

ON MULTI-INDEX WHITTAKER FUNCTION, RELATED INTEGRALS AND INEQUALITIES

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Abstract. A new generalization of Whittaker function $M_{\lambda,\mu}(z)$ is introduced and studied by means of the extended multi-index confluent hypergeometric function of the first kind $\Phi_{(\alpha_i, \beta_i)}^{(\gamma); P}$ introduced in [1]. The related Euler-type integral representation and the Laplace–Mellin and Hankel integral transforms are also presented. Functional two-sided bounding inequality is established for the multi-index Mittag-Leffler function, and in continuation functional lower bound is derived for the associated ML-extended Whittaker function.

1. Introduction and preliminaries

The Whittaker functions $M_{\lambda,\pm\mu}(z)$ are the linearly independent solutions of the Whittaker differential equation [2, 3, 4]

$$w'' + \left(\frac{\frac{1}{4} - \mu^2}{z^2} + \frac{\lambda}{z} - \frac{1}{4} \right) w = 0, \quad w = M_{\lambda,\pm\mu}(z),$$

where $z = 0$ is a branching point for $M_{\lambda,\mu}(z)$, and $z = \infty$ is an essential singularity. These functions can be represented *via* the confluent hypergeometric function (Kummer function)

$${}_1F_1[a; c; z] = \Phi(a; c; z) = \sum_{n \geq 0} \frac{(a)_n z^n}{(c)_n n!}, \quad c \in \mathbb{C} \setminus \mathbb{Z}_0^-, \quad (1.1)$$

namely, there holds

$$M_{\lambda,\mu}(z) = z^{\mu + \frac{1}{2}} e^{-\frac{z}{2}} \Phi\left(\mu - \lambda + \frac{1}{2}; 2\mu + 1; z\right), \quad (1.2)$$

where $\lambda \in \mathbb{C}$, $\min\{\Re(\mu), \Re(\mu - \lambda)\} > -\frac{1}{2}$. These kind of functions have important roles in applications of mathematics to physical and technical problems, and they are

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firmly identified *via* (1.2) by the confluent hypergeometric functions, having roles in several branches of applied mathematics and theoretical physics for example, fluid mechanics, atomic structure theory and electromagnetic diffraction.

In the sequel we need the Euler function of the first kind (or beta function)

$$B(s, t) = \int_0^1 x^{s-1} (1-x)^{t-1} dx, \quad \min\{\Re(s), \Re(t)\} > 0, \quad (1.3)$$

and the Euler function of the second kind (gamma function)

$$\Gamma(r) = \int_0^\infty x^{r-1} e^{-x} dx, \quad \Re(r) > 0.$$

These are connected *via* the formula

$$B(s, t) = \frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)}.$$

The quotient $\Gamma(r+\alpha)/\Gamma(\alpha) = (\alpha)_r$, $\Re(\alpha) > 0$ (where it is defined that $(0)_0 = 1$), is known as the Pochhammer symbol. It is used in the hypergeometric functions' series definition e.g. ${}_1F_1$ in (1.1).

As for any $a, b \in \mathbb{C}$, $\Re(c) > \Re(a) > 0$ the transformation

$$\frac{(a)_n}{(c)_n} = \frac{B(a+n, c-a)}{B(a, c-a)}$$

enables to rewrite the Kummer function into

$$\Phi(a; c; z) = \sum_{n \geq 0} \frac{B(a+n, c-a)}{B(a, c-a)} \frac{z^n}{n!}, \quad \Re(c) > \Re(a) > 0, \quad (1.4)$$

and by the same manner the familiar Gauss hypergeometric function series definition reads

$${}_2F_1(a, b; c; z) = \sum_{n \geq 0} (a)_n \frac{B(b+n, c-b)}{B(b, c-b)} \frac{z^n}{n!}, \quad \Re(a) > 0, \Re(c) > \Re(b) > 0. \quad (1.5)$$

Euler's integral representation formulae for Φ and ${}_2F_1$ are (see [5, 2])

$$\begin{aligned} \Phi(a; c; z) &= \frac{1}{B(a, c-a)} \int_0^1 x^{a-1} (1-x)^{c-a-1} e^{zx} dx \\ {}_2F_1(a, b; c; z) &= \frac{1}{B(b, c-b)} \int_0^1 x^{b-1} (1-x)^{c-b-1} (1-zx)^{-a} dx, \end{aligned} \quad (1.6)$$

respectively. The parameters' ranges are the same as in the series representations of these special functions.

2. Generalization model of Whittaker function $M_{\lambda,\mu}$

The model of generalization of the Whittaker function $M_{\lambda,\mu}(z)$ starts with inserting a function $h(x)$ in the integrand of beta function integral (1.3) so, that the resulting integral

$$B_h(s,t) = \int_0^1 x^{s-1}(1-x)^{t-1}h(x) dx \quad (2.1)$$

converges in some sense. Replacing now the beta function $B(a+n, c-a)$ which contains the summation index in the numerator in (1.4) with $B_h(a+n, c-a)$ we obtain a h -extended Kummer-type function

$$\Phi_h(a; c; z) = \sum_{n \geq 0} \frac{B_h(a+n, c-a)}{B(a, c-a)} \frac{z^n}{n!},$$

and after that by the same way from (1.5) we arrive at the associated h -extended Gaussian hypergeometric series

$$F_h(a, b; c; z) = \sum_{n \geq 0} (a)_n \frac{B_h(b+n, c-b)}{B(b, c-b)} \frac{z^n}{n!}.$$

Next, using the integral form of $B_h(\cdot, \cdot)$ in both previous sums by the integration–summation order change we infer the integral representations formulae coupled to that series. Precisely, we arrive at

$$\begin{aligned} \Phi_h(a; c; z) &= \frac{1}{B(a, c-a)} \int_0^1 x^{a-1}(1-x)^{c-a-1} e^{zx} h(x) dx \quad (2.2) \\ F_h(a, b; c; z) &= \frac{1}{B(b, c-b)} \int_0^1 x^{b-1}(1-x)^{c-b-1} (1-zx)^{-a} h(x) dx. \end{aligned}$$

Finally, by virtue of (2.2) and the definition (1.2), we resume the definition of the h -extended Whittaker function:

$$M_{\lambda,\mu;h}(z) = z^{\mu+\frac{1}{2}} e^{-\frac{z}{2}} \Phi_h\left(\mu - \lambda + \frac{1}{2}; 2\mu + 1; z\right) \quad (2.3)$$

$$= \frac{z^{\mu+\frac{1}{2}} e^{-\frac{z}{2}}}{B\left(\mu - \lambda + \frac{1}{2}, \mu + \lambda + \frac{1}{2}\right)} \int_0^1 x^{\mu-\lambda-\frac{1}{2}} (1-x)^{\mu+\lambda-\frac{1}{2}} e^{zx} h(x) dx. \quad (2.4)$$

The parameter space of the model Φ_h (and *a fortiori* for $M_{\lambda,\mu;h}$) contains $\Re(c) > \Re(a) > 0$ and depends on the behavior of h .

In the sequel we give a brief overview of the evolution of this h -model. A list of Whittaker functions by the above exposed and another methods for a few sort of special functions have been explored in [6, 7, 8, 9] and the references cited therein.

In 1997, Chaudhry *et al.* [10] presented the so-called p -extension of the beta function by introducing the exponential factor $h_p(x) = \exp\left(\frac{-p}{x(1-x)}\right)$ in the integrand of (1.3), denoting the related h -functions by B_p, Φ_p, F_p . In 2013, Nagar *et al.* [8]

generalized the Whittaker function of first kind by utilizing the extended confluent hypergeometric function Φ_p defined by Chaudhry *et al.* [11] as follows:

$$M_{\lambda,\mu;p}(z) = z^{\mu+\frac{1}{2}} e^{-\frac{z}{2}} \Phi_p\left(\mu - \lambda + \frac{1}{2}; 2\mu + 1; z\right).$$

Here Φ_p is the outcome of the exponential kernel function's use, that is when $h_p(x) = \exp\left(\frac{-p}{x(1-x)}\right)$. The parameters involved are the same as in (1.1), additionally $p \geq 0$. By setting $p = 0$, this definition clearly reduces to the classical Whittaker function (1.2).

In 2018, Shadab *et al.* [12] presented the case of the beta function specifying

$$h(x) = E_\alpha\left(-\frac{p}{x(1-x)}\right),$$

where E_α stands for the classical Mittag-Leffler function with one parameter [13, 14]

$$E_\alpha(z) = \sum_{n \geq 0} \frac{z^n}{\Gamma(1 + \alpha n)}, \quad z \in \mathbb{C}, \alpha > 0.$$

Obviously, their specification covers the one by Nagar *et al.* in [8]. In [12] the related extended Kummer $\Phi_{p,\alpha}$ and Gaussian $F_{p,\alpha}$ functions, in both-series and integral form are listed (compare and follow the h -modeling for their case stepwise).

More recently, Ali *et al.* [1] presented the generalization of beta function of the type (2.1) with $h(x)$ being the multi-index ($3s$ -parametric) Mittag-Leffler function $E_{(\alpha_i),(\beta_i)}^{(\gamma_i),s}(x)$. This function, introduced and mainly studied by Paneva-Konovska [19, 17, 18], is represented by the series

$$E_{(\alpha_i),(\beta_i)}^{(\gamma_i),s}(z) = \sum_{n \geq 0} \frac{(\gamma_1)_n \cdots (\gamma_s)_n}{\Gamma(\beta_1 + \alpha_1 n) \cdots \Gamma(\beta_s + \alpha_s n)} \frac{z^n}{(n!)^s}, \quad z \in \mathbb{C}, \quad (2.5)$$

for all $s \in \mathbb{N}_2 = \{2, 3, \dots\}$, $1 \leq i \leq s$ (see also Kiryakova [16, 15] for the particular case of $2s$ -parametric Mittag-Leffler function $E_{(\alpha_i),(\beta_i)}(z)$ with $\gamma_i = 1$). When $s = 1$ and $\alpha_1 = \alpha$, $\beta_1 = \beta$, $\gamma_1 = \gamma$, the function (2.5) is the 3-parametric Mittag-Leffler function $E_{\alpha,\beta}^\gamma(z)$, if additionally $\gamma = 1$ then (2.5) reduces to the 2-parametric Mittag-Leffler function $E_{\alpha,\beta}(z)$, and at last $E_{\alpha,1}(z)$ is the classical Mittag-Leffler function $E_\alpha(z)$ with 1 parameter (for details see e.g. [18]). The parameters are under the conditions $\alpha_i, \beta_i, \gamma_i \in \mathbb{C}$, $\Re(\alpha_i) > 0$ for $i = 1, \dots, s$. In all the results listed below the h -extended beta function, where h is the $3s$ -parameter multi-index Mittag-Leffler function, is

$$B_{(\alpha_i),(\beta_i)}^{(\gamma_i),p}(q,r) = \int_0^1 x^{q-1} (1-x)^{r-1} E_{(\alpha_i),(\beta_i)}^{(\gamma_i),s}\left(-\frac{p}{x(1-x)}\right) dx,$$

provided $\Re(q) > 0$, $\Re(r) > 0$; $p \geq 0$. Accordingly, the generalizations of extended Kummer and Gauss functions, respectively are [1]

$$\Phi_{(\alpha_i),(\beta_i)}^{(\gamma_i),p}(a;c;z) = \sum_{n \geq 0} \frac{B_{(\alpha_i),(\beta_i)}^{(\gamma_i),p}(a+n, c-a)}{B(a, c-a)} \frac{z^n}{n!}, \quad (2.6)$$

$$F_{(\alpha_i, \beta_i)}^{(\gamma_i), p}(a, b; c; z) = \sum_{n \geq 0} \frac{(a)_n \mathbf{B}_{(\alpha_i, \beta_i)}^{(\gamma_i), p}(b+n, c-b) z^n}{\mathbf{B}(b, c-b) n!}, \quad (2.7)$$

where $p \geq 0$, $\Re(c) > \Re(a)$, $\Re(b) > 0$, $\alpha_j > 0$, $\beta_j, \gamma_j \in \mathbb{R}$ and in both cases $|z| < 1$, while outside of the unit disk we take the analytic continuation.

REMARK 2.1. It is observed that, for $\gamma_1 = \dots = \gamma_s = 1$ and further $s = 2$, if we set $(\alpha_1, \alpha_2) = (1, 0)$, and $(\beta_1, \beta_1) = (1, 1)$, then (2.6) and (2.7) reduce to the extensions of Gauss and confluent hypergeometric functions defined by Chaudhry *et al.* [11].

The related integral forms for $\Phi_{(\alpha_i, \beta_i)}^{(\gamma_i), p}(a; c; z)$ and $F_{(\alpha_i, \beta_i)}^{(\gamma_i), p}(a, b; c; z)$ [1] are also the outcomes of the h -model when h is the multi-index Mittag-Leffler function (2.5). Thus,

$$\Phi_{(\alpha_i, \beta_i)}^{(\gamma_i), p}(a; c; z) = \int_0^1 \frac{x^{a-1} (1-x)^{c-a-1} e^{zx}}{\mathbf{B}(a, c-a)} E_{(\alpha_i), (\beta_i)}^{(\gamma_i), s} \left(-\frac{p}{x(1-x)} \right) dx, \quad (2.8)$$

$$F_{(\alpha_i, \beta_i)}^{(\gamma_i), p}(a, b; c; z) = \int_0^1 \frac{x^{b-1} (1-x)^{c-b-1}}{\mathbf{B}(b, c-b) (1-zx)^a} E_{(\alpha_i), (\beta_i)}^{(\gamma_i), s} \left(-\frac{p}{x(1-x)} \right) dx, \quad (2.9)$$

provided $|\arg(1-z)| < \pi$, while the ranges of other involved parameters remain the same as above.

REMARK 2.2. It is easily seen that in (2.5), for $\gamma_1 = \dots = \gamma_s = 1$ we get the known extension of beta function defined by Ghayasuddin *et al.* [20]. Further, the case when $s = 2$, on setting $(\alpha_1, \alpha_2) = (\alpha, 0)$, and $(\beta_1, \beta_2) = (1, 1)$, yields other earlier considered extensions of the Euler beta function. Next, $\alpha = 1$ reduces the Mittag-Leffler function to exponential function e^z , and taking $p = 0$ we get (1.3).

Furthermore, different extensions of beta, Gauss and confluent hypergeometric functions have been presented and explored by several researchers, see for details [1, 10, 11, 12, 20] among others.

3. New generalization of extended Whittaker function

In this section, we study a new generalization of extended Whittaker function regarding generalized extended Kummer function considered in [1]. Some basic properties of this extended Whittaker function are proposed.

The Whittaker function $M_{\lambda, \mu}(z)$ is defined in terms of the Kummer function, power and exponential functions product, see (1.2). So, it is completely natural to consider the h -extension of the Whittaker function *via* the h -extended Kummer function, consult (2.3) and (2.4). The most general variant of the input h is the $3s$ -parameter multi-index extended Kummer-type function [1], which results are appropriate to (2.6) and (2.8), are listed in brief manner in the previous section. So we have following definition.

DEFINITION 3.1. The ML-extended Whittaker function $\mathbb{M}_{(\alpha_i, \beta_i), \lambda, \mu}^{(\gamma_i), p}(z)$ we define as

$$\begin{aligned} \mathbb{M}_{(\alpha_i, \beta_i), \lambda, \mu}^{(\gamma_i), p}(z) &= \mathbb{M}_{(\alpha_1, \dots, \alpha_s, \beta_1, \dots, \beta_s), \lambda, \mu}^{(\gamma_1, \dots, \gamma_s), p}(z) \\ &= z^{\mu + \frac{1}{2}} e^{-\frac{z}{2}} \Phi_{(\alpha_i, \beta_i)}^{(\gamma_i), p}(\mu - \lambda + \frac{1}{2}; 2\mu + 1; z), \end{aligned} \quad (3.1)$$

provided $\Re(\mu) > -\frac{1}{2}$, $\Re(\mu \pm \lambda) > -\frac{1}{2}$, $\lambda \in \mathbb{C}$; $2\mu \in \mathbb{C} \setminus \mathbb{Z}^-$, where $s \in \mathbb{N}_2$ and $1 \leq i \leq s$. Here $\Phi_{(\alpha_i, \beta_i)}^{(\gamma_i), p}$ means the extended Kummer function (2.6).

We note that, If $\gamma_1 = \dots = \gamma_s = 1$ and further for $s = 2$, by setting $(\alpha_1, \alpha_2) = (1, 0)$, and $\beta_1 = \beta_2 = 1$ in (3.1) this definition reduces to the extension of Whittaker function defined by Nagar *et al.* [8], which further for $p = 0$ gives the classical Whittaker function $M_{\lambda, \mu}(z)$.

The integral representation of the ML-extended Whittaker function we obtain by inserting expression (2.6) into (3.1). Thus,

$$\begin{aligned} \mathbb{M}_{(\alpha_i, \beta_i), \lambda, \mu}^{(\gamma_i), p}(z) &= \frac{z^{\mu + \frac{1}{2}} e^{-\frac{z}{2}}}{\mathbb{B}(\mu - \lambda + \frac{1}{2}, \mu + \lambda + \frac{1}{2})} \int_0^1 x^{\mu - \lambda - \frac{1}{2}} (1-x)^{\mu + \lambda - \frac{1}{2}} e^{zx} \\ &\quad \cdot E_{(\alpha_i), (\beta_i)}^{(\gamma_i), s}\left(-\frac{p}{x(1-x)}\right) dx, \end{aligned} \quad (3.2)$$

where $\Re(\mu) > \Re(\mu \pm \lambda) > -\frac{1}{2}$, $p \geq 0$, and $\alpha_i > 0$, $\beta_i, \gamma_i \in \mathbb{R}$.

The substitution $\frac{x-a}{b-a} \mapsto x$ in (3.2) results in alternative formula, viz.

$$\begin{aligned} \mathbb{M}_{(\alpha_i, \beta_i), \lambda, \mu}^{(\gamma_i), p}(z) &= \frac{z^{\mu + \frac{1}{2}} e^{-\frac{z}{2}}}{(b-a)^{2\mu}} \int_a^b \frac{(x-a)^{\mu - \lambda - \frac{1}{2}} (b-x)^{\mu + \lambda - \frac{1}{2}}}{\mathbb{B}(\mu - \lambda + \frac{1}{2}, \mu + \lambda + \frac{1}{2})} e^{\frac{z(x-a)}{b-a}} \\ &\quad \cdot E_{(\alpha_i), (\beta_i)}^{(\gamma_i), s}\left(-\frac{p(b-a)^2}{(x-a)(b-x)}\right) dx. \end{aligned}$$

A more elegant formula follows by setting $a = -1, b = 1$:

$$\mathbb{M}_{(\alpha_i, \beta_i), \lambda, \mu}^{(\gamma_i), p}(z) = \frac{z^{\mu + \frac{1}{2}}}{4\mu} \int_{-1}^1 \frac{(1+x)^{\mu - \lambda - \frac{1}{2}} (1-x)^{\mu + \lambda - \frac{1}{2}}}{\mathbb{B}(\mu - \lambda + \frac{1}{2}, \mu + \lambda + \frac{1}{2})} e^{\frac{zx}{2}} E_{(\alpha_i), (\beta_i)}^{(\gamma_i), s}\left(-\frac{4p}{1-x^2}\right) dx.$$

Further, on substituting $\frac{x}{1+x} \mapsto x$ in (3.2), we obtain another form integral representation:

$$\begin{aligned} \mathbb{M}_{(\alpha_i, \beta_i), \lambda, \mu}^{(\gamma_i), p}(z) &= \frac{z^{\mu + \frac{1}{2}} e^{-\frac{z}{2}}}{\mathbb{B}(\mu - \lambda + \frac{1}{2}, \mu + \lambda + \frac{1}{2})} \int_0^\infty \frac{x^{\mu - \lambda - \frac{1}{2}}}{(1+x)^{2\mu + 1}} e^{\frac{zx}{1+x}} \\ &\quad \cdot E_{(\alpha_i), (\beta_i)}^{(\gamma_i), s}\left(-\frac{p(1+x)^2}{x}\right) dx. \end{aligned}$$

We note that, if $\gamma_1 = \dots = \gamma_s = 1$ and on taking $s = 2$, $(\alpha_1, \alpha_2) = (1, 0)$ and $\beta_1 = \beta_2 = 1$, all variants of (3.2) we re-obtain the integral representations by Nagar *et al.* [8]. Further, $p = 0$ yields the integral representations for the classical Whittaker function $M_{\lambda, \mu}(z)$.

THEOREM 3.2. For all $p \geq 0$ we have

$$\mathbb{M}_{(\alpha_i, \beta_i), \lambda, \mu}^{(\gamma_i), p}(-z) = (-1)^{\mu + \frac{1}{2}} \mathbb{M}_{(\alpha_i, \beta_i), -\lambda, \mu}^{(\gamma_i), p}(z).$$

Proof. The Kummer-type transformation of the confluent hypergeometric function, given in [22, p. 44, Eq. (1.18)] (see also [21]) reads,

$$\Phi(a; c; z) = e^z \Phi(c - a; c; -z),$$

and it is completely transmitted to all the successors that arose by h -modeling. Therefore, there holds

$$\Phi_{(\alpha_i, \beta_i)}^{(\gamma_i), p}(a; c; z) = e^z \Phi_{(\alpha_i, \beta_i)}^{(\gamma_i), p}(c - a; c; -z).$$

Now, having in mind the definition of ML-extended Whittaker function, we conclude that

$$\begin{aligned} \mathbb{M}_{(\alpha_i, \beta_i), \lambda, \mu}^{(\gamma_i), p}(-z) &= (-z)^{\mu + \frac{1}{2}} e^{\frac{z}{2}} \Phi_{(\alpha_i, \beta_i)}^{(\gamma_i), p}\left(\mu - \lambda + \frac{1}{2}; 2\mu + 1; -z\right) \\ &= (-1)^{\mu + \frac{1}{2}} z^{\mu + \frac{1}{2}} e^{-\frac{z}{2}} \Phi_{(\alpha_i, \beta_i)}^{(\gamma_i), p}\left(\mu - (-\lambda) + \frac{1}{2}; 2\mu + 1; z\right). \end{aligned}$$

The rest is obvious. \square

Our next goal is to present integral transforms images of ML-extended Whittaker function. Firstly let us consider the integral

$$\varphi[f; k, t] = \int_0^\infty x^{k-1} e^{-tx} f(x) dx$$

for some convenient input function $f(x)$. The resulting function $\varphi[f; k, t]$ is the so-called Laplace–Mellin transform of f , as both the Laplace exponential, and the Mellin power kernel is contained in the integrand, provided the integral converges.

Obviously, we could consider this integral as the Laplace transform of $x^{k-1} f(x)$, and also as the Mellin transform of the function $e^{-tx} f(x)$ separately. Therefore, the Laplace transform $\mathcal{L}[f](t) = \varphi[f; 1, t]$, and the Mellin transform $\mathcal{M}[f](k) = \varphi[f; k, 0]$ for certain suitable input function f .

THEOREM 3.3. For $p \geq 0$, $2t - a > 0$ and $\Re(k + \mu) > -\frac{1}{2}$, we have

$$\begin{aligned} \varphi\left[\mathbb{M}_{(\lambda_i, \beta_i), \lambda, \mu}^{(\gamma_i), p}(ax); k, t\right] &= \int_0^\infty x^{k-1} e^{-tx} \mathbb{M}_{(\alpha_i, \beta_i), \lambda, \mu}^{(\gamma_i), p}(ax) dx \\ &= \frac{a^{\mu + \frac{1}{2}} \Gamma(k + \mu + \frac{1}{2})}{(t + \frac{a}{2})^{k + \mu + \frac{1}{2}}} F_{(\alpha_i, \beta_i)}^{(\gamma_i), p}\left(\mu + \lambda + \frac{1}{2}, \mu - \lambda + \frac{1}{2}; 2\mu + 1; \frac{2a}{2t + a}\right), \quad (3.3) \end{aligned}$$

where $\left|\arg \frac{2t - a}{2t + a}\right| < \pi$.

Proof. Applying the integral representation (3.2) of ML–expanded Whittaker function, we obtain

$$\begin{aligned} \varphi \left[\mathbb{M}_{(\lambda_i, \beta_i), \lambda, \mu}^{(\gamma), p}(ax); k, t \right] &= \frac{a^{\mu + \frac{1}{2}}}{\mathbf{B}(\mu - \lambda + \frac{1}{2}, \mu + \lambda + \frac{1}{2})} \int_0^\infty x^{\mu + k - \frac{1}{2}} e^{-(t + \frac{a}{2})x} dx \\ &\quad \cdot \int_0^1 u^{\mu - \lambda - \frac{1}{2}} (1 - u)^{\mu + \lambda - \frac{1}{2}} e^{axu} E_{(\alpha_i), (\beta_i)}^{(\gamma), s} \left(-\frac{P}{u(1 - u)} \right) du \\ &= \frac{a^{\mu + \frac{1}{2}} \Gamma(k + \mu + \frac{1}{2})}{(t + \frac{a}{2})^{k + \mu + \frac{1}{2}} \mathbf{B}(\mu - \lambda + \frac{1}{2}, \mu + \lambda + \frac{1}{2})} \int_0^1 u^{\mu - \lambda - \frac{1}{2}} (1 - u)^{\mu + \lambda - \frac{1}{2}} \\ &\quad \cdot \left(1 - \frac{2au}{2t + a} \right)^{-(k + \mu + \frac{1}{2})} E_{(\alpha_i), (\beta_i)}^{(\gamma), s} \left(-\frac{P}{u(1 - u)} \right) du. \end{aligned}$$

Finally, applying the integral representation (2.9) we obtain (3.3). \square

The Laplace transform result follows by setting $k = 1$ in (3.3). In turn, for $t = 0$ the formula (3.3) becomes the Mellin transform of $\mathbb{M}_{(\lambda_i, \beta_i), \lambda, \mu}^{(\gamma), p}(ax)$. Therefore, the following corollaries can be written, respectively.

COROLLARY 3.4. For $p \geq 0, 2t - a > 0$ and $\Re(\mu) > -\frac{3}{2}$, the following identity holds true:

$$\begin{aligned} \mathcal{L} \left[\mathbb{M}_{(\alpha_i, \beta_i), \lambda, \mu}^{(\gamma), p}(ax) \right] (t) &= 2(2a)^{\mu + \frac{1}{2}} \Gamma\left(\mu + \frac{3}{2}\right) (2t + a)^{-\mu - \frac{3}{2}} \\ &\quad \cdot F_{(\alpha_i, \beta_i)}^{(\gamma), p} \left(\mu + \lambda + \frac{1}{2}, \mu - \lambda + \frac{1}{2}; 2\mu + 1; \frac{2a}{2t + a} \right), \end{aligned}$$

where $\left| \arg \frac{2t - a}{2t + a} \right| < \pi$.

COROLLARY 3.5. For $p \geq 0, 0 < a < 2$ and $\Re(k + \mu) > -\frac{1}{2}$, we have

$$\begin{aligned} \mathcal{M} \left[\mathbb{M}_{(\alpha_i, \beta_i), \lambda, \mu}^{(\gamma), p}(ax) \right] (k) &= 2^{k + \mu + \frac{1}{2}} a^{-k} \Gamma\left(k + \mu + \frac{1}{2}\right) \\ &\quad \cdot F_{(\alpha_i, \beta_i)}^{(\gamma), p} \left(\mu + \lambda + \frac{1}{2}, \mu - \lambda + \frac{1}{2}; 2\mu + 1; 2 \right). \end{aligned}$$

Finally, putting $a = 2t$ in (3.3), we produce the corollary given below.

COROLLARY 3.6. For $p \geq 0$ and $\Re(k + \mu) > -\frac{1}{2}$, there follows:

$$\varphi \left[\mathbb{M}_{(\alpha_i, \beta_i), \lambda, \mu}^{(\gamma), p}(ax); k, \frac{a}{2} \right] = a^{-k} \Gamma\left(k + \mu + \frac{1}{2}\right) F_{(\alpha_i, \beta_i)}^{(\gamma), p} \left(\mu + \lambda + \frac{1}{2}, \mu - \lambda + \frac{1}{2}; 2\mu + 1; 1 \right).$$

Let us consider now the real hypergeometric representation of the Legendre function of the first kind [23, p. 43, Eq. (29)]

$$P_v^\mu(z) = \frac{1}{\Gamma(1 - \mu)} \left(\frac{z + 1}{z - 1} \right)^{\frac{\mu}{2}} {}_2F_1 \left(-v, v + 1; 1 - \mu; \frac{1 - z}{2} \right),$$

where the parametric space is the one determined by the Gaussian function, that is $\mu, \nu \in \mathbb{R}$ and $z > 1$. Next, we call Hankel transform of the order ν of some suitable function f , the convergent integral [24, p. 3]

$$\mathcal{H}_\nu[f](t) = \int_0^\infty x J_\nu(tx) f(x) dx,$$

where the transform kernel function is the Bessel function of the first kind $J_\nu(z)$ of the order ν :

$$J_\nu(z) = \sum_{n \geq 0} \frac{(-1)^n \left(\frac{z}{2}\right)^{2n+\nu}}{\Gamma(n+\nu+1)n!}.$$

Now, we present Hankel transform results for the ML-extended Whittaker function.

THEOREM 3.7. For $\lambda, \mu \in \mathbb{C}; \Re(\mu \pm \lambda) > -\frac{1}{2}$ and $\Re(\mu + \nu) > -\frac{5}{2}$ there holds true:

$$\begin{aligned} \mathcal{H}_\nu \left[\mathbb{M}_{(\alpha_i, \beta_i), \lambda, \mu}^{(\gamma_i), p}(x) \right] (a) &= \sum_{n \geq 0} \frac{\mathbb{B}_{(\alpha_i, \beta_i)}^{(\gamma_i), p}(\mu - \lambda + \frac{1}{2} + n, \mu + \lambda + \frac{1}{2})}{\mathbb{B}(\mu - \lambda + \frac{1}{2}, \mu + \lambda + \frac{1}{2})} \\ &\cdot \frac{\Gamma(\mu + \nu + n + \frac{5}{2})}{(a^2 + \frac{1}{4})^{\frac{\mu}{2} + \frac{n}{2} + \frac{5}{4}} n!} P_{\mu+n+\frac{3}{2}}^{-\nu} \left(\frac{1}{\sqrt{4a^2+1}} \right). \end{aligned} \quad (3.4)$$

Proof. Considering (3.1) and (2.6), expanding $\mathbb{M}_{(\alpha_i, \beta_i), \lambda, \mu}^{(\gamma_i), p}(x)$ in terms of generalized extended beta function and by the legitimate change of integration and summation order we acquire

$$\begin{aligned} \int_0^\infty x \mathbb{M}_{(\alpha_i, \beta_i), \lambda, \mu}^{(\gamma_i), p}(x) J_\nu(ax) dx &= \sum_{n \geq 0} \frac{\mathbb{B}_{(\alpha_i, \beta_i)}^{(\gamma_i), p}(\mu - \lambda + \frac{1}{2} + n, \mu + \lambda + \frac{1}{2})}{\mathbb{B}(\mu - \lambda + \frac{1}{2}, \mu + \lambda + \frac{1}{2}) n!} \\ &\cdot \int_0^\infty x^{\mu+n+\frac{3}{2}} e^{-\frac{x}{2}} J_\nu(ax) dx \\ &= \sum_{n \geq 0} \frac{\mathbb{B}_{(\alpha_i, \beta_i)}^{(\gamma_i), p}(\mu - \lambda + \frac{1}{2} + n, \mu + \lambda + \frac{1}{2})}{\mathbb{B}(\mu - \lambda + \frac{1}{2}, \mu + \lambda + \frac{1}{2}) n!} \\ &\cdot \mathcal{L}[x^{\mu+n+\frac{3}{2}} J_\nu(ax)] \left(\frac{1}{2} \right). \end{aligned}$$

Applying the known Laplace transform result [25, p. 182, Eq. (9)]

$$\mathcal{L}[x^\mu J_\nu(ax)](s) = \frac{\Gamma(\mu + \nu + 1)}{(a^2 + s^2)^{\frac{\mu+1}{2}}} P_\mu^{-\nu} \left(\frac{s}{\sqrt{a^2 + s^2}} \right),$$

where $\Re(\mu + \nu) > -1$ and $\Re(s) > |\Im(a)|$ we obtain our stated result (3.4). \square

REMARK 3.8. If $s = 2$, $\gamma_1 = \gamma_2 = 1$, $(\alpha_1, \alpha_2) = (1, 0)$ and $\beta_1 = \beta_2 = 1$, we deduce from (3.4) the Hankel transform result for the extended Whittaker function defined by Nagar *et al.* [8], which reads

$$\mathcal{H}_\nu [M_{\lambda, \mu, p}(x)](a) = \sum_{n \geq 0} \frac{B_p(\mu - \lambda + \frac{1}{2} + n, \mu + \lambda + \frac{1}{2})}{B(\mu - \lambda + \frac{1}{2}, \mu + \lambda + \frac{1}{2})} \cdot \frac{\Gamma(\mu + \nu + n + \frac{5}{2})}{(a^2 + \frac{1}{4})^{\frac{\mu}{2} + \frac{\nu}{2} + \frac{5}{4}} n!} P^{-\nu} \left(\frac{1}{\sqrt{4a^2 + 1}} \right).$$

4. Functional bounding inequalities upon $\mathbb{M}_{(\alpha_i, \beta_i), \lambda, \mu}^{(\gamma), p}(x)$

In this part of the article we establish functional bounding inequalities upon the ML-extended Whittaker function $\mathbb{M}_{(\alpha_i, \beta_i), \lambda, \mu}^{(\gamma), p}(x)$ which starting point is the Fox–Wright representation of this special function. For these considerations we need the precise definition of the Fox–Wright function, which is a generalized hypergeometric function [26], [27, p. 4, Eq. (2.4)]:

$${}_m\Psi_n \left(\begin{matrix} (a_1, A_1), \dots, (a_m, A_m) \\ (b_1, B_1), \dots, (b_n, B_n) \end{matrix} \middle| z \right) = {}_m\Psi_n \left(\begin{matrix} (a_i, A_i) \\ (b_i, B_i) \end{matrix} \middle| z \right) = \sum_{k \geq 0} \frac{\prod_{j=1}^m \Gamma(a_j + kA_j)}{\prod_{j=1}^n \Gamma(b_j + kB_j)} \frac{z^k}{k!}, \quad (4.1)$$

where $A_j \geq 0$, $j = 1, \dots, m$; $B_l \geq 0$, $l = 1, \dots, n$. The convergence conditions and the radius of convergence of the series (4.1) follow from the asymptotic of involved gamma functions. So, (4.1) converges for all $z \in \mathbb{C}$ when $\Delta = 1 + \sum_{j=1}^n B_j - \sum_{j=1}^m A_j > 0$. If $\Delta = 0$, then the series converges for $|z| < \rho$, and $|z| = \rho$ when $\Re(\mu) > \frac{1}{2}$ where

$$\rho = \left(\prod_{j=1}^m A_j^{-A_j} \right) \left(\prod_{j=1}^n B_j^{B_j} \right), \quad \mu = \sum_{j=1}^n b_j - \sum_{j=1}^m a_j + \frac{m-n}{2}.$$

The Fox–Wright function extends the generalized hypergeometric function ${}_pF_q$

$${}_mF_n \left(\begin{matrix} (a_i) \\ (b_i) \end{matrix} \middle| z \right) = \sum_{k \geq 0} \frac{\prod_{j=1}^m (a_j)_k}{\prod_{j=1}^n (b_j)_k} \frac{z^k}{k!},$$

namely, if $A_r = B_s = 1$ then the Fox–Wright function ${}_m\Psi_n$ reduces to the generalized hypergeometric function (up to the multiplicative constant), i. e.

$${}_m\Psi_n \left(\begin{matrix} (a_i, 1) \\ (b_i, 1) \end{matrix} \middle| z \right) = \Gamma \left[\begin{matrix} (a_i) \\ (b_i) \end{matrix} \right] {}_mF_n \left(\begin{matrix} (a_i) \\ (b_i) \end{matrix} \middle| z \right),$$

with

$$\Gamma \left[\begin{matrix} (a_i) \\ (b_i) \end{matrix} \right] = \frac{\Gamma(a_1) \cdots \Gamma(a_m)}{\Gamma(b_1) \cdots \Gamma(b_n)}.$$

Define the parameter domain [28, p. 134, Eq. (35)]

$$\begin{aligned}
 {}_m\mathbb{D}'_n = \left\{ (a_i, A_i, b_i, B_i) : \prod_{j=1}^n \left(1 + \frac{B_j}{b_j}\right)^{B_j} \leq \prod_{j=1}^m \left(1 + \frac{A_j}{a_j}\right)^{2A_j} \left(1 + \frac{1}{a_j}\right)^{-A_j^2} \right. \\
 \left. \& \prod_{j=1}^m a_j^{A_j} \left(1 + \frac{A_j}{a_j}\right)^{A_j - \frac{1}{2}} \leq \prod_{j=1}^n \left(b_j - \frac{1 - B_j}{2}\right)^{B_j} \right\}. \tag{4.2}
 \end{aligned}$$

For all $(a_i, A_i, b_i, B_i) \in {}_m\mathbb{D}'_n$ there holds [28, p. 134, Corollary]

$$\Psi_0 e^{\Psi_1 \Psi_0^{-1} |x|} \leq {}_m\Psi_n \left(\begin{matrix} (a_i, A_i) \\ (b_i, B_i) \end{matrix} \middle| x \right) \leq \Psi_0 - \Psi_1 (1 - e^{-|x|}), \quad x \in \mathbb{R}, \tag{4.3}$$

when $2a_j + A_j > 1, 2b_k + B_k > 1; j = 1, \dots, m; k = 1, \dots, n$ and $A_j, B_k \in [0, 1]$; the bilateral equality occurs for $x = 0$. Here the constants Ψ_0 and Ψ_1 are respectively

$$\Psi_0 = \Gamma \left[\begin{matrix} (a_i) \\ (b_i) \end{matrix} \right]; \quad \Psi_1 = \Gamma \left[\begin{matrix} (a_i + A_i) \\ (b_i + B_i) \end{matrix} \right].$$

Now, we re-call the Fox–Wright function representation of the multi-index Mittag-Leffler function, in our setting written [18, p. 146, Theorem 8.6, Eq. (8.36)]

$$E_{(\alpha_i), (\beta_i)}^{(\gamma_i), s}(z) = \Gamma \left[\begin{matrix} (1)_s \\ (\gamma_i) \end{matrix} \right] {}_s\Psi_{2s-1} \left(\begin{matrix} (\gamma_i, 1) \\ (\beta_i, \alpha_i)_s, (1, 1)_{s-1} \end{matrix} \middle| z \right), \tag{4.4}$$

where $\alpha_j > 0; \beta_j, \gamma_j \in \mathbb{C}$ and $\Re(\gamma_i) > 0$, while $(\eta)_s = (\eta_1, \dots, \eta_s)$ with $\eta_i = \eta$ for $i = 1, \dots, s$.

Our next step is to use the exponential bounding inequality (4.3) for a subclass of Fox–Wright functions ${}_m\Psi_n, m \leq n$, established in [28]. Collecting all these facts, we formulate our first inequality result for a specific real argument multi-index Mittag-Leffler function.

THEOREM 4.1. *For all the positive integers s and $(\alpha_i, \beta_i, \gamma_i) \in {}_s\mathbb{D}''_{2s-1}$ and all $x \in \mathbb{R}$, where*

$${}_s\mathbb{D}''_{2s-1} = \left\{ (\alpha_i, \beta_i, \gamma_i) : \prod_{j=1}^s \left(1 + \frac{\alpha_j}{\beta_j}\right)^{\alpha_j} \frac{2\gamma_j}{1+\gamma_j} \leq 2; \prod_{j=1}^s \gamma_j (1 + \gamma_j) \leq \prod_{j=1}^s \left(\beta_j - \frac{1 - \alpha_j}{2}\right)^{2\alpha_j} \right\}, \tag{4.5}$$

when additionally $\beta_j \in [0, 1], \gamma_j > 0, 2\beta_j + \alpha_j > 1; j = 1, \dots, s$, we have

$$\Gamma \left[\begin{matrix} (\gamma_i) \\ (\beta_i) \end{matrix} \right] e^{\frac{\Gamma[(\beta_i), (\gamma_i+1)]}{\Gamma[(\alpha_i + \beta_i), (\gamma_i)]} |x|} \leq \Gamma \left[\begin{matrix} (\gamma_i) \\ (1)_s \end{matrix} \right] E_{(\alpha_i), (\beta_i)}^{(\gamma_i), s}(x) \leq \Gamma \left[\begin{matrix} (\gamma_i) \\ (\beta_i) \end{matrix} \right] - \Gamma \left[\begin{matrix} (\gamma_i + 1) \\ (\alpha_i + \beta_i) \end{matrix} \right] (1 - e^{-|x|}). \tag{4.6}$$

Proof. The parameter domain ${}_s\mathbb{D}''_{2s-1}$ one restricts to a simpler form when in (4.2) we use the substitution $(\gamma_j, 1, \alpha_j, \beta_j) \mapsto (a_j, A_j, b_j, B_j)$ and $m = s, n = 2s - 1$. Now, collecting the previously exposed results we immediately arrive the statement of the theorem. \square

REMARK 4.2. In particular, under the conditions of Theorem 4.1, the case $\gamma_i = 1$ (for all $i = 1, \dots, s$) gives the corresponding corollary referring to the $2s$ -parametric ML function $E_{(\alpha_i),(\beta_i)}(z)$. In this case the inequality (4.6) is reduced to the following:

$$\Gamma\left[\begin{matrix} (1)_s \\ (\beta_i) \end{matrix}\right] e^{\Gamma\left[\begin{matrix} (\beta_i) \\ (\alpha_i + \beta_i) \end{matrix}\right] |x|} \leq E_{(\alpha_i),(\beta_i)}(x) \leq \Gamma\left[\begin{matrix} (1)_s \\ (\beta_i) \end{matrix}\right] - \Gamma\left[\begin{matrix} (2)_s \\ (\alpha_i + \beta_i) \end{matrix}\right] (1 - e^{|x|}).$$

The case $s = 1$ is connected with the 3-parametric ML function $E_{\alpha,\beta}^\gamma(z)$. Then the inequality (4.6) takes the following simple form:

$$\frac{1}{\Gamma(\beta)} e^{\frac{\gamma\Gamma(\beta)}{\Gamma(\alpha+\beta)} |x|} \leq E_{\alpha,\beta}^\gamma(x) \leq \frac{1}{\Gamma(\beta)} - \frac{\gamma}{\Gamma(\alpha+\beta)} (1 - e^{|x|}).$$

If additionally $\gamma = 1$, the inequality (4.6) becomes

$$\frac{1}{\Gamma(\beta)} e^{\frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} |x|} \leq E_{\alpha,\beta}(x) \leq \frac{1}{\Gamma(\beta)} - \frac{1}{\Gamma(\alpha+\beta)} (1 - e^{|x|}),$$

and if also $\beta = 1$, the inequality (4.6) turns out to be

$$e^{\frac{1}{\Gamma(\alpha+1)} |x|} \leq E_\alpha(x) \leq 1 - \frac{1}{\Gamma(\alpha+1)} (1 - e^{|x|}).$$

In the most particular case $\alpha = \beta = \gamma = 1$ the inequality would be connected with the exponential function, but in this case the parameters do not satisfy the condition (4.5).

Now, it remains to apply the left-hand-side of the delivered two-sided inequality to the multi-index ML-extended Whittaker function.

THEOREM 4.3. For all the positive integers s and $(\alpha_i, \beta_i, \gamma_i) \in {}_s\mathbb{D}'_{2s-1}$, $x > 0$ we have

$$\mathbb{M}_{(\alpha_i, \beta_i), \lambda, \mu}^{(\gamma_i), p}(x) \geq C_p x^{\mu + \frac{1}{2}} e^{-\frac{x}{2}} \Phi\left(\mu - \lambda + \frac{1}{2}; 2\mu + 1; x\right), \tag{4.7}$$

where the constant

$$C_p = \frac{B(\mu - \lambda + \frac{1}{2}, 2\mu + 1)}{B(\mu - \lambda + \frac{1}{2}, \mu + \lambda + \frac{1}{2})} \Gamma\left[\begin{matrix} (1)_s \\ (\beta_i) \end{matrix}\right] \exp\left\{4p \frac{\Gamma[(\beta_i), (\gamma_i + 1)]}{\Gamma[(\alpha_i + \beta_i), (\gamma_i)]}\right\},$$

provided $\beta_j \in [0, 1]$, $\gamma_j > 0$, $2\beta_j + \alpha_j > 1$; $j = 1, \dots, s$.

Proof. Let us apply the lower bound in (4.6) to the multi-index Mittag-Leffler function in the integrand in the integral representation, viz.

$$\begin{aligned} \mathbb{M}_{(\alpha_i, \beta_i), \lambda, \mu}^{(\gamma_i), p}(x) &= x^{\mu + \frac{1}{2}} e^{-\frac{x}{2}} \int_0^1 \frac{t^{\mu - \lambda - \frac{1}{2}} (1-t)^{\mu + \lambda - \frac{1}{2}} e^{xt}}{B(\mu - \lambda + \frac{1}{2}, \mu + \lambda + \frac{1}{2})} E_{(\alpha_i), (\beta_i)}^{(\gamma_i), s}\left(-\frac{p}{t(1-t)}\right) dt \\ &\geq \frac{\Gamma\left[\begin{matrix} (\gamma_i) \\ (1)_s \end{matrix}\right]^{-1} \Gamma\left[\begin{matrix} (\gamma_i) \\ (\beta_i) \end{matrix}\right] x^{\mu + \frac{1}{2}} e^{-\frac{x}{2}}}{B(\mu - \lambda + \frac{1}{2}, \mu + \lambda + \frac{1}{2})} \int_0^1 t^{\mu - \lambda - \frac{1}{2}} (1-t)^{\mu + \lambda - \frac{1}{2}} e^{xt} \end{aligned}$$

$$\begin{aligned} & \cdot \exp \left\{ \frac{\Gamma[(\beta_i), (\gamma_i + 1)]}{\Gamma[(\alpha_i + \beta_i), (\gamma_i)]} \frac{p}{t(1-t)} \right\} dt \\ & \geq \Gamma \left[\begin{matrix} (1)_s \\ (\beta_i) \end{matrix} \right] \frac{\exp \left\{ 4p \frac{\Gamma[(\beta_i), (\gamma_i + 1)]}{\Gamma[(\alpha_i + \beta_i), (\gamma_i)]} \right\} x^{\mu + \frac{1}{2}} e^{-\frac{x}{2}}}{B(\mu - \lambda + \frac{1}{2}, \mu + \lambda + \frac{1}{2})} \int_0^1 \frac{t^{\mu - \lambda - \frac{1}{2}} e^{-xt}}{(1-t)^{\frac{1}{2} - \mu - \lambda}} dt, \end{aligned}$$

where the estimate

$$\frac{p}{t(1-t)} \geq 4p, \quad p \geq 0, t \in (0, 1),$$

was used. The last integral we clearly recognize as the Kummer function's integral form (up to a multiplicative constant), compare (1.6). However, this is equivalent to the theorem's assertion. \square

The uniform lower bound of the weight function $x^{\mu + \frac{1}{2}} e^{-\frac{x}{2}}$ is zero for positive values of the argument earned by the inequality (4.6). However, a functional upper bound cannot be established by virtue of the right-hand-side estimate in (4.6).

5. Concluding remarks

In this research note we present an extension of Whittaker function based on the use of the multi-index Mittag-Leffler extended confluent hypergeometric function of the Kummer type, which was investigated in [1]. First we derive some basic properties of the extended Whittaker function, then we focus to a so-called h -extension model of the hypergeometric type functions which series and integral representations are obtained. The Whittaker functions also belong to this function class, therefore taking h to be the multi-index Mittag-Leffler function, we study the extensions of that kind of extension for real argument Whittaker function $\mathbb{M}_{(\alpha_i, \beta_i), \lambda, \mu}^{(\gamma_i), p}(x)$. Next, Laplace-Mellin and Hankel integral transform results are presented with some associated corollaries.

We close the exposition with a separate section devoted to functional inequalities: firstly, a bilateral bounding inequality is established for the real argument multi-index Mittag-Leffler function $E_{(\alpha_i), (\beta_i)}^{(\gamma_i), s}(x)$. In accordance with our knowledge the only such upper bound result is [17, p. 1090, Theorem 2.1., Eq. (2.3)] exposed also in the monograph [18, p. 140, Theorem 8.4.].

Secondly, with the aid of the obtained inequality for the ML-function, we finally infer a functional lower bound for the multi-index ML-extended Whittaker function $\mathbb{M}_{(\alpha_i, \beta_i), \lambda, \mu}^{(\gamma_i), p}(x)$.

Thirdly, the readers' attention is referred to the structure of the parameters' domain ${}_s\mathbb{D}_{2s-1}''$. Namely, the exhaustive discussion of the shape of the parameters space in [28, pp. 131–133] is based on the Gautschi quotient and Gurland's ratio, which gives precise, but hardly handleable constraint inequalities. These inequalities are modestly weakened getting ${}_m\mathbb{D}'_n$ by [28, p. 134, Lemma 2] together with the bilateral inequality [28, p. 134, Lemma 3], which is a consequence of a Lazarević–Lupaş inequality [29] from one, and of a Chao Ping Chen–Feng Qi's result [30] from the other side.

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