_{m,m,a,b}(y) - \frac{\lambda}{m+1} q_{m+1,m,a,b}(y) \\
 &= \sum_{k=0}^m \frac{k(m+a)}{m(m+b)} q_{m,k,a,b}(y) + \lambda \left( \sum_{k=0}^m \frac{k(m+a)}{m(m+b)} \frac{m-2k+1}{m^2-1} q_{m+1,k,a,b}(y) \right. \\
 &\quad \left. - \sum_{k=1}^{m-1} \frac{k(m+a)}{m(m+b)} \frac{m-2k-1}{m^2-1} q_{m+1,k+1,a,b}(y) \right) \\
 &= \sum_{k=0}^m \frac{k(m+a)}{m(m+b)} q_{m,k,a,b}(y) + \lambda [\phi_1(m, a, b, y) - \phi_2(m, a, b, y)],
 \end{aligned}$$

where

$$\begin{aligned}
 \phi_1(m, a, b, y) &= \sum_{k=0}^m \frac{k(m+a)}{m(m+b)} \frac{m-2k+1}{m^2-1} q_{m+1,k,a,b}(y); \\
 \phi_2(m, a, b, y) &= \sum_{k=1}^{m-1} \frac{k(m+a)}{m(m+b)} \frac{m-2k-1}{m^2-1} q_{m+1,k+1,a,b}(y).
 \end{aligned}$$

Now, we calculate the identities  $\phi_1(m, a, b, y)$  and  $\phi_2(m, a, b, y)$ .

$$\begin{aligned}
 \phi_1(m, a, b, y) &= \sum_{k=0}^m \frac{k(m+a)}{m(m+b)} \frac{m-2k+1}{m^2-1} q_{m+1,k,a,b}(y) \\
 &= \frac{(m+a)}{(m-1)(m+b)} \sum_{k=0}^m \frac{k}{m} q_{m+1,k,a,b}(y) - \frac{2(m+a)}{(m^2-1)(m+b)} \sum_{k=0}^m \frac{k^2}{m} q_{m+1,k,a,b}(y) \\
 &= \frac{y(m+1)}{m(m-1)} \sum_{k=0}^{m-1} q_{m,k,a,b}(y) - \frac{2y}{m(m-1)} \sum_{k=0}^{m-1} q_{m,k,a,b}(y) \\
 &\quad - \frac{2y^2(m+b)}{(m-1)(m+a)} \sum_{k=0}^{m-2} q_{m-1,k,a,b}(y) \\
 &= \frac{y \left[ 1 - \left( \frac{m+b}{m+a} y \right)^m \right]}{m} - \frac{2(m+b)y^2 \left[ 1 - \left( \frac{m+b}{m+a} y \right)^{m-1} \right]}{(m+a)(m-1)}.
 \end{aligned}$$

$$\begin{aligned}
 &\phi_2(m, a, b, y) \\
 &= \sum_{k=1}^{m-1} \frac{k(m+a)}{m(m+b)} \frac{m-2k-1}{m^2-1} q_{m+1, k+1, a, b}(y) \\
 &= \frac{y}{m} \sum_{k=1}^{m-1} q_{m, k, a, b}(y) + \frac{2y}{m(m-1)} \sum_{k=1}^{m-1} q_{m, k, a, b}(y) \\
 &\quad - \frac{(m+a)}{m(m+1)(m+b)} \sum_{k=1}^{m-1} q_{m+1, k+1, a, b}(y) \\
 &\quad - \frac{2y^2(m+b)}{(m-1)(m+a)} \sum_{k=0}^{m-2} q_{m-1, k, a, b}(y) - \frac{2(m+a)}{m(m^2-1)(m+b)} \sum_{k=1}^{m-1} q_{m+1, k+1, a, b}(y) \\
 &= \frac{y \left[ 1 - \left(1 - \frac{m+b}{m+a}y\right)^m - \left(\frac{m+b}{m+a}y\right)^m \right]}{m} + \frac{2y \left[ 1 - \left(1 - \frac{m+b}{m+a}y\right)^m - \left(\frac{m+b}{m+a}y\right)^m \right]}{m(m-1)} \\
 &\quad - \frac{(m+a) \left[ 1 - \left(1 - \frac{m+b}{m+a}y\right)^{m+1} - (m+1)y \left(\frac{m+b}{m+a}\right) \left(1 - \frac{m+b}{m+a}y\right)^m - \left(\frac{m+b}{m+a}y\right)^{m+1} \right]}{(m+b)m(m+1)} \\
 &\quad - \frac{2(m+b)y^2 \left[ 1 - \left(\frac{m+b}{m+a}y\right)^{m-1} \right]}{(m+a)(m-1)} \\
 &\quad - \frac{2(m+a) \left[ 1 - \left(1 - \frac{m+b}{m+a}y\right)^{m+1} - (m+1)y \left(\frac{m+b}{m+a}\right) \left(1 - \frac{m+b}{m+a}y\right)^m - \left(\frac{m+b}{m+a}y\right)^{m+1} \right]}{(m+b)m(m^2-1)}.
 \end{aligned}$$

If we combine  $\phi_1(m, a, b, y)$  and  $\phi_2(m, a, b, y)$ , hence we arrive (2.1).

Again, taking into consideration the definition of operators (1.5) and  $\tilde{q}_{m, j, a, b}(\lambda; y)$  (1.6), it follows

$$\begin{aligned}
 F_{m, a, b}^\lambda(t^2; y) &= \sum_{k=0}^m \frac{k^2(m+a)^2}{m^2(m+b)^2} \tilde{q}_{m, k, a, b}(\lambda; y) \\
 &= \sum_{k=0}^{m-1} \frac{k^2(m+a)^2}{m^2(m+b)^2} \left[ q_{m, k, a, b}(y) + \lambda \left( \frac{m-2k+1}{m^2-1} q_{m+1, k, a, b}(y) \right. \right. \\
 &\quad \left. \left. - \frac{m-2k-1}{m^2-1} q_{m+1, k+1, a, b}(y) \right) \right] + q_{m, m, a, b}(y) - \frac{\lambda}{m+1} q_{m+1, m, a, b}(y) \\
 &= \sum_{k=0}^m \frac{k^2(m+a)^2}{m^2(m+b)^2} q_{m, k, a, b}(y) + \lambda \left( \sum_{k=0}^m \frac{k^2(m+a)^2}{m^2(m+b)^2} \frac{m-2k+1}{m^2-1} q_{m+1, k, a, b}(y) \right. \\
 &\quad \left. - \sum_{k=1}^{m-1} \frac{k^2(m+a)^2}{m^2(m+b)^2} \frac{m-2k-1}{m^2-1} q_{m+1, k+1, a, b}(y) \right) \\
 &= \sum_{k=0}^m \frac{k^2(m+a)^2}{m^2(m+b)^2} q_{m, k, a, b}(y) + \lambda [\phi_3(m, a, b, y) - \phi_4(m, a, b, y)],
 \end{aligned}$$

where

$$\begin{aligned}\phi_3(m, a, b, y) &= \sum_{k=0}^m \frac{k^2(m+a)^2}{m^2(m+b)^2} \frac{m-2k+1}{m^2-1} q_{m+1,k,a,b}(y); \\ \phi_4(m, a, b, y) &= \sum_{k=1}^{m-1} \frac{k^2(m+a)^2}{m^2(m+b)^2} \frac{m-2k-1}{m^2-1} q_{m+1,k+1,a,b}(y).\end{aligned}$$

Now, we calculate the identities  $\phi_3(m, a, b, y)$  and  $\phi_4(m, a, b, y)$ .

$$\begin{aligned}\phi_3(m, a, b, y) &= \sum_{k=0}^m \frac{k^2(m+a)^2}{m^2(m+b)^2} \frac{m-2k+1}{m^2-1} q_{m+1,k,a,b}(y) \\ &= \frac{(m+a)^2}{(m-1)(m+b)^2} \sum_{k=0}^m \frac{k^2}{m^2} q_{m+1,k,a,b}(y) - \frac{2(m+a)^2}{(m^2-1)(m+b)^2} \sum_{k=0}^m \frac{k^3}{m^2} q_{m+1,k,a,b}(y) \\ &= \frac{(m+1)y^2}{m(m-1)} \sum_{k=0}^{m-2} q_{m-1,k,a,b}(y) + \frac{(m+a)(m+1)y}{(m+b)m^2(m-1)} \sum_{k=0}^{m-1} q_{m,k,a,b}(y) \\ &\quad - \frac{2(m+b)y^3}{(m+a)m} \sum_{k=0}^{m-1} q_{m,k,a,b}(y) - \frac{6y^2}{m(m-1)} \sum_{k=0}^{m-2} q_{m-1,k,a,b}(y) \\ &\quad - \frac{2(m+a)y}{(m+b)m^2(m-1)} \sum_{k=0}^{m-1} q_{m,k,a,b}(y) \\ &= \frac{(m+1)y^2 \left[1 - \left(\frac{m+b}{m+a}y\right)^{m-1}\right]}{m(m-1)} + \frac{(m+a)(m+1)y \left[1 - \left(\frac{m+b}{m+a}y\right)^m\right]}{(m+b)m^2(m-1)} \\ &\quad - \frac{2(m+b)y^3 \left[1 - \left(\frac{m+b}{m+a}y\right)^{m-2}\right]}{(m+a)m} - \frac{6y^2 \left[1 - \left(\frac{m+b}{m+a}y\right)^{m-1}\right]}{m(m-1)} \\ &\quad - \frac{2(m+a)y \left[1 - \left(\frac{m+b}{m+a}y\right)^m\right]}{(m+b)m^2(m-1)} \\ &= \frac{(m-5)y^2 \left[1 - \left(\frac{m+b}{m+a}y\right)^{m-1}\right]}{m(m-1)} + \frac{(m+a)y \left[1 - \left(\frac{m+b}{m+a}y\right)^m\right]}{(m+b)m^2} \\ &\quad - \frac{2(m+b)y^3 \left[1 - \left(\frac{m+b}{m+a}y\right)^{m-2}\right]}{(m+a)m}.\end{aligned}$$

$$\begin{aligned}\phi_4(m, a, b, y) &= \sum_{k=1}^{m-1} \frac{k^2(m+a)^2}{m^2(m+b)^2} \frac{m-2k-1}{m^2-1} q_{m+1,k+1,a,b}(y) \\ &= \frac{(m+a)^2}{(m+1)(m+b)^2} \sum_{k=1}^{m-1} \frac{k^2}{m^2} q_{m+1,k+1,a,b}(y) - \frac{2(m+a)^2}{(m^2-1)(m+b)^2} \sum_{k=1}^{m-1} \frac{k^3}{m^2} q_{m+1,k+1,a,b}(y)\end{aligned}$$

$$\begin{aligned}
 &= \frac{y^2}{m} \sum_{k=0}^{m-2} q_{m-1,k,a,b}(y) - \frac{(m+a)y}{m^2(m+b)} \sum_{k=1}^{m-1} q_{m,k,a,b}(y) \\
 &\quad + \frac{(m+a)^2}{(m+b)^2 m^2(m+1)} \sum_{k=1}^{m-1} q_{m+1,k+1,a,b}(y) \\
 &\quad - \frac{2(m+b)y^3}{m(m+a)} \sum_{k=0}^{m-3} q_{m-2,k,a,b}(y) - \frac{2(m+a)y}{m^2(m-1)(m+b)} \sum_{k=1}^{m-1} q_{m,k,a,b}(y) \\
 &\quad + \frac{2(m+a)^2}{m^2(m^2-1)(m+b)^2} \sum_{k=1}^{m-1} q_{m+1,k+1,a,b}(y) \\
 &= \frac{y^2 \left[ 1 - \left(\frac{m+b}{m+a}\right)^{m-1} \right]}{m} - \frac{(m+a)y \left[ 1 - \left(\frac{m+b}{m+a}\right)^m - \left(1 - \frac{m+b}{m+a}\right)^m \right]}{(m+b)m^2} \\
 &\quad + \frac{(m+a)^2 \left[ 1 - (m+1) \left(\frac{m+b}{m+a}\right) y \left(1 - \frac{m+b}{m+a}\right)^m - \left(1 - \frac{m+b}{m+a}\right)^{m+1} - \left(\frac{m+b}{m+a}\right)^{m+1} \right]}{(m+b)^2 m^2(m+1)} \\
 &\quad - \frac{2(m+b)y^3 \left[ 1 - \left(\frac{m+b}{m+a}\right)^{m-2} \right]}{m(m+a)} - \frac{2(m+a)y \left[ 1 - \left(\frac{m+b}{m+a}\right)^m - \left(1 - \frac{m+b}{m+a}\right)^m \right]}{m^2(m-1)(m+b)} \\
 &\quad + \frac{2(m+a)^2 \left[ 1 - (m+1) \left(\frac{m+b}{m+a}\right) y \left(1 - \frac{m+b}{m+a}\right)^m - \left(1 - \frac{m+b}{m+a}\right)^{m+1} - \left(\frac{m+b}{m+a}\right)^{m+1} \right]}{(m+b)^2 m^2(m^2-1)}.
 \end{aligned}$$

If we combine  $\phi_3(m, a, b, y)$  and  $\phi_4(m, a, b, y)$ , hence we obtain (2.3), thus we reach the proof of Lemma 2.  $\square$

COROLLARY 1. Let  $y \in [0, \frac{m+a}{m+b}]$  and  $\lambda \in [-1, 1]$ . As a consequence of Lemma 2, we obtain following relations

(i)

$$\begin{aligned}
 &F_{m,a,b}^\lambda(t-y; y) \\
 &= \frac{\left(\frac{m+a}{m+b}\right) \left[ 1 + \left(\frac{m+b}{m+a}\right)^{m+1} - \left(1 - \frac{m+b}{m+a}\right)^{m+1} - 2\frac{m+b}{m+a}y \right]}{m(m-1)} \lambda \\
 &\leq \frac{\left(\frac{m+a}{m+b}\right) \left[ 1 + \left(\frac{m+b}{m+a}\right)^{m+1} + \left(1 - \frac{m+b}{m+a}\right)^{m+1} + 2\frac{m+b}{m+a}y \right]}{m(m-1)} := \beta_{m,a,b}(y)
 \end{aligned}$$

(ii)

$$\begin{aligned}
 &F_{m,a,b}^\lambda((t-y)^2; y) \\
 &= \frac{y[(m+a) - (m+b)y]}{m(m+b)} + \left( \frac{2\left(\frac{m+a}{m+b}\right) \left[ y \left(1 - \frac{m+b}{m+a}\right)^{m+1} + \left(\frac{m+b}{m+a}\right)^{m+1} \left(\frac{m+a}{m+b} - y\right) \right]}{m(m-1)} \right. \\
 &\quad \left. + \frac{\left(\frac{m+a}{m+b}\right)^2 \left[ \left(1 - \frac{m+b}{m+a}\right)^{m+1} + \left(\frac{m+b}{m+a}\right)^{m+1} - 1 \right]}{m^2(m-1)} \right) \lambda
 \end{aligned}$$

$$\begin{aligned} &\leq \frac{y[(m+a) - (m+b)y]}{m(m+b)} + \frac{2\left(\frac{m+a}{m+b}\right) \left[ y\left(1 - \frac{m+b}{m+a}y\right)^{m+1} + \left(\frac{m+b}{m+a}y\right)^{m+1} \left(\frac{m+a}{m+b} - y\right) \right]}{m(m-1)} \\ &\quad + \frac{\left(\frac{m+a}{m+b}\right)^2 \left[ \left(1 - \frac{m+b}{m+a}y\right)^{m+1} + \left(\frac{m+b}{m+a}y\right)^{m+1} + 1 \right]}{m^2(m-1)} := \gamma_{m,a,b}(y) \end{aligned}$$

REMARK 1. For the operators defined by (1.5), we have following results

▷ In case  $\lambda = 0$ , the operators given by (1.5) reduce to the class of new type Bernstein polynomials defined by Izgi [10].

▷ In case  $a = b$ , the operators given by (1.5) reduce to the  $\lambda$ -Bernstein operators defined by Cai et al. [4].

▷ In case  $\lambda = 0$  and  $a = b$ , the operators given by (1.5) reduce to classical Bernstein operators defined by [3].

▷ In case  $\lambda = 0 = a$  and  $b = 1$ , the operators given by (1.5) reduce to the new class of Bernstein type operators defined by Deo et al. [7].

### 3. Convergence results of $F_{m,a,b}^\lambda$

In the next theorem, we introduce the Korovkin type approximation theorem. As it is known, the space  $C[0, \frac{m+a}{m+b}]$  denote the real-valued continuous function on  $[0, \frac{m+a}{m+b}]$  and it is equipped with the norm for a function  $\mu$  as follows:

$$\|\mu\|_{C[0, \frac{m+a}{m+b}]} = \sup_{y \in [0, \frac{m+a}{m+b}]} |\mu(y)|.$$

THEOREM 1. If  $\mu \in C[0, \frac{m+a}{m+b}]$ , then we have

$$\lim_{m \rightarrow \infty} F_{m,a,b}^\lambda(\mu; y) = \mu(y),$$

uniformly on  $[0, \frac{m+a}{m+b}]$ .

*Proof.* According to the Bohman-Korovkin theorem [13], it is sufficient to verify

$$\lim_{m \rightarrow \infty} \max_{y \in [0, \frac{m+a}{m+b}]} \left| F_{m,a,b}^\lambda(t^s; y) - y^s \right| = 0, \text{ for } s = 0, 1, 2.$$

Using (2.1), for  $s = 0$ , it is clear.

For  $s = 1$ , in view of (2.2), we have

$$\begin{aligned} &\lim_{m \rightarrow \infty} \max_{y \in [0, \frac{m+a}{m+b}]} \left| F_{n,a,b}^\lambda(t; y) - y \right| \\ &= \lim_{m \rightarrow \infty} \max_{y \in [0, \frac{m+a}{m+b}]} = \left| \frac{\left(\frac{m+a}{m+b}\right) \left[ 1 + \left(\frac{m+b}{m+a}y\right)^{m+1} - \left(1 - \frac{m+b}{m+a}y\right)^{m+1} - 2\frac{m+b}{m+a}y \right]}{m(m-1)} \lambda \right| = 0. \end{aligned}$$

Similarly, by (2.3), then

$$\begin{aligned} & \lim_{m \rightarrow \infty} \max_{y \in [0, \frac{m+a}{m+b}]} \left| F_{m,a,b}^\lambda(t^2; y) - y^2 \right| \\ &= \lim_{m \rightarrow \infty} \max_{y \in [0, \frac{m+a}{m+b}]} \left| \frac{y[(m+a) - (m+b)y]}{m(m+b)} + \left( \frac{2\left(\frac{m+a}{m+b}\right) - 4y^2 + 2\left(\frac{m+a}{m+b}\right)^2 \left(\frac{m+b}{m+a}\right)^{m+1}}{m(m-1)} \right. \right. \\ & \quad \left. \left. + \frac{\left(\frac{m+a}{m+b}\right)^2 \left[ \left(1 - \frac{m+b}{m+a}y\right)^{m+1} + \left(\frac{m+b}{m+a}y\right)^{m+1} - 1 \right]}{m^2(m-1)} \right) \lambda \right| \\ &= \lim_{m \rightarrow \infty} \left\{ \left( \frac{m+a}{m+b} \right)^2 \frac{1}{4m} + \frac{2\left(\frac{m+a}{m+b}\right) \left[ 1 - \left(\frac{m+a}{m+b}\right) \right]}{m(m-1)} \lambda \right\} = 0. \end{aligned}$$

Hence, we get the required sequel.  $\square$

Further, we discuss the order of convergence with regard to the ordinary modulus of continuity, an element of Lipschitz type continuous function and Peetre’s  $K$ -functional.

The Peetre’s  $K$ -functional is given by

$$K_2(\mu, \eta) = \inf_{v \in C^2[0, \frac{m+a}{m+b}]} \{ \|\mu - v\| + \eta \|v''\| \},$$

where  $\eta > 0$  and  $C^2[0, \frac{m+a}{m+b}] = \{ v \in C[0, \frac{m+a}{m+b}] : v', v'' \in C[0, \frac{m+a}{m+b}] \}$ .

Taking into account [8], there exists an absolute constant  $C > 0$  such that

$$K_2(\mu; \eta) \leq C \omega_2(\mu; \sqrt{\eta}), \quad \eta > 0 \tag{3.1}$$

where

$$\omega_2(\mu; \eta) = \sup_{0 < \alpha \leq \eta} \sup_{y \in [0, \frac{m+a}{m+b}]} |\mu(y + 2\alpha) - 2\mu(y + \alpha) + \mu(y)|,$$

is the second order modulus of smoothness of the function  $\mu \in C[0, \frac{m+a}{m+b}]$ . Further, by

$$\omega(\mu; \eta) := \sup_{0 < \alpha \leq \eta} \sup_{y \in [0, \frac{m+a}{m+b}]} |\mu(y + \alpha) - \mu(y)|,$$

we denote the ordinary modulus of continuity of  $\mu \in C[0, \frac{m+a}{m+b}]$ . Since  $\eta > 0$ ,  $\omega(\mu; \eta)$  has some useful properties see: [2].

Also, we give an element of Lipschitz type continuous function with  $Lip_L(\zeta)$ , where  $L > 0$  and  $0 < \zeta \leq 1$ . If the expression below:

$$|\mu(t) - \mu(y)| \leq L|t - y|^\zeta, \quad (t, y \in \mathbb{R}),$$

holds, then one can say a function  $\mu$  is belong to  $Lip_L(\zeta)$ .

**THEOREM 2.** *Let  $\mu \in C[0, \frac{m+a}{m+b}]$ ,  $y \in [0, \frac{m+a}{m+b}]$  and  $\lambda \in [-1, 1]$ . Then, the following inequality verify*

$$\left| F_{m,a,b}^\lambda(\mu; y) - \mu(y) \right| \leq 2\omega(\mu; \sqrt{\gamma_{m,a,b}(y)}),$$

where  $\gamma_{m,a,b}(y)$  defined in Corollary 1.

*Proof.* Taking  $|\mu(t) - \mu(y)| \leq \left(1 + \frac{|t-y|}{\delta}\right) \omega(\mu; \delta)$  into account and operating  $F_{m,a,b}^\lambda(\cdot; y)$ , we obtain

$$\left| F_{m,a,b}^\lambda(\mu; y) - \mu(y) \right| \leq \left(1 + \frac{1}{\delta} F_{m,a,b}^\lambda(|t-y|; y)\right) \omega(\mu; \delta).$$

Utilizing the Cauchy-Bunyakovsky-Schwarz inequality and from Lemma 2, it gives

$$\begin{aligned} \left| F_{m,a,b}^\lambda(\mu; y) - \mu(y) \right| &\leq \left(1 + \frac{1}{\delta} \sqrt{F_{m,a,b}^\lambda((t-y)^2; y)}\right) \omega(\mu; \delta) \\ &\leq \left(1 + \frac{1}{\delta} \sqrt{\gamma_{m,a,b}(y)}\right) \omega(\mu; \delta). \end{aligned}$$

Choosing  $\delta = \sqrt{\gamma_{m,a,b}(y)}$ , hence the proof is completed.  $\square$

**THEOREM 3.** *Let  $y \in [0, \frac{m+a}{m+b}]$  and  $\lambda \in [-1, 1]$ . Then, for  $\mu \in Lip_L(\zeta)$  we obtain*

$$\left| F_{m,a,b}^\lambda(\mu; y) - \mu(y) \right| \leq L(\gamma_{m,a,b}(y))^{\frac{\zeta}{2}}.$$

*Proof.* By the linearity and monotonicity of the operators (1.5), it becomes

$$\begin{aligned} \left| F_{m,a,b}^\lambda(\mu; y) - \mu(y) \right| &\leq F_{m,a,b}^\lambda(|\mu(t) - \mu(y)|; y) \\ &= \sum_{k=0}^m \tilde{q}_{m,k,a,b}(y) \left| \mu\left(\frac{k(m+a)}{m(m+b)}\right) - \mu(y) \right| \\ &\leq M \sum_{k=0}^m \tilde{q}_{m,k,a,b}(y) \left| \frac{k(m+a)}{m(m+b)} - y \right|^\zeta. \end{aligned}$$

Utilizing the Hölder’s inequality with  $p_1 = \frac{2}{\zeta}$  and  $p_2 = \frac{2}{2-\zeta}$  and in view of Corollary 1 and Lemma 2, we arrive

$$\begin{aligned} \left| F_{m,a,b}^\lambda(\mu; y) - \mu(y) \right| &\leq L \sum_{k=0}^m \tilde{q}_{m,k,a,b}(y) (y) \left\{ \left(\frac{k(m+a)}{m(m+b)} - y\right)^2 \right\}^{\frac{\zeta}{2}} \left\{ \sum_{k=0}^m \tilde{q}_{m,k,a,b}(y) \right\}^{\frac{2-\zeta}{2}} \\ &= L \left\{ F_{m,a,b}^\lambda((t-y)^2; y) \right\}^{\frac{\zeta}{2}} \left\{ F_{m,a,b}^\lambda(1; y) \right\}^{\frac{2-\zeta}{2}} \\ &\leq L(\gamma_{m,a,b}(y))^{\frac{\zeta}{2}}. \end{aligned}$$

Thus, the proof is completed.  $\square$

**THEOREM 4.** For all  $\mu \in C[0, \frac{m+a}{m+b}]$ ,  $y \in [0, \frac{m+a}{m+b}]$  and  $\lambda \in [-1, 1]$ , the following inequality holds:

$$\left| F_{m,a,b}^\lambda(\mu; y) - \mu(y) \right| \leq C\omega_2\left(\mu; \frac{1}{2}\sqrt{\gamma_{m,a,b}(y) + (\beta_{m,a,b}(y))^2} + \omega(\mu; \beta_{m,a,b}(y))\right),$$

where  $C > 0$  is a constant,  $\beta_{m,a,b}(y)$ ,  $\gamma_{m,a,b}(y)$  are same as in Corollary 1.

*Proof.* Let  $\mu \in C[0, \frac{m+a}{m+b}]$ . We denote

$$\alpha_{m,a,b}(y) := y + \frac{\left(\frac{m+a}{m+b}\right) \left[1 + \left(\frac{m+b}{m+a}y\right)^{m+1} - \left(1 - \frac{m+b}{m+a}y\right)^{m+1} - 2\frac{m+b}{m+a}y\right]}{m(m-1)}\lambda,$$

it is obvious that  $\alpha_{m,a,b}(y) \in [0, \frac{m+a}{m+b}]$  for sufficiently large  $m$ . We define the following auxiliary operators:

$$\widehat{F}_{m,a,b}^\lambda(\mu; y) = F_{m,a,b}^\lambda(\mu; y) - \mu(\alpha_{m,a,b}(y)) + \mu(y). \tag{3.2}$$

In view of (2.1) and (2.2), it follows that

$$\widehat{F}_{m,a,b}^\lambda(t - y; y) = 0.$$

By Taylor’s formula, one has

$$\xi(t) = \xi(y) + (t - y)\xi'(y) + \int_y^t (t - u)\xi''(u)du, \quad \left(\xi \in C^2\left[0, \frac{m+a}{m+b}\right]\right). \tag{3.3}$$

After operating  $\widehat{F}_{m,a,b}^\lambda(\cdot; y)$  to (3.3), we obtain

$$\begin{aligned} \widehat{F}_{m,a,b}^\lambda(\xi; y) - \xi(y) &= \widehat{F}_{m,a,b}^\lambda((t - y)\xi'(y); y) + \widehat{F}_{m,a,b}^\lambda\left(\int_y^t (t - u)\xi''(u)du; y\right) \\ &= \xi'(y)\widehat{F}_{m,a,b}^\lambda(t - y; y) + F_{m,a,b}^\lambda\left(\int_y^t (t - u)\xi''(u)du; y\right) - \int_y^{\alpha_{m,a,b}(y)} (\alpha_{m,a,b}(y) - u)\xi''(u)du \\ &= F_{m,a,b}^\lambda\left(\int_y^t (t - u)\xi''(u)du; y\right) - \int_y^{\alpha_{m,a,b}(y)} (\alpha_{m,a,b}(y) - u)\xi''(u)du. \end{aligned}$$

Taking Lemma 2 and (3.2) into the account,

$$\begin{aligned} &\left| \widehat{F}_{m,a,b}^\lambda(\xi; y) - \xi(y) \right| \\ &\leq \left| F_{m,a,b}^\lambda\left(\int_y^t (t - u)\xi''(u)du; y\right) \right| + \left| \int_y^{\alpha_{m,a,b}(y)} (\alpha_{m,a,b}(y) - u)\xi''(u)du \right| \end{aligned}$$

$$\begin{aligned} &\leq F_{m,a,b}^\lambda \left( \left| \int_y^t (t-u) |\xi''(u)| du; y \right| + \int_y^{\alpha_{m,a,b}(y)} |\alpha_{m,a,b}(y) - u| |\xi''(u)| du \right) \\ &\leq \|\xi''\| \left\{ F_{m,a,b}^\lambda((t-y)^2; y) + (\alpha_{m,a,b}(y) - y)^2 \right\} \\ &\leq \{ \gamma_{m,a,b}(y) + (\beta_{m,a,b}(y))^2 \} \|\xi''\|. \end{aligned}$$

Also from (2.1), (2.2) and (3.2), it deduce the following

$$\begin{aligned} \left| \widehat{F}_{m,a,b}^\lambda(\mu; y) \right| &\leq \left| F_{m,a,b}^\lambda(\mu; y) \right| + 2 \|\mu\| \\ &\leq \|\mu\| F_{m,a,b}^\lambda(1; y) + 2 \|\mu\| \leq 3 \|\mu\|. \end{aligned} \tag{3.4}$$

On the other hand, by (3.3) and (3.4) imply

$$\begin{aligned} \left| F_{m,a,b}^\lambda(\mu; y) - \mu(y) \right| &\leq \left| \widehat{F}_{m,a,b}^\lambda(\mu - \xi; y) - (\mu - \xi)(y) \right| \\ &\quad + \left| \widehat{F}_{m,a,b}^\lambda(\xi; y) - \xi(y) \right| + \left| \mu(y) - \mu(\alpha_{m,a,b}(y)) \right| \\ &\leq 4 \|\mu - \xi\| + \{ \gamma_{m,a,b}(y) + (\beta_{m,a,b}(y))^2 \} \|\xi''\| + \omega(\mu; \beta_{m,a,b}(y)). \end{aligned}$$

On account of this, if we take the infimum on the right-hand side over all  $\xi \in C^2[0, \frac{m+a}{m+b}]$  and by (3.1), we arrive

$$\begin{aligned} \left| F_{m,a,b}^\lambda(\mu; y) - \mu(y) \right| &\leq 4K_2 \left( \mu; \frac{\{ \gamma_{m,a,b}(y) + (\beta_{m,a,b}(y))^2 \}}{4} \right) + \omega(\mu; \beta_{m,a,b}(y)) \\ &\leq C\omega_2 \left( \mu; \frac{1}{2} \sqrt{\gamma_{m,a,b}(y) + (\beta_{m,a,b}(y))^2} \right) + \omega(\mu; \beta_{m,a,b}(y)). \end{aligned}$$

Hence, we obtain the proof of this Theorem.  $\square$

**THEOREM 5.** *If  $\mu \in C^1[0, \frac{m+a}{m+b}]$ , then for all  $y \in [0, \frac{m+a}{m+b}]$ ,  $\lambda \in [-1, 1]$ , we arrive:*

$$\left| F_{m,a,b}^\lambda(\mu; y) - \mu(y) \right| \leq \beta_{m,a,b}(y) |\mu'(y)| + 2\sqrt{\gamma_{m,a,b}(y)} \omega(\mu'; \sqrt{\gamma_{m,a,b}(y)}),$$

where  $\beta_{m,a,b}(y)$ ,  $\gamma_{m,a,b}(y)$  are same as in Corollary 1.

*Proof.* Let  $\mu \in C^1[0, \frac{m+a}{m+b}]$ . For any  $y, t \in [0, \frac{m+a}{m+b}]$ , we get

$$\mu(t) - \mu(y) = \mu'(y)(t-y) + \int_y^t (\mu'(u) - \mu'(y)) du.$$

After operating  $F_{m,a,b}^\lambda(\cdot; y)$  to the both sides of foregoing expression, then

$$F_{m,a,b}^\lambda(\mu(t) - \mu(y); y) = \mu'(y) F_{m,a,b}^\lambda(t-y; y) + F_{m,a,b}^\lambda \left( \int_y^t (\mu'(u) - \mu'(y)) du; y \right).$$

In view of the following well-known property

$$|\mu(u) - \mu(y)| \leq \left(1 + \frac{|u-y|}{\delta}\right) \omega(\mu; \delta), \quad \delta > 0,$$

then

$$\left| \int_y^t |\mu'(u) - \mu'(y)| du \right| \leq \left( \frac{(t-y)^2}{\delta} + |t-y| \right) \omega(\mu'; \delta).$$

Hence,

$$\begin{aligned} |F_{m,a,b}^\lambda(\mu; y) - \mu(y)| &\leq |F_{m,a,b}^\lambda(t-y; y)| |\mu'(y)| \\ &\quad + \left[ \frac{F_{m,a,b}^\lambda((t-y)^2; y)}{\delta} + F_{m,a,b}^\lambda(|t-y|; y) \right] \omega(\mu'; \delta). \end{aligned}$$

Utilizing the Cauchy-Bunyakovsky-Schwarz inequality on the right-hand side of foregoing inequality and taking into consideration Corollary 1, it becomes

$$\begin{aligned} |F_{m,a,b}^\lambda(\mu; y) - \mu(y)| &\leq |F_{m,a,b}^\lambda(t-y; y)| |\mu'(y)| \\ &\quad + \omega(\mu'; \delta) \left[ \frac{\sqrt{F_{m,a,b}^\lambda((t-y)^2; y)}}{\delta} + 1 \right] \sqrt{F_{m,a,b}^\lambda((t-y)^2; y)} \\ &\leq \beta_{m,a,b}(y) |\mu'(y)| + \omega(\mu'; \delta) \left[ \frac{\sqrt{\gamma_{m,a,b}(y)}}{\delta} + 1 \right] \sqrt{\gamma_{m,a,b}(y)}. \end{aligned}$$

Taking  $\delta = \sqrt{\gamma_{m,a,b}(y)}$ , which gives the required result.  $\square$

### 4. Graphical and numerical results

In this section, with the aid of Maple software, we present some graphics and error estimation tables to demonstrate the convergence of operators (1.5) to certain functions with the different values of  $a, b, m$  and  $\lambda$  parameters. Moreover, we compare the convergence of operators (1.5) to the certain function  $\mu(y)$  with the operators given by (1.1), (1.3) and (1.4).

EXAMPLE 1. Let  $\mu(y) = 1 - \sin(2\pi y)$  (yellow),  $\lambda = 1, a = 0.1$  and  $b = 0.7$ . In Figure 1, for  $m = 15$  (red),  $m = 30$  (green),  $m = 75$  (purple), we demonstrate the convergence of operators  $F_{m,a,b}^\lambda(\mu; y)$  to  $\mu(y)$ . In Table 1, with choosing  $a = 0.2, b = 0.8, y = 0.25$  and for the certain values of  $-1 \leq \lambda \leq 1$ , we estimate the error of approximation operators  $F_{m,a,b}^\lambda(\mu; y)$  to  $\mu(y)$  for  $m = 15, 30, 75, 150$ , respectively. It is obvious from Table 1 that, as the value of  $m$  increases than the error of approximation operators  $F_{m,a,b}^\lambda(\mu; y)$  to  $\mu(y)$  is decreases. Also, for  $\lambda > 0$ , the absolute difference between operators  $F_{m,a,b}^\lambda(\mu; y)$  and  $\mu(y)$  is smaller than between  $F_{m,a,b}(\mu; y)$  and  $\mu(y)$ .

EXAMPLE 2. Let  $\mu(y) = y \sin(2\pi y)/2$  (yellow),  $\lambda = -1$ ,  $a = 0.1$  and  $b = 0.7$ . In Figure 2, for  $m = 15$  (red),  $m = 30$  (green),  $m = 75$  (purple), we demonstrate the convergence of operators  $F_{m,a,b}^\lambda(\mu; y)$  to  $\mu(y)$ . In Table 2, with choosing  $a = 1$ ,  $b = 3$ ,  $y = 0.8$  and for the certain values of  $-1 \leq \lambda \leq 1$ , we estimate the error of approximation operators  $F_{m,a,b}^\lambda(\mu; y)$  to  $\mu(y)$  for  $m = 15, 30, 75, 150$ , respectively. It is clear from Table 2 that, as the value of  $m$  increases than the error of approximation operators  $F_{m,a,b}^\lambda(\mu; y)$  to  $\mu(y)$  is decreases. Also, for  $\lambda > 0$ , the absolute difference between operators  $F_{m,a,b}^\lambda(\mu; y)$  and  $\mu(y)$  is smaller than between  $F_{m,a,b}(\mu; y)$  and  $\mu(y)$ .

EXAMPLE 3. Let  $\mu(y) = y \cos(3e^y)/(1 + y^2)$  (yellow),  $\lambda = 1$ ,  $a = 0$  and  $b = 0.2$ . In Figure 3, for  $m = 10$ , we show the convergence of operators  $F_{m,a,b}^\lambda(\mu; y)$  (red),  $F_{m,a,b}(\mu; y)$  (green),  $B_m^\lambda(\mu; y)$  (purple) and  $B_m(\mu; y)$  (blue) to  $\mu(y)$ . In Table 3, with choosing  $\lambda = 1$  and for the certain values of  $0 \leq y \leq 1$ , we estimate the error of approximation operators  $F_{m,a,b}^\lambda(\mu; y)$ ,  $F_{m,a,b}(\mu; y)$ ,  $B_m^\lambda(\mu; y)$  and  $B_m(\mu; y)$  to  $\mu(y)$  for  $m = 990$ . It is clear from Table 3 that, the absolute difference between operators  $F_{m,a,b}^\lambda(\mu; y)$  and  $\mu(y)$  is smaller than between  $F_{m,a,b}(\mu; y)$  and  $\mu(y)$ , between  $B_m^\lambda(\mu; y)$  and  $\mu(y)$ , between  $B_m(\mu; y)$  and  $\mu(y)$ . Namely, the error of approximation of operators  $F_{m,a,b}^\lambda(\mu; y)$  is better than operators  $F_{m,a,b}(\mu; y)$ ,  $B_m^\lambda(\mu; y)$  and  $B_m(\mu; y)$ .

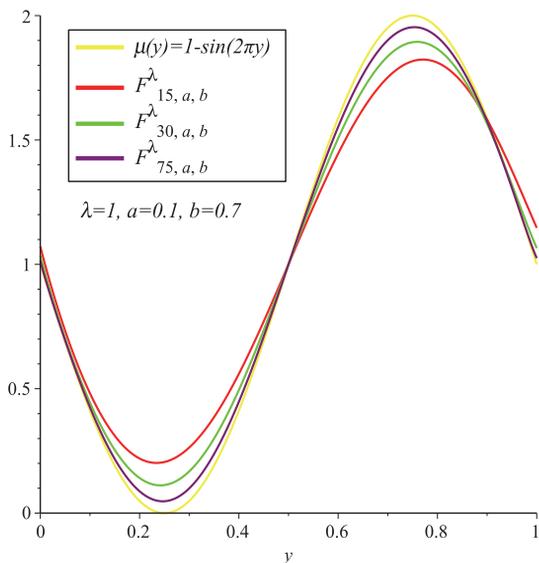


Figure 1: The convergence of operators  $F_{m,0.1,0.7}^1(\mu; y)$  to  $\mu(y) = 1 - \sin(2\pi y)$  (yellow) for  $m = 15$  (red),  $m = 30$  (green),  $m = 75$  (purple)

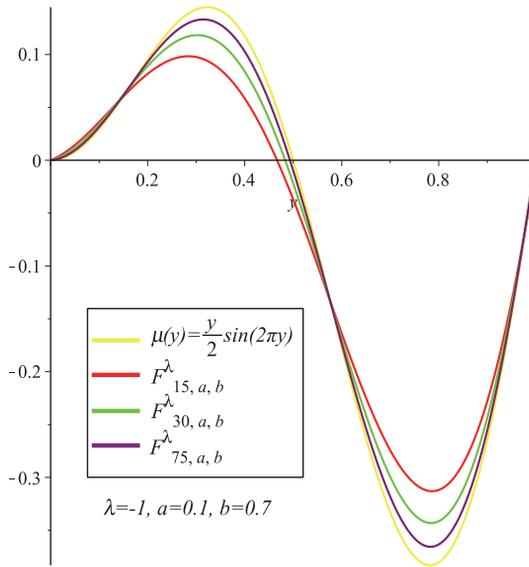


Figure 2: The convergence of operators  $F_{m,0.1,0.7}^{-1}(\mu; y)$  to  $\mu(y) = y \sin(2\pi y) / 2$  (yellow) for  $m = 15$  (red),  $m = 30$  (green),  $m = 75$  (purple)

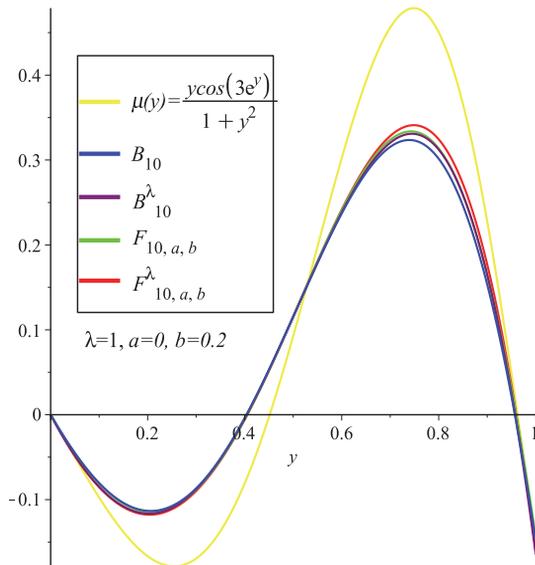


Figure 3: The convergence of operators  $F_{10,0,0.2}^1(\mu; y)$  (red),  $F_{10,0,0.2}(\mu; y)$  (green),  $B_{10}^1(\mu; y)$  (purple) and  $B_{10}(\mu; y)$  (blue) to  $\mu(y) = y \cos(3e^y) / (1 + y^2)$  (yellow)

Table 1: Error of approximation operators  $F_{m,a,b}^\lambda(\mu;y)$  to  $\mu(y) = 1 - \sin(2\pi y)$  for  $m = 15, 30, 75, 150$

$\lambda$	$ \mu(y) - F_{m,a,b}^\lambda(\mu;y) $		$y = 0.25$	$a = 0.2, b = 0.8$
	$m = 15$	$m = 30$	$m = 75$	$m = 150$
-1	0.213218040	0.113939155	0.047702899	0.024250568
-0.75	0.212240374	0.113775828	0.047691683	0.024249137
0	0.209307374	0.113285842	0.047658034	0.024244841
0.75	0.206374375	0.112795858	0.047624384	0.024240545
1	0.205396709	0.112632529	0.047613167	0.024239113

Table 2: Error of approximation operators  $F_{m,a,b}^\lambda(\mu;y)$  to  $\mu(y) = y\sin(2\pi y)/2$  for  $m = 15, 30, 75, 150$

$\lambda$	$ \mu(y) - F_{m,a,b}^\lambda(\mu;y) $		$y = 0.8$	$a = 1, b = 3$
	$m = 15$	$m = 30$	$m = 75$	$m = 150$
-1	0.041965113	0.030488789	0.015446375	0.008356444
-0.75	0.040779260	0.030354903	0.015432666	0.008353694
0	0.037221701	0.029953245	0.015391540	0.008345446
0.75	0.033664141	0.029551587	0.015350415	0.008337198
1	0.032478288	0.029417700	0.015336706	0.008334449

Table 3: Error of approximation operators  $F_{m,a,b}^\lambda(\mu;y)$ ,  $F_{m,a,b}(\mu;y)$ ,  $B_m^\lambda(\mu;y)$  and  $B_m(\mu;y)$  to  $\mu(y) = y\cos(3e^y)/(1+y^2)$  for  $m = 990$

$y$	$ \mu(y) - F_{m,a,b}^\lambda(\mu;y) $	$ \mu(y) - F_{m,a,b}(\mu;y) $	$ \mu(y) - B_m^\lambda(\mu;y) $	$ \mu(y) - B_m(\mu;y) $
0.1	0.000127691	0.000128426	0.000127700	0.000128435
0.2	0.000545082	0.000545340	0.000545187	0.000545444
0.5	0.000494954	0.000494957	0.000495172	0.000495174
0.8	0.001836189	0.001836891	0.001837952	0.001838653
0.9	0.000755498	0.000758146	0.000756985	0.000759632

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*Authors contributions.* The authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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*Reşat Aslan*  
*Department of Mathematics*  
*Faculty of Sciences and Arts*  
*Harran University*  
*63100, Haliliye, Şanlıurfa, Turkey*  
*ORCID ID: 0000-0002-8180-9199*  
*e-mail: resat63@hotmail.com*

*Mohammad Mursaleen*  
*Department of Mathematics*  
*Aligarh Muslim University*  
*Aligarh 202002, India*  
*and*  
*Department of Medical Research*  
*China Medical University Hospital*  
*China Medical University (Taiwan)*  
*Taichung, Taiwan*  
*ORCID ID: 0000-0003-4128-0427*  
*e-mail: mursaleenm@gmail.com*