

## WEIGHTED A-STATISTICAL CONVERGENCE AND BÖGEL APPROXIMATION BY OPERATORS OF EXPONENTIAL TYPE

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*Abstract.* We give a proper definition of an exponential type operator proposed by Ismail and May [21] so that it acts from  $C(S)$  into  $C(S)$  and explore its non-multiplicativity,  $r^{\text{th}}$  order generalization and weighted A-statistical convergence in the univariate case. Next, we define properly the associated tensor product of the operators and investigate its approximation properties. Lastly, we introduce the associated Generalized Boolean Sum (GBS) operators and present error estimates using mixed modulus of smoothness for Bögel continuous functions.

### 1. Introduction

For  $f \in C[0, 1)$  and each  $\sigma \in \mathbb{N}$ , Ismail and May ([21], p. 457) defined an exponential type operator as

$$\mathfrak{R}_\sigma(f; x) = e^{-\sigma x} \sum_{\ell=0}^{\infty} \frac{\sigma(\sigma + \ell)^{\ell-1}}{\ell!} (xe^{-x})^\ell f\left(\frac{\ell}{\sigma + \ell}\right), \quad 0 \leq x < 1 \quad (1.1)$$

and studied some local direct and inverse theorems. Gupta and Agrawal [19] introduced a link operator of an alternate form ([21], p. 457, Eq. (3.14)) of the operator (1.1) by means of Paltanea basis functions and studied some direct results and Voronovskaja type asymptotic theorems. Considering the definition (1.1) of  $\mathfrak{R}_\sigma$  to be true for the space  $C(S) := \{\psi : S \rightarrow \mathbb{R} : \psi \text{ is continuous}\}$ ,  $S = [0, 1]$ . Lipi and Deo [28] studied the Bezier variant and the tensor product of these operators for the functions in  $C(S)$  and  $C(S^2)$ ,  $S^2 = S \times S$  respectively. Mishra and Deo [29] defined a Kantorovich variant of the operators (1.1) on  $C(S)$  and proved some direct theorems for the univariate and bivariate cases by means of the modulus of continuity and the Peetre's K-functional.

We observe here that for the operator (1.1) to be defined from  $C(S)$  into  $C(S)$ , a proper modification in its definition needs to be made as follows:

$$\mathfrak{R}_\sigma(f; x) = e^{-\sigma x} \sum_{\ell=0}^{\infty} \frac{\sigma(\sigma + \ell)^{\ell-1}}{\ell!} (xe^{-x})^\ell f\left(\frac{\ell}{\sigma + \ell}\right), \quad 0 \leq x < 1 \quad (1.2)$$

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and  $\mathfrak{R}_\sigma(f; 1) = f(1)$ ,  $x = 1$ . Then it turns out that  $\lim_{\sigma \rightarrow \infty} \mathfrak{R}_\sigma(f; x) = f(x)$ , uniformly in  $x \in S$  for all  $f \in C(S)$  and so  $\mathfrak{R}_\sigma$  is well defined for each positive integer  $\sigma$ .

In the present paper, first we study a Grüss Voronovskaja type theorem and then introduce a  $r^{\text{th}}$  order generalization of the operators (1.2) to study the approximation behaviour for the elements of the sufficiently smooth Lipschitz class functions. Next, we study the Korovkin type theorem and Voronovskaja type theorem for the weighted A-statistical convergence of these operators. Further, we investigate the approximation degree of the tensor product of the operators (1.2) with the aid of the total and partial modulus of continuity, Voronovskaja type theorem and the Peetre's K-functional. Lastly, we define the associated GBS operator and determine the error in the approximation by means of the mixed modulus of smoothness for functions in the Bögel space.

In what follows, let  $C(S)$  be endowed with the norm  $\|f\|_{C(S)} = \sup_{x \in S} |f(x)|$ .

LEMMA 1. [28] For the operator  $\mathfrak{R}_\sigma(\cdot; x)$  given by (1.2) and  $0 \leq x < 1$ , we have

- (i)  $\mathfrak{R}_\sigma(1; x) = 1$ ;
- (ii)  $\mathfrak{R}_\sigma(s; x) = \frac{\sigma}{\sigma + 1}x$ ;
- (iii)  $\mathfrak{R}_\sigma(s^2; x) = \frac{\sigma^2}{(\sigma + 1)(\sigma + 2)}x^2 + \frac{\sigma}{(\sigma + 1)^2}x$ ;
- (iv)  $\mathfrak{R}_\sigma(s^3; x) = \frac{\sigma^3}{(\sigma + 1)(\sigma + 2)(\sigma + 3)}x^3 + \frac{\sigma^2(3\sigma + 4)}{(\sigma + 1)^2(\sigma + 2)^2}x^2 + \frac{\sigma}{(\sigma + 1)^3}x$ ;
- (v)  $\mathfrak{R}_\sigma(s^4; x) = \frac{\sigma^4}{(\sigma + 1)(\sigma + 2)(\sigma + 3)(\sigma + 4)}x^4 + \frac{2\sigma^3(3\sigma^2 + 11\sigma + 9)}{(\sigma + 1)^2(\sigma + 2)^2(\sigma + 3)^2}x^3$   
 $+ \frac{\sigma^2(7\sigma^2 + 18\sigma + 12)}{(\sigma + 1)^3(\sigma + 2)^3}x^2 + \frac{\sigma}{(\sigma + 1)^4}x$ .

LEMMA 2. [28] For  $0 \leq x < 1$ , the central moments for the operator  $\mathfrak{R}_\sigma$  are given by:

- (i)  $\mathfrak{R}_\sigma((s - x); x) = \frac{-x}{\sigma + 1}$ ;
- (ii)  $\mathfrak{R}_\sigma((s - x)^2; x) = \frac{-(\sigma - 2)}{(\sigma + 1)(\sigma + 2)}x^2 + \frac{\sigma}{(\sigma + 1)^2}x$ ;
- (iii)  $\mathfrak{R}_\sigma((s - x)^4; x) = \frac{3\sigma^2 - 46\sigma + 24}{(\sigma + 1)(\sigma + 2)(\sigma + 3)(\sigma + 4)}x^4$   
 $+ \frac{(-6\sigma^3 + 36\sigma^2 + 216\sigma + 216)\sigma x^3}{(\sigma + 1)^2(\sigma + 2)^2(\sigma + 3)^2}$   
 $+ \sigma \frac{(3\sigma^3 - 6\sigma^2 - 36\sigma - 32)}{(\sigma + 1)^3(\sigma + 2)^3}x^2 + \frac{\sigma x}{(\sigma + 1)^2}$ .

REMARK 1. [28] For  $0 \leq x < 1$ , there hold the following results:

- (i)  $\lim_{\sigma \rightarrow \infty} \sigma \mathfrak{R}_\sigma((s-x); x) = -x$ ;
- (ii)  $\lim_{\sigma \rightarrow \infty} \sigma \mathfrak{R}_\sigma((s-x)^2; x) = x(1-x)$ ;
- (iii)  $\lim_{\sigma \rightarrow \infty} \sigma^2 \mathfrak{R}_\sigma((s-x)^4; x) = 3x^2(x-1)^2$ .

LEMMA 3. [28] Let  $f \in C(S)$ . Then we have

$$\|\mathfrak{R}_\sigma(f)\|_{C(S)} \leq \|f\|_{C(S)}.$$

## 2. $r^{\text{th}}$ generalization of the operator

From the investigations of Voronovskaja [36] and Korovkin [27], it turned out that the linear positive operators possess a very slow rate of convergence  $O(n^{-1})$ , as  $n \rightarrow \infty$ , even when the function is sufficiently smooth. To address this problem, Kirov and Popova [26], considered a generalization of linear positive operators by means of a Taylor polynomial and established some approximation results. Inspired by their method, we introduce a  $r^{\text{th}}$  order ( $r \in \mathbb{N}$ ) generalization of (1.2) as follows:

$$\mathfrak{R}_{\sigma,r}(f; x) = e^{-\sigma x} \sum_{\ell=0}^{\infty} \frac{\sigma(\sigma+\ell)^{\ell-1}}{\ell!} (xe^{-x})^\ell \left( \sum_{j=0}^r f^{(j)} \left( \frac{x}{\sigma+\ell} \right) \frac{\left(x - \frac{\ell}{\sigma+\ell}\right)^j}{j!} \right), \quad 0 \leq x < 1 \quad (2.1)$$

and  $\mathfrak{R}_{\sigma,r}(f; 1) = f(1)$ .

THEOREM 1. Let  $f \in C^r(S)$  and  $f^{(r)} \in Lip_M(\zeta)$ ,  $0 \leq \zeta < 1$  then for any  $\sigma \in \mathbb{N}$

$$\|f - \mathfrak{R}_{\sigma,r}(f)\|_{C(S)} \leq \frac{M}{(r-1)!} \frac{\zeta}{\zeta+r} B(\zeta, r) \left\| \mathfrak{R}_\sigma(|x-t|^{r+\zeta}) \right\|_{C(S)},$$

where  $B(\zeta, r)$  is Beta function.

*Proof.* From equation (2.1), we may write

$$\begin{aligned} & f(x) - \mathfrak{R}_{\sigma,r}(f; x) \\ &= e^{-\sigma x} \sum_{\ell=0}^{\infty} \frac{\sigma(\sigma+\ell)^{\ell-1}}{\ell!} (xe^{-x})^\ell \left( f(x) - \sum_{j=0}^r f^{(j)} \left( \frac{\ell}{\sigma+\ell} \right) \frac{\left(x - \frac{\ell}{\sigma+\ell}\right)^j}{j!} \right). \end{aligned} \quad (2.2)$$

In view of Taylor’s formula [26], we get

$$\begin{aligned}
 f(x) - \sum_{j=0}^r f^{(j)}\left(\frac{\ell}{\sigma+\ell}\right) \frac{\left(x - \frac{\ell}{\sigma+\ell}\right)^j}{j!} & \tag{2.3} \\
 = \frac{\left(x - \frac{\ell}{\sigma+\ell}\right)^r}{(r-1)!} \int_0^1 (1-v)^{r-1} \left( f^{(r)}\left(\frac{\ell}{\sigma+\ell} + v\left(x - \frac{\ell}{\sigma+\ell}\right)\right) - f^{(r)}\left(\frac{\ell}{\sigma+\ell}\right) \right) dv.
 \end{aligned}$$

Since  $f^{(r)} \in Lip_M(\zeta)$ ,

$$\left| f^{(r)}\left(\frac{\ell}{\sigma+\ell} + v\left(x - \frac{\ell}{\sigma+\ell}\right)\right) - f^{(r)}\left(\frac{\ell}{\sigma+\ell}\right) \right| \leq Mv^\zeta \left| x - \frac{\ell}{\sigma+\ell} \right|^\zeta. \tag{2.4}$$

Using the properties of Beta function, we can write

$$\int_0^1 v^\zeta (1-v)^{r-1} dv = B(1+\zeta, r) = \frac{\zeta}{\zeta+r} B(\zeta, r). \tag{2.5}$$

Using (2.4) and (2.5) in (2.3), we get

$$\begin{aligned}
 \left| f(x) - \sum_{j=0}^r f^{(j)}\left(\frac{\ell}{\sigma+\ell}\right) \frac{\left(x - \frac{\ell}{\sigma+\ell}\right)^j}{j!} \right| & \leq M \frac{\left| x - \frac{\ell}{\sigma+\ell} \right|^{r+\zeta}}{(r-1)!} \int_0^1 v^\zeta (1-v)^{r-1} dv \\
 & = M \frac{\left| x - \frac{\ell}{\sigma+\ell} \right|^{r+\zeta}}{(r-1)!} \frac{\zeta}{\zeta+r} B(\zeta, r).
 \end{aligned}$$

Therefore from (2.2), we have

$$\|f - \mathfrak{A}_{\sigma,r}(f)\|_{C(S)} \leq \frac{\zeta}{\zeta+r} B(\zeta, r) \|\mathfrak{A}_\sigma(|x-t|^{\eta+\zeta})\|_{C(S)}. \quad \square$$

REMARK 2. Consider the function  $f^*$  defined as

$$f^*(t) = |t-x|^{r+\zeta}.$$

Using the well known inequality

$$|a^\alpha - b^\alpha| \leq \alpha|a-b|, \text{ for } 0 \leq a, b \leq 1 \text{ and } \alpha \geq 1,$$

it follows that  $f^* \in Lip_{r+\zeta} 1$ . Since  $f^*(x) = 0$ , we have

$$\|\mathfrak{A}_\sigma(f^*)\| \leq \left(1 + \frac{\eta_{\sigma,2}}{\delta^2}\right) \omega(f^*; \delta)$$

where  $\eta_{\sigma,2} = \|\mathfrak{A}_\sigma(t-x)^2\|_{C(S)}$ . Choosing  $\delta = \sqrt{\eta_{\sigma,2}}$ , from Theorem 1, we obtain

$$\|f - \mathfrak{A}_{\sigma,r}(f)\|_{C(S)} \leq \frac{2M}{(r-1)!} \frac{\zeta}{\zeta+r} B(\zeta, r) (r+\zeta) \sqrt{\eta_{\sigma,2}}.$$

### 3. Weighted A-statistical convergence

In 1959, the concept of the statistical convergence was given by Zygmund [38]. In the same year, Fast [17] and Steinhaus [34] independently introduced statistical convergence to assign a limit to the sequences which are not convergent in the usual sense.

In 2009, the concept of weighted statistical convergence was defined by Karakaya and Chishti [24] and further studied by Mursaleen et al. [31]. Mohiuddine [30] introduced the notion of statistical weighted A-summability of a sequence and established its relation with the weighted A-statistical convergence.

Our aim in this section is to study the weighted A-statistical approximation properties of the operator  $\mathfrak{R}_\sigma$ . Let us begin with some notations and definitions as follows:

The notion of the asymptotic (or natural) density of a set  $J \subset \mathbb{N}$  is defined as:

$$\delta(J) = \lim_{\sigma} \frac{1}{\sigma} |\{\ell \leq \sigma : \ell \in J\}|,$$

whenever the limit exists. A sequence  $(x_\ell)$  is called statistically convergent to a number  $v$  if for every  $\varepsilon > 0$ ,

$$\lim_{\sigma} \frac{1}{\sigma} |\{\ell \leq \sigma : |x_\ell - v| \geq \varepsilon\}| = 0.$$

Let  $A = (a_{\sigma\ell})$  be a non-negative infinite summability matrix. For a given sequence  $(x_\ell)$ , the A-transform  $(Ax)_\sigma$  is defined as

$$(Ax)_\sigma = \sum_{\ell=1}^{\infty} a_{\sigma\ell} x_\ell$$

provided the series converges for each  $\sigma$ .

In particular, if  $A = C_1$ , the Cesàro matrix of order one, the A-statistical convergence reduces to the statistical convergence. However, if we consider  $A = I$ , the identity matrix, then A-statistical convergence is same as the usual convergence. For further details, one can refer to ([15], [16], [18] and [23] etc.).

Let  $t = (t_\ell)$  be a sequence of non-negative real numbers such that  $t_1 > 0$  and  $T_\sigma = \sum_{\ell=1}^{\sigma} t_\ell \rightarrow \infty$ , as  $\sigma \rightarrow \infty$ . An infinite matrix  $A = (a_{\sigma\ell})$  is called weighted regular matrix if

$$\lim_{\sigma} \frac{1}{T_\sigma} \sum_{m=1}^{\sigma} \sum_{\ell=1}^{\infty} t_m a_{m\ell} x_\ell = l,$$

whenever  $\lim_{\sigma} x_\sigma = l$ .

If  $A = (a_{\sigma\ell})$  be a non-negative weighted regular matrix then a sequence  $x = (x_\ell)$  of real or complex numbers is said to be weighted A-statistically convergent to a number  $L$ , if for every  $\varepsilon > 0$ ,

$$\lim_{\sigma} \frac{1}{T_\sigma} \sum_{m=1}^{\sigma} \sum_{\ell \in E_\varepsilon} t_m a_{m\ell} = 0,$$

where  $E_\varepsilon = \{\ell \in \mathbb{N} : |x_\ell - L| \geq \varepsilon\}$ . In this case, we write  $\tilde{s}_A^W - \lim_\sigma x_\sigma = L$ . In particular, if  $t_m = 1$ , for all  $m = 1, \dots, \sigma$  then the weighted A-statistical convergence reduces to the A-statistical convergence.

Let  $A = (a_{\sigma\ell})$  be a non negative weighted regular matrix and  $(b_\sigma)$  be a positive non increasing sequence. Then the sequence  $(x_\sigma)$  is said to converge weighted A-statistically to the number  $l$  with the rate  $o(b_\sigma)$  provided for every  $\varepsilon > 0$ ,

$$\lim_\sigma \frac{1}{b_\sigma} \left\{ \frac{1}{T_\sigma} \sum_{m=1}^\sigma \sum_{\ell \in E_\varepsilon} t_m a_{m\ell} \right\} = 0.$$

We denote it as  $x_\sigma - l = \tilde{s}_A^W - o(b_\sigma)$ . The sequence  $(x_\sigma)$  is called weighted A-statistically bounded with the rate  $O(b_\sigma)$ , if for every  $\varepsilon > 0$ ,  $\sup_\sigma \frac{1}{b_\sigma} \left\{ \frac{1}{T_\sigma} \sum_{m=1}^\sigma \sum_{\ell \in E_\varepsilon} t_m a_{m\ell} \right\} < \infty$ , and it is denoted by  $x_\sigma = \tilde{s}_A^W - O(b_\sigma)$ , as  $\sigma \rightarrow \infty$ .

Throughout this section, let us assume that  $A = (a_{\sigma\ell})$  is a non-negative weighted regular matrix and  $t = (t_\ell)$  is a given sequence of non-negative integers such that  $t_1 > 0$  and  $T_\sigma = \sum_{\ell=1}^\sigma t_\ell \rightarrow \infty$ , as  $\sigma \rightarrow \infty$ .

First, we establish the basic convergence theorem for the operators (1.2) in weighted A-statistical approximation.

**THEOREM 2.** *For any function  $f \in C(S)$ , we have*

$$\tilde{s}_A^W - \lim_\sigma \| \mathfrak{R}_\sigma f - f \|_{C(S)} = 0.$$

*Proof.* Following ([18], Theorem 1), it is enough to show that  $\tilde{s}_A^W - \lim_\sigma \| \mathfrak{R}_\sigma(e_i) - e_i \|_{C(S)} = 0$ , where  $e_i(s) = s^i$ ,  $i = 0, 1, 2$ . Applying Lemma 1, we have

$$\| \mathfrak{R}_\sigma(e_0) - e_0 \|_{C(S)} = 0.$$

Hence,

$$\tilde{s}_A^W - \lim_\sigma \| \mathfrak{R}_\sigma(e_0) - e_0 \|_{C(S)} = 0.$$

Again from Lemma 1,

$$\begin{aligned} \| \mathfrak{R}_\sigma(e_1) - e_1 \|_{C(S)} &= \sup_{x \in S} \left| \frac{\sigma x}{\sigma + 1} - x \right| \\ &= \frac{1}{\sigma + 1} = A(\sigma). \end{aligned}$$

Now, let us define the sets:

$$\Pi = \{ \sigma \in \mathbb{N} : \| \mathfrak{R}_\sigma(e_1) - e_1 \|_{C(S)} \geq \varepsilon \}$$

and

$$\Pi_1 = \left\{ \sigma \in \mathbb{N} : \frac{1}{\sigma+1} \geq \varepsilon \right\}.$$

Then,  $\Pi \subseteq \Pi_1$  implies that

$$\frac{1}{T_\sigma} \sum_{m=1}^{\sigma} t_m \sum_{\ell \in \Pi} a_{m\ell} \leq \frac{1}{T_\sigma} \sum_{m=1}^{\sigma} t_m \sum_{\ell \in \Pi_1} a_{m\ell}.$$

Since

$$\tilde{s}t_A^W - \lim_{\sigma} \frac{1}{\sigma+1} = 0,$$

we have

$$\tilde{s}t_A^W - \lim_{\sigma} \| \mathfrak{R}_\sigma(e_1) - e_1 \|_{C(S)} = 0.$$

Similarly,

$$\begin{aligned} \| \mathfrak{R}_\sigma(e_2) - e_2 \|_{C(S)} &= \sup_{x \in S} \left| \frac{\sigma^2 x^2}{(\sigma+1)(\sigma+2)} + \frac{\sigma x}{(\sigma+1)^2} - x^2 \right|; \\ &\leq \frac{3\sigma+2}{(\sigma+1)(\sigma+2)} + \frac{\sigma}{(\sigma+1)^2}. \end{aligned}$$

Let us consider the sets:

$$\Pi_2 = \{ \sigma \in \mathbb{N} : \lim_{\sigma} \| \mathfrak{R}_\sigma(e_2) - e_2 \|_{C(S)} \geq \varepsilon \}$$

$$\Pi_3 = \left\{ \sigma \in \mathbb{N} : \frac{3\sigma+2}{(\sigma+1)(\sigma+2)} \geq \frac{\varepsilon}{2} \right\}$$

$$\Pi_4 = \left\{ \sigma \in \mathbb{N} : \frac{\sigma}{(\sigma+1)^2} \geq \frac{\varepsilon}{2} \right\}.$$

Then, we can write  $\Pi_2 \subseteq \Pi_3 \cup \Pi_4$ , which leads us to

$$\frac{1}{T_\sigma} \sum_{m=1}^{\sigma} t_m \sum_{\ell \in \Pi_2} a_{m\ell} \leq \frac{1}{T_\sigma} \sum_{m=1}^{\sigma} t_m \sum_{\ell \in \Pi_3} a_{m\ell} + \frac{1}{T_\sigma} \sum_{m=1}^{\sigma} t_m \sum_{\ell \in \Pi_4} a_{m\ell}.$$

Now, since

$$\tilde{s}t_A^W - \lim_{\sigma} \frac{3\sigma+2}{(\sigma+1)(\sigma+2)} = 0,$$

and

$$\tilde{s}t_A^W - \lim_{\sigma} \frac{\sigma}{(\sigma+1)^2} = 0,$$

it follows that

$$\tilde{st}_A^W - \lim_{\sigma} \| \mathfrak{R}_{\sigma}(e_2) - e_2 \|_{C(S)} = 0. \quad \square$$

Next, we establish the Voronovskaja type theorem for the operators given by (1.2) in weighted A-statistical approximation.

**THEOREM 3.** *For every  $f'' \in C(S)$ , we have*

$$\tilde{st}_A^W - \lim_{\sigma} \sigma(\mathfrak{R}_{\sigma}(f; x) - f(x)) = -xf'(x) + \frac{x(1-x)}{2} f''(x),$$

uniformly with respect to  $x \in S$ .

*Proof.* For  $f'' \in C(S)$ , we can write

$$f(s) = f(x) + (s-x)f'(x) + \frac{(s-x)^2}{2!} f''(x) + (s-x)^2 \Theta(s, x),$$

where  $\Theta(s, x) \in C(S)$  and  $\Theta(s, x) \rightarrow 0$  as  $s \rightarrow x$ . Operating by the operator  $\mathfrak{R}_{\sigma}(\cdot; x)$  on the above equation, we get

$$\begin{aligned} & \sigma(\mathfrak{R}_{\sigma}(f; x) - f(x)) \\ &= f'(x) \sigma \mathfrak{R}_{\sigma}((s-x); x) + \sigma \frac{\mathfrak{R}_{\sigma}((s-x)^2; x)}{2!} f''(x) + \sigma \mathfrak{R}_{\sigma}((s-x)^2 \Theta(s, x); x). \end{aligned}$$

In view of Remark 1, we get

$$\begin{aligned} & \tilde{st}_A^W - \lim_{\sigma} \sigma(\mathfrak{R}_{\sigma}(f; x) - f(x)) \\ &= -xf'(x) + \frac{x(1-x)}{2} f''(x) + \tilde{st}_A^W - \lim_{\sigma} \sigma \mathfrak{R}_{\sigma}((s-x)^2 \Theta(s, x); x). \end{aligned}$$

Using Cauchy-Schwarz inequality

$$\sigma | \mathfrak{R}_{\sigma}((s-x)^2 \Theta(s, x); x) | \leq \sqrt{\sigma^2 \mathfrak{R}_{\sigma}((s-x)^4; x)} \sqrt{\mathfrak{R}_{\sigma}(\Theta^2((s, x); x))},$$

it is clear that

$$\tilde{st}_A^W - \lim_{\sigma} \sigma \mathfrak{R}_{\sigma}((s-x)^2 \Theta(s, x); x) = 0,$$

uniformly in  $x \in S$ , as by Theorem 2,

$$\tilde{st}_A^W - \lim_{\sigma} \mathfrak{R}_{\sigma}(\Theta^2(s, x); x) = \Theta^2(x, x) = 0,$$

uniformly in  $x \in S$ , since  $\Theta^2(s, x) \in C(S)$ , and  $\tilde{st}_A^W - \lim_{\sigma} \sigma^2 \mathfrak{R}_{\sigma}((s-x)^4; x) = 3x^2(x-1)^2$ , uniformly in  $x \in S$ , from Remark 1.  $\square$

The following theorem determines the rate of weighted A-statistical convergence by  $\mathfrak{R}_{\sigma}(f)$ , for  $f \in C(S)$ .

**THEOREM 4.** *Let  $f \in C(S)$  and  $(b_\sigma)$  be a positive non-increasing sequence. Assume that,*

$$\omega(f; \sqrt{\eta_{\sigma,2}}) = \tilde{st}_A^W - o(b_\sigma), \text{ as } \sigma \rightarrow \infty.$$

*Then the operators  $\mathfrak{R}_\sigma$  verify*

$$\|\mathfrak{R}_\sigma(f) - f\|_{C(S)} = \tilde{st}_A^W - o(b_\sigma), \text{ as } \sigma \rightarrow \infty,$$

*where  $\eta_{\sigma,2}$  is defined as in Remark 3.*

*Proof.* In view of Lemma 1,

$$\begin{aligned} \|\mathfrak{R}_\sigma(f) - f\|_{C(S)} &\leq \left\{ 1 + \frac{1}{\delta^2} \|\mathfrak{R}_\sigma((t-x)^2)\|_{C(S)} \right\} \omega(f; \delta), \delta > 0 \\ &= 2\omega(f; \sqrt{\eta_{\sigma,2}}), \end{aligned}$$

where  $\delta = \sqrt{\eta_{\sigma,2}}$ .

Hence, for any  $\varepsilon > 0$

$$\frac{1}{b_\sigma} \left\{ \frac{1}{T_\sigma} \sum_{m=1}^\sigma \sum_{\ell: \|R_\ell(f) - f\|_{C(S)} \geq \varepsilon} t_m a_{m\ell} \right\} \leq \frac{1}{b_\sigma} \left\{ \frac{1}{T_\sigma} \sum_{m=1}^\sigma \sum_{\ell: 2\omega(f; \sqrt{\eta_{\ell,2}}) \geq \varepsilon} t_m a_{m\ell} \right\}. \quad (3.1)$$

Since

$$\omega(f; \sqrt{\eta_{\sigma,2}}) = \tilde{st}_A^W - o(b_\sigma),$$

from (3.1) it follows that

$$\lim_\sigma \frac{1}{b_\sigma} \left\{ \frac{1}{T_\sigma} \sum_{m=1}^\sigma \sum_{\ell: \|R_\ell(f) - f\|_{C(S)} \geq \varepsilon} t_m a_{m\ell} \right\} = 0,$$

hence

$$\|\mathfrak{R}_\sigma(f) - f\|_{C(S)} = \tilde{st}_A^W - o(b_\sigma), \text{ as } \sigma \rightarrow \infty. \quad \square$$

Our next result is an asymptotic convergence theorem in weighted A-statistical approximation by means of the modulus of continuity.

**THEOREM 5.** *Let  $f'' \in C(S)$  and  $(c_\sigma)$  be a positive non-increasing sequence such that  $\omega(f; \sigma^{-1}) = \tilde{st}_A^W - o(c_\sigma)$ , as  $\sigma \rightarrow \infty$  then for each  $x \in S$ ,*

$$\begin{aligned} &\sigma \left| \mathfrak{R}_\sigma(f; x) - f(x) + \frac{x}{\sigma+1} f'(x) - \frac{f''(x)}{2!} \left\{ \frac{-(\sigma-2)x^2}{(\sigma+1)(\sigma+2)} + \frac{\sigma}{(\sigma+1)^2 x} \right\} \right| \\ &= \tilde{st}_A^W - o(c_\sigma), \text{ as } \sigma \rightarrow \infty. \end{aligned}$$

*Proof.* Since  $f'' \in C(S)$ , we have

$$f(t) = f(x) + (t-x)f'(x) + \frac{(t-x)^2}{2!}f''(x) + \int_x^t (t-u)(f''(u) - f''(x))du.$$

We apply the operator  $\mathfrak{R}_\sigma(\cdot; x)$  to the above equation and use Lemma 2. Then,

$$\begin{aligned} & \left| \mathfrak{R}_\sigma(f; x) - f(x) + \frac{x}{\sigma+1}f'(x) - \frac{f''(x)}{2!} \left\{ \frac{-(\sigma-2)x^2}{(\sigma+1)(\sigma+2)} + \frac{\sigma}{(\sigma+1)^2x} \right\} \right| \\ & \leq \mathfrak{R}_\sigma \left( \left| \int_t^x |t-u| |f''(u) - f''(x)| du \right| ; x \right). \end{aligned} \tag{3.2}$$

Using the elementary inequality

$$|f''(u) - f''(x)| \leq \left\{ 1 + \frac{(u-x)^2}{\delta^2} \right\} \omega(f''; \delta), \quad \delta > 0, \quad u, x \in S,$$

and the Remark 1,

$$\begin{aligned} & \mathfrak{R}_\sigma \left( \left| \int_t^x |t-u| |f''(u) - f''(x)| du \right| ; x \right) \\ & \leq \mathfrak{R}_\sigma \left( \left| \int_t^x |t-u| \left\{ 1 + \frac{(u-x)^2}{\delta^2} \right\} \omega(f''; \delta) du \right| ; x \right) \\ & \leq \mathfrak{R}_\sigma \left( \left| \int_t^x |t-u| \left\{ 1 + \frac{(t-x)^2}{\delta^2} \right\} du \right| ; x \right) \omega(f''; \delta) \\ & = \frac{1}{2} \omega(f''; \delta) \mathfrak{R}_\sigma \left( (t-x)^2 + \frac{(t-x)^4}{\delta^2} ; x \right) \\ & \leq \frac{M}{\sigma} \omega(f''; \sigma^{-\frac{1}{2}}), \end{aligned}$$

for some constant  $M > 0$  and  $\delta = \sigma^{-\frac{1}{2}}$ . Hence,

$$\begin{aligned} & \sigma \left| \mathfrak{R}_\sigma(f; x) - f(x) + \frac{x}{\sigma+1}f'(x) - \frac{f''(x)}{2!} \left\{ \frac{-(\sigma-2)x^2}{(\sigma+1)(\sigma+2)} + \frac{\sigma}{(\sigma+1)^2x} \right\} \right| \\ & \leq M \omega(f''; \sigma^{-\frac{1}{2}}). \end{aligned} \tag{3.3}$$

For any  $\varepsilon > 0$ , let us consider the sets:

$$\begin{aligned} U_1 = \left\{ \sigma \in \mathbb{N} : \sigma \left| \mathfrak{R}_\sigma(f; x) - f(x) + \frac{x}{\sigma+1}f'(x) \right. \right. \\ \left. \left. - \frac{f''(x)}{2!} \left\{ \frac{-(\sigma-2)x^2}{(\sigma+1)(\sigma+2)} + \frac{\sigma}{(\sigma+1)^2x} \right\} \right| \geq \varepsilon \right\} \end{aligned}$$

and

$$U_2 = \{ \sigma \in \mathbb{N} : M \omega(f''; \sigma^{-\frac{1}{2}}) \geq \varepsilon \}.$$

Then, from (3.3) it is clear that  $U_1 \subset U_2$  and hence

$$\frac{1}{c_\sigma} \left\{ \frac{1}{T_\sigma} \sum_{m=1}^{\sigma} \sum_{\ell \in U_1} t_m a_{m\ell} \right\} \leq \frac{1}{c_\sigma} \left\{ \frac{1}{T_\sigma} \sum_{m=1}^{\sigma} \sum_{\ell \in U_2} t_m a_{m\ell} \right\}. \quad (3.4)$$

By our hypothesis,  $\omega(f''; \sigma^{-\frac{1}{2}}) = \tilde{st}_A^W - o(c_\sigma)$ , therefore from (3.4), we obtain the assertion.  $\square$

Grüss [20] obtained an estimate of the difference

$$T(f, g) = \frac{1}{(b-a)} \int_a^b f(t)g(t)dt - \frac{1}{(b-a)^2} \int_a^b f(t)dt \int_a^b g(t)dt.$$

Our aim is to look at this result from an another angle. We wish to see how non-multiplicative can the linear operator (1.2) be. For this purpose, we establish the Grüss Voronovskaja type theorem in weighted A-statistical approximation which shows the non-multiplicative character of the operator  $\mathfrak{R}_\sigma$ .

**THEOREM 6.** *Let  $f'', g'' \in C(S)$  then there holds:*

$$\tilde{st}_A^W - \lim_{\sigma} \sigma \{ \mathfrak{R}_\sigma(fg; x) - \mathfrak{R}_\sigma(f; x)\mathfrak{R}_\sigma(g; x) \} = x(1-x)f'(x)g'(x),$$

uniformly in  $x \in S$ .

*Proof.* Using the identity

$$(fg)''(x) = f''(x)g(x) + 2g'(x)f'(x) + g''(x)f(x), \quad x \in S$$

by simple calculations, we may write

$$\begin{aligned} & \sigma \{ \mathfrak{R}_\sigma(fg; x) - \mathfrak{R}_\sigma(f; x)\mathfrak{R}_\sigma(g; x) \} \\ = & \sigma \left( \mathfrak{R}_\sigma(fg; x) - f(x)g(x) - (fg)'(x)\mathfrak{R}_\sigma((s-x); x) - \frac{\mathfrak{R}_\sigma((s-x)^2; x)(fg)''(x)}{2} \right. \\ & - g(x) \left( \mathfrak{R}_\sigma(f; x) - f(x) - f'(x)\mathfrak{R}_\sigma((s-x); x) - \frac{\mathfrak{R}_\sigma((s-x)^2; x)f''(x)}{2} \right) \\ & - \mathfrak{R}_\sigma(f; x) \left( \mathfrak{R}_\sigma(g; x) - g(x) - g'(x)\mathfrak{R}_\sigma((s-x); x) - \frac{\mathfrak{R}_\sigma((s-x)^2; x)g''(x)}{2} \right) \\ & + \mathfrak{R}_\sigma((s-x)^2; x)f'(x)g'(x) + g''(x) \frac{\mathfrak{R}_\sigma((s-x)^2; x)}{2!} (f(x) - \mathfrak{R}_\sigma(f; x)) \\ & \left. + g'(x)\mathfrak{R}_\sigma((s-x); x)(f(x) - \mathfrak{R}_\sigma(f; x)) \right). \quad (3.5) \end{aligned}$$

Now, from Remark 1,

$$\tilde{st}_A^W - \lim_{\sigma} \sigma \mathfrak{R}_\sigma((s-x); x) = -x,$$

and

$$\tilde{st}_A^W - \lim_{\sigma} \sigma \mathfrak{R}_{\sigma}((s-x)^2; x) = x(1-x),$$

hence, applying Theorem 2 and Theorem 5, we have

$$\begin{aligned} \tilde{st}_A^W - \lim_{\sigma} \sigma \{ \mathfrak{R}_{\sigma}(fg; x) - \mathfrak{R}_{\sigma}(f; x)\mathfrak{R}_{\sigma}(g; x) \} &= \{ \tilde{st}_A^W - \lim_{\sigma} \sigma \mathfrak{R}_{\sigma}((s-x)^2; x) \} f'(x)g'(x) \\ &= x(1-x)f'(x)g'(x), \end{aligned}$$

uniformly in  $x \in S$ .  $\square$

#### 4. Bivariate case of the operator $\mathfrak{R}_{\sigma}$

Kingsley [25] initiated the study of Bernstein operators for the functions of two variables of class  $C^k$  (the class of  $k$  times continuously differentiable functions) on a closed and bounded rectangle region. Butzer [13] investigated some approximation properties for these operators. Volkov [35] established the convergence theorem for the sequence of linear positive operators for the continuous functions of two variables. Stancu [33] proposed another kind of generalization of Bernstein operators on the isosceles right triangle  $\Delta := \{(u, v) : u + v \leq 1, u \geq 0, v \geq 0\}$ . Zhou [37] introduced multidimensional Bernstein-Durrmeyer operators in the  $L_p$  space and studied some approximation properties. For further studies in this direction, authors refer the readers to ([1], [6], [7], [14], [2] and [9] etc.).

For  $f \in C(S^2)$ , endowed with the supnorm  $\|f\|_{C(S^2)} = \sup_{(x_1, x_2) \in S^2} |f(x_1, x_2)|$ , the tensor product of (1.2) is defined as

$$\mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}(f(s, t); x_1, x_2) = \sum_{\ell_1=0}^{\infty} \sum_{\ell_2=0}^{\infty} \mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}(x_1, x_2) f\left(\frac{\ell_1}{\sigma_1 + \ell_1}, \frac{\ell_2}{\sigma_2 + \ell_2}\right), \tag{4.1}$$

where

$$\begin{aligned} \mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}(x_1, x_2) &= e^{-(\sigma_1 x_1 + \sigma_2 x_2)} \frac{\sigma_1 \sigma_2 (\sigma_1 + \ell_1)^{\ell_1} (\sigma_2 + \ell_2)^{\ell_2}}{\ell_1! \ell_2!} (x_1 e^{-x_1})^{\ell_1} (x_2 e^{-x_2})^{\ell_2}, \\ &(x_1, x_2) \in [0, 1]^2 \end{aligned}$$

and  $\mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}(f; x_1, x_2) = f(x_1, x_2)$ , for all  $(x_1, x_2) \in S^2 \setminus [0, 1]^2$ .

Let  $e_{i,j} = s^i t^j$ ,  $0 \leq i + j \leq 2$ . Then, in view of Lemma 1 and (4.1), by simple calculations we have:

LEMMA 4. For the operator defined by (4.1), there hold the identities:

- (i)  $\mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}(e_{0,0}; x_1, x_2) = 1$ ;
- (ii)  $\mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}(e_{1,0}; x_1, x_2) = \frac{\sigma_1}{\sigma_1 + 1} x_1$ ;

$$(iii) \mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}(e_{0,1}; x_1, x_2) = \frac{\sigma_2}{\sigma_2 + 1} x_2;$$

$$(iv) \mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}(e_{2,0}; x_1, x_2) = \frac{\sigma_1^2}{(\sigma_1 + 1)(\sigma_1 + 2)} x_1^2 + \frac{\sigma_1}{(\sigma_1 + 1)^2} x_1;$$

$$(v) \mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}(e_{0,2}; x_1, x_2) = \frac{\sigma_2^2}{(\sigma_2 + 1)(\sigma_2 + 2)} x_2^2 + \frac{\sigma_2}{(\sigma_2 + 1)^2} x_2;$$

$$(vi) \mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}(e_{3,0}; x_1, x_2) = \frac{\sigma_1^3}{(\sigma_1 + 1)(\sigma_1 + 2)(\sigma_1 + 3)} x_1^3 + \frac{\sigma_1^2(3\sigma_1 + 4)}{(\sigma_1 + 1)^2(\sigma_1 + 2)^2} x_1^2 + \frac{\sigma_1}{(\sigma_1 + 1)^3} x_1;$$

$$(vii) \mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}(e_{0,3}; x_1, x_2) = \frac{\sigma_2^3}{(\sigma_2 + 1)(\sigma_2 + 2)(\sigma_2 + 3)} x_2^3 + \frac{\sigma_2^2(3\sigma_2 + 4)}{(\sigma_2 + 1)^2(\sigma_2 + 2)^2} x_2^2 + \frac{\sigma_2}{(\sigma_2 + 1)^3} x_2;$$

$$(viii) \mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}(e_{4,0}; x_1, x_2) = \frac{\sigma_1^4}{(\sigma_1 + 1)(\sigma_1 + 2)(\sigma_1 + 3)(\sigma_1 + 4)} x_1^4 + \frac{2\sigma_1^3(3\sigma_1^2 + 11\sigma_1 + 9)}{(\sigma_1 + 1)^2(\sigma_1 + 2)^2(\sigma_1 + 3)^2} x_1^3 + \frac{\sigma_1^2(7\sigma_1^2 + 18\sigma_1 + 12)}{(\sigma_1 + 1)^3(\sigma_1 + 2)^3} x_1^2 + \frac{\sigma_1}{(\sigma_1 + 1)^4} x_1;$$

$$(ix) \mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}(e_{0,4}; x_1, x_2) = \frac{\sigma_2^4}{(\sigma_2 + 1)(\sigma_2 + 2)(\sigma_2 + 3)(\sigma_2 + 4)} x_2^4 + \frac{2\sigma_2^3(3\sigma_2^2 + 11\sigma_2 + 9)}{(\sigma_2 + 1)^2(\sigma_2 + 2)^2(\sigma_2 + 3)^2} x_2^3 + \frac{\sigma_2^2(7\sigma_2^2 + 18\sigma_2 + 12)}{(\sigma_2 + 1)^3(\sigma_2 + 2)^3} x_2^2 + \frac{\sigma_2}{(\sigma_2 + 1)^4} x_2.$$

LEMMA 5. [28] For  $0 \leq x_1, x_2 < 1$ , the central moments for the operator  $\mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}$  are given by:

$$(i) \mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}((s - x_1)^2; x_1, x_2) = \frac{-(\sigma_1 - 2)}{(\sigma_1 + 1)(\sigma_1 + 2)} x_1^2 + \frac{\sigma_1}{(\sigma_1 + 1)^2} x_1;$$

$$(ii) \mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}((t - x_2)^2; x_1, x_2) = \frac{-(\sigma_2 - 2)}{(\sigma_2 + 1)(\sigma_2 + 2)} x_2^2 + \frac{\sigma_2}{(\sigma_2 + 1)^2} x_2;$$

$$\begin{aligned}
 (iii) \quad \mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}((s-x_1)^4; x_1, x_2) &= \frac{3\sigma_1^2 - 46\sigma_1 + 24}{(\sigma_1 + 1)(\sigma_1 + 2)(\sigma_1 + 3)(\sigma_1 + 4)} x_1^4 \\
 &+ \frac{(-6\sigma_1^3 + 36\sigma_1^2 + 216\sigma_1 + 216)\sigma_1 x_1^3}{(\sigma_1 + 1)^2(\sigma_1 + 2)^2(\sigma_1 + 3)^2} \\
 &+ \sigma_1 \frac{(3\sigma_1^3 - 6\sigma_1^2 - 36\sigma_1 - 32)}{(\sigma_1 + 1)^3(\sigma_1 + 2)^3} x_1^2 + \frac{\sigma_1 x_1}{(\sigma_1 + 1)^2};
 \end{aligned}$$

$$\begin{aligned}
 (iv) \quad \mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}((t-x_2)^4; x_1, x_2) &= \frac{3\sigma_2^2 - 46\sigma_2 + 24}{(\sigma_2 + 1)(\sigma_2 + 2)(\sigma_2 + 3)(\sigma_2 + 4)} x_2^4 \\
 &+ \frac{(-6\sigma_2^3 + 36\sigma_2^2 + 216\sigma_2 + 216)\sigma_2 x_2^3}{(\sigma_2 + 1)^2(\sigma_2 + 2)^2(\sigma_2 + 3)^2} \\
 &+ \sigma_2 \frac{(3\sigma_2^3 - 6\sigma_2^2 - 36\sigma_2 - 32)}{(\sigma_2 + 1)^3(\sigma_2 + 2)^3} x_2^2 + \frac{\sigma_2 x_2}{(\sigma_2 + 1)^2}.
 \end{aligned}$$

For  $f \in C(S^2)$ , the first order total modulus of continuity for the two dimensional case is defined as follows:

$$\bar{\omega}(f; \delta_1, \delta_2) = \sup \left\{ |f(s, t) - f(x_1, x_2)| : |s - x_1| \leq \delta_1, |t - x_2| \leq \delta_2 \right\},$$

where  $\delta_1, \delta_2 > 0$ . It is known that:

(a)  $\bar{\omega}(f; \delta_1, \delta_2) \rightarrow 0$ , if  $\delta_1 \rightarrow 0$  and  $\delta_2 \rightarrow 0$ ,

(b)  $|f(s, t) - f(x_1, x_2)| \leq \bar{\omega}(f; \delta_1, \delta_2) \left(1 + \frac{|s - x_1|}{\delta_1}\right) \left(1 + \frac{|t - x_2|}{\delta_2}\right)$ . in our further consideration, let us assume

$$\delta_{\sigma_1, i} = \|\mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}((s-x_1)^i)\|_{C(S^2)}$$

and

$$\delta_{\sigma_2, i} = \|\mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}((t-x_2)^i)\|_{C(S^2)},$$

for  $i = 1, 2, \dots$

Now, we give an estimate of the rate of approximation for the two dimensional case of  $\mathfrak{R}_\sigma$  in terms of the total modulus of continuity.

**THEOREM 7.** *Let  $f \in C(S^2)$ . Then, we have*

$$\|\mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}(f) - f\|_{C(S^2)} \leq 4\bar{\omega}(f; \sqrt{\delta_{\sigma_1, 2}}, \sqrt{\delta_{\sigma_2, 2}}).$$

*Proof.* By using the property of  $\mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}(\cdot; x_1, x_2)$  and the modulus of continuity,

$$\begin{aligned}
 &|\mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}(f; x_1, x_2) - f(x_1, x_2)| \\
 &\leq \mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}(|f(s, t) - f(x_1, x_2)|; x_1, x_2),
 \end{aligned}$$

$$\begin{aligned} &\leq \bar{\omega}(f; \delta_1, \delta_2) \left( \mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}(1; x_1, x_2) + \frac{1}{\delta_1} \mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}(|s - x_1|; x_1, x_2) \right. \\ &\quad + \frac{1}{\delta_2} \mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}(|t - x_2|; x_1, x_2) + \frac{1}{\delta_1 \delta_2} \mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}(|s - x_1|; x_1, x_2) \\ &\quad \left. \times \mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}(|t - x_2|; x_2) \right). \end{aligned}$$

Applying Cauchy-Schwarz inequality and Lemma 1

$$\begin{aligned} &|\mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}(f; x_1, x_2) - f(x_1, x_2)| \\ &\leq \bar{\omega}(f; \delta_1, \delta_2) \left( 1 + \frac{1}{\delta_1} \sqrt{\mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}((s - x_1)^2; x_1, x_2)} \right. \\ &\quad + \frac{1}{\delta_2} \sqrt{\mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}((t - x_2)^2; x_1, x_2)} \\ &\quad \left. + \frac{1}{\delta_1 \delta_2} \sqrt{\mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}((s - x_1)^2; x_1, x_2)} \sqrt{\mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}((t - x_2)^2; x_1, x_2)} \right). \end{aligned}$$

Now, choose  $\delta_1 = \sqrt{\delta_{\sigma_1, 2}}$  and  $\delta_2 = \sqrt{\delta_{\sigma_2, 2}}$ , to complete the proof.  $\square$

For  $f \in C(S^2)$  and  $\delta > 0$ , the partial moduli of continuity with respect to  $x_1$  and  $x_2$  is given by

$$\omega_1(f; \delta) = \sup \left\{ |f(x_1, x_2) - f(x_2, x_2)| : x_2 \in S \text{ and } |x_1 - x_2| \leq \delta \right\}$$

and

$$\omega_2(f; \delta) = \sup \left\{ |f(x_1, y_1) - f(x_1, y_2)| : x_1 \in S \text{ and } |y_1 - y_2| \leq \delta \right\}.$$

It is well known that they satisfy the properties of the usual modulus of continuity.

In the following result, we determine the convergence estimate for the operators (4.1) by means of the partial moduli of continuity.

**THEOREM 8.** *Let  $f \in C(S^2)$ . Then, we have*

$$\|\mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}(f) - f\|_{C(S^2)} \leq 2\{\omega_2(f; \sqrt{\delta_{\sigma_2, 2}}) + \omega_1(f; \sqrt{\delta_{\sigma_1, 2}})\}.$$

*Proof.* In view of the definition of partial moduli of continuity, we have

$$\begin{aligned} &|\mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}(f; x_1, x_2) - f(x_1, x_2)| \\ &\leq \mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}(|f(s, t) - f(s, x_2)|; x_1, x_2) \\ &\quad + \mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}(|f(s, x_2) - f(x_1, x_2)|; x_1, x_2) \\ &\leq \omega_2(f; \delta_2) \left( \mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}(1; x_1, x_2) + \frac{1}{\delta_2} (\mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2} |t - x_2|; x_1, x_2) \right) \\ &\quad + \omega_1(f; \delta_1) \left( \mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}(1; x_1, x_2) + \frac{1}{\delta_1} \mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}(|s - x_1|; x_1, x_2) \right). \end{aligned}$$

Applying the Cauchy-Schwarz inequality and Lemma 1,

$$\begin{aligned} |\mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}(f; x_1, x_2) - f(x_1, x_2)| &\leq \omega_2(f; \delta_2) \left( 1 + \frac{1}{\delta_2} \sqrt{\mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}((t - x_2)^2; x_2)} \right) \\ &\quad + \omega_1(f; \delta_1) \left( 1 + \frac{1}{\delta_1} \sqrt{\mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}((s - x_1)^2; x_1)} \right) \\ &= 2\{\omega_2(f; \delta_2) + \omega_1(f; \delta_1)\}. \end{aligned}$$

Hence on choosing  $\delta_1 = \sqrt{\delta_{\sigma_{1,2}}}$  and  $\delta_2 = \sqrt{\delta_{\sigma_{2,2}}}$ , we get the required result.  $\square$

Let  $C^2(S^2) := \{f \in C(S^2) : \frac{\partial^{i+j} f}{\partial x^i \partial y^j} \in C(S^2), \text{ for } 0 \leq i + j \leq 2\}$  with the norm defined as:

$$\|f\|_{C^2(S^2)} = \|f\|_{C(S^2)} + \sum_{i=1}^2 \left( \left\| \frac{\partial^i f}{\partial x_1^i} \right\|_{C(S^2)} + \left\| \frac{\partial^i f}{\partial x_2^i} \right\|_{C(S^2)} \right) + \left\| \frac{\partial^2 f}{\partial x_1 \partial x_2} \right\|_{C(S^2)}.$$

The Peetre’s K-functional of the function  $f \in C(S^2)$  is defined as:

$$\mathcal{K}(f; \delta) = \inf_{g \in C^2(S^2)} \{ \|f - g\|_{C(S^2)} + \delta \|g\|_{C^2(S^2)} \}, \quad \delta > 0.$$

Also by [12], it follows that

$$\mathcal{K}(f; \delta) \leq M \left\{ \tilde{\omega}_2(f; \sqrt{\delta}) + \min(1, \delta) \|f\|_{C(S^2)} \right\}, \tag{4.2}$$

holds for all  $\delta > 0$  and  $M$  does not depend on  $\delta$  and  $f$ , where  $\tilde{\omega}_2(f; \sqrt{\delta})$  is the second order modulus of continuity for the two dimensional case.

In the following theorem, we investigate the approximation degree for the operators (4.1) with the aid of the Peetre’s K-functional.

**THEOREM 9.** For  $f \in C(S^2)$ , we have

$$\begin{aligned} &|\mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}(f; x_1, x_2) - f(x_1, x_2)| \\ &\leq M \left\{ \tilde{\omega}_2\left(f; \frac{1}{2\sqrt{2}} \sqrt{(\sqrt{\delta_{\sigma_{1,2}}} + \sqrt{\delta_{\sigma_{2,2}}})^2 + (\delta_{\sigma_{1,1}} + \delta_{\sigma_{2,1}})^2}\right) \right. \\ &\quad \left. + \min\{1, (\sqrt{\delta_{\sigma_{1,2}}} + \sqrt{\delta_{\sigma_{2,2}}})^2 + (\delta_{\sigma_{1,1}} + \delta_{\sigma_{2,1}})^2\} \|f\|_{C(S^2)} \right\} \\ &\quad + \bar{\omega}\left(f; \sqrt{(\delta_{\sigma_{1,1}})^2 + (\delta_{\sigma_{2,1}})^2}\right). \end{aligned}$$

*Proof.* Consider an auxiliary operator as follows:

$$\mathfrak{R}_{\sigma_1, \sigma_2}^{*\ell_1, \ell_2}(f; x_1, x_2) = \mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}(f; x_1, x_2) - f(\mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}(s; x_1, x_2), \mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}(t; x_1, x_2)) + f(x_1, x_2). \tag{4.3}$$

In view of Lemma 1,  $R_{\sigma_1, \sigma_2}^{*\ell_1, \ell_2}(1; x_1, x_2) = 1$ ,

$$\mathfrak{R}_{\sigma_1, \sigma_2}^{*\ell_1, \ell_2}((s - x_1); x_1, x_2) = 0$$

and

$$\mathfrak{R}_{\sigma_1, \sigma_2}^{*\ell_1, \ell_2}((t - x_2); x_1, x_2) = 0.$$

Let  $g \in C^2(S^2)$  and  $x_1, x_2 \in S$  be arbitrary. Using the Taylor's formula,

$$\begin{aligned} g(s, t) - g(x_1, x_2) &= g(s, y) - g(x_1, x_2) + g(s, t) - g(s, y) \\ &= \frac{\partial g(x_1, x_2)}{\partial x}(s - x_1) + \int_x^s (s - \eta) \frac{\partial^2 g(\eta, y)}{\partial \eta^2} d\eta \\ &\quad + \frac{\partial g(x_1, x_2)}{\partial y}(t - x_2) + \int_y^t (t - \zeta) \frac{\partial^2 g(x, \zeta)}{\partial \zeta^2} d\zeta \\ &\quad + \int_y^t \int_x^s \frac{\partial^2 g(\eta, \zeta)}{\partial \eta \partial \zeta} d\eta d\zeta. \end{aligned}$$

Applying  $R_{\sigma_1, \sigma_2}^{*\ell_1, \ell_2}(\cdot; x_1, x_2)$  on both sides of the above equation and using (4.3) we find

$$\begin{aligned} &\left| \mathfrak{R}_{\sigma_1, \sigma_2}^{*\ell_1, \ell_2}(g(s, t); x_1, x_2) - g(x_1, x_2) \right| \\ &= \left| \mathfrak{R}_{\sigma_1, \sigma_2}^{*\ell_1, \ell_2} \left( \int_x^s (s - \eta) \frac{\partial^2 g(\eta, y)}{\partial \eta^2} d\eta; x_1, x_2 \right) \right. \\ &\quad + \mathfrak{R}_{\sigma_1, \sigma_2}^{*\ell_1, \ell_2} \left( \int_y^t (t - \zeta) \frac{\partial^2 g(x, \zeta)}{\partial \zeta^2} d\zeta; x_1, x_2 \right) \\ &\quad \left. + \mathfrak{R}_{\sigma_1, \sigma_2}^{*\ell_1, \ell_2} \left( \int_y^t \int_x^s \frac{\partial^2 g(\eta, \zeta)}{\partial \eta \partial \zeta} d\eta d\zeta; x_1, x_2 \right) \right| \\ &\leq \left| \mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2} \left( \int_x^s (s - \eta) \frac{\partial^2 g(\eta, y)}{\partial \eta^2} d\eta; x_1, x_2 \right) \right| \\ &\quad + \left| \int_x^{\mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}(s; x_1, x_2)} (\mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}(s; x_1, x_2) - \eta) \frac{\partial^2 g(\eta, y)}{\partial \eta^2} d\eta \right| \\ &\quad + \left| \mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2} \left( \int_y^t (t - \zeta) \frac{\partial^2 g(x, \zeta)}{\partial \zeta^2} d\zeta; x_1, x_2 \right) \right| \\ &\quad + \left| \int_y^{\mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}(t; x_1, x_2)} (\mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}(t; x_1, x_2) - \zeta) \frac{\partial^2 g(x, \zeta)}{\partial \zeta^2} d\zeta \right| \\ &\quad + \left| \mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2} \left( \int_y^t \int_x^s \frac{\partial^2 g(\eta, \zeta)}{\partial \eta \partial \zeta} d\eta d\zeta; x_1, x_2 \right) \right| \\ &\quad + \left| \int_y^{\mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}(t; x_1, x_2)} \int_x^{\mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}(s; x_1, x_2)} \frac{\partial^2 g(\eta, \zeta)}{\partial \eta \partial \zeta} d\eta d\zeta \right|. \end{aligned}$$

Hence,

$$\begin{aligned}
 & \left| \mathfrak{R}_{\sigma_1, \sigma_2}^{*\ell_1, \ell_2}(g(s, t); x_1, x_2) - g(x_1, x_2) \right| \\
 & \leq \frac{1}{2} \{ \|\mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}(s - x_1)\|_{C(S^2)}^2 + \|\mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}(s - x_1)\|_{C(S^2)}^2 \\
 & \quad + \|\mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}(t - x_2)\|_{C(S^2)}^2 + \|\mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}(s - x_1)\|_{C(S^2)}^2 \\
 & \quad + \sqrt{\|\mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}(s - x_1)\|_{C(S^2)}} \sqrt{\|\mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}(t - x_2)\|_{C(S^2)}} \\
 & \quad + \|\mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}(s - x_1)\|_{C(S^2)} \|\mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}(t - x_2)\|_{C(S^2)} \} \|g\|_{C^2(S^2)} \\
 & = \frac{1}{2} \{ (\sqrt{\delta_{\sigma_1, 2}} + \sqrt{\delta_{\sigma_2, 2}})^2 + (\delta_{\sigma_1, 1} + \delta_{\sigma_2, 1})^2 \} \|g\|_{C^2(S^2)}.
 \end{aligned}
 \tag{4.4}$$

Also, using Lemma 3

$$\begin{aligned}
 & |\mathfrak{R}_{\sigma_1, \sigma_2}^{*\ell_1, \ell_2}(f; x_1, x_2)| \\
 & \leq |\mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}(f; x_1, x_2)| + |f(\mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}(s; x_1, x_2), \mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}(t; x_1, x_2))| + |f(x_1, x_2)| \\
 & \leq 3\|f\|_{C(S^2)}.
 \end{aligned}
 \tag{4.5}$$

Hence, using (4.5) and (4.4), for any  $g \in C^2(S^2)$

$$\begin{aligned}
 & |\mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}(f; x_1, x_2) - f(x_1, x_2)| \\
 & \leq |R_{\sigma_1, \sigma_2}^{*\ell_1, \ell_2}(f - g; x_1, x_2)| + |R_{\sigma_1, \sigma_2}^{*\ell_1, \ell_2}(g; x_1, x_2) - g(x_1, x_2)| + |g(x_1, x_2) - f(x_1, x_2)| \\
 & \quad + \left| f(\mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}(s; x_1, x_2), R_{\sigma_1, \sigma_2}^{*\ell_1, \ell_2}(t; x_1, x_2)) - f(x_1, x_2) \right| \\
 & \leq 4\|f - g\|_{C(S^2)} + |R_{\sigma_1, \sigma_2}^{*\ell_1, \ell_2}(g; x_1, x_2) - g(x_1, x_2)| \\
 & \quad + \left| f(\mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}(s; x_1, x_2), \mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}(t; x_1, x_2)) - f(x_1, x_2) \right| \\
 & \leq \left( 4\|f - g\|_{C(S^2)} + \frac{1}{2} \{ (\sqrt{\delta_{\sigma_1, 2}} + \sqrt{\delta_{\sigma_2, 2}})^2 + (\delta_{\sigma_1, 1} + \delta_{\sigma_2, 1})^2 \} \|g\|_{C^2(S^2)} \right. \\
 & \quad \left. + \bar{\omega} \left( f; \sqrt{(\delta_{\sigma_1, 1})^2 + (\delta_{\sigma_2, 1})^2} \right) \right).
 \end{aligned}$$

Now, we use the definition of the K-functional and the relation (4.2) to establish the assertion.  $\square$

### 5. GBS case of the operator $\mathfrak{R}_\sigma$

The study of GBS operators was initiated by Bögel ([10], [11]) who introduced the idea of Bögel continuous (B-continuous) and Bögel differentiable (B-differentiable) functions. Badea et al. [4] proved the very famous “Test function theorem” for B-continuous functions. Badea et al. [5] gave the Korovkin-type theorem in the quantitative form. For more insight on this topic, we refer the readers to some interesting articles ([1], [9], [22], [8] and [32]) and the book [19] etc.

In this section, we propose to define the GBS operator related to the operator  $\mathfrak{R}_\sigma$  defined in (1.2) and study its approximation properties. We begin with some definitions and notations as described below:

Let  $\mathfrak{U}_1$  and  $\mathfrak{U}_2$  be the compact real intervals.

A function  $f : \mathfrak{U}_1 \times \mathfrak{U}_2 \rightarrow \mathfrak{R}$  is called a B-continuous function in  $\mathfrak{U}_1 \times \mathfrak{U}_2$  iff

$$\lim_{(s,t) \rightarrow (x_1, x_2)} \Delta_{(s,t)} f(x_1, x_2) = 0,$$

for each  $(x_1, x_2) \in \mathfrak{U}_1 \times \mathfrak{U}_2$ , where

$$\Delta_{(s,t)} f(x_1, x_2) = f(s, t) - f(s, x_2) - f(x_1, t) + f(x_1, x_2)$$

is called the mixed difference of  $f$ . The set of all B-continuous functions is denoted by  $C_b(\mathfrak{U}_1 \times \mathfrak{U}_2)$ .

A function  $f$  is called a B-differentiable function in  $\mathfrak{U}_1 \times \mathfrak{U}_2$  iff

$$\lim_{(s,t) \rightarrow (x_1, x_2)} \frac{\Delta_{(s,t)} f(x_1, x_2)}{(s - x_1)(t - x_2)},$$

exists and is finite for every  $(x_1, x_2) \in \mathfrak{U}_1 \times \mathfrak{U}_2$ . We denote the set of all B-differentiable functions by  $D_b(\mathfrak{U}_1 \times \mathfrak{U}_2)$ .

A function  $f$  is called B-bounded on  $\mathfrak{U}_1 \times \mathfrak{U}_2$  iff there exists some  $k > 0$  such that  $|\Delta_{(s,t)} f(x_1, x_2)| \leq k$  for any  $(s, t), (x_1, x_2) \in \mathfrak{U}_1 \times \mathfrak{U}_2$ . Let  $B(S^2)$  denote the space of bounded functions (in the usual sense) on  $S^2$  with the norm  $\|\cdot\|_\infty$  and  $C(S^2) = \{f \in B(S^2) : f \text{ is continuous}\}$ .

The mixed modulus of smoothness of  $f \in B_b(\mathfrak{U}_1 \times \mathfrak{U}_2)$  is defined as

$$\omega_{mixed}(f; \delta_1, \delta_2) := \sup \{ |\Delta_{(s,t)} f(x_1, x_2)| : |s - x_1| < \delta_1, |t - x_2| < \delta_2 \},$$

for all  $(x_1, x_2), (s, t) \in S^2$  and for any  $\delta_1, \delta_2 > 0$ .

A function  $f$  is called uniformly B-continuous on  $\mathfrak{U}_1 \times \mathfrak{U}_2$  iff for any  $\varepsilon > 0$ ,  $\exists \delta = \delta(\varepsilon) > 0$  such that

$$|\Delta_{(s,t)} f(x_1, x_2)| < \varepsilon,$$

whenever  $\max\{|s - x_1|, |t - x_2|\} < \delta$ . It is known [3] that  $\omega_{mixed}(f, \delta_1, \delta_2) \rightarrow 0$ , as  $\delta_1, \delta_2 \rightarrow 0$  iff  $f$  is uniformly B-continuous on  $\mathfrak{U}_1 \times \mathfrak{U}_2$  and

$$\omega_{mixed}(f; \lambda_1 \delta_1, \lambda_2 \delta_2) \leq (1 + \lambda_1)(1 + \lambda_2) \omega_{mixed}(f; \delta_1, \delta_2), \quad \lambda_1, \lambda_2 > 0. \quad (5.1)$$

The GBS operator  $\mathbb{G}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2} : C_b(S^2) \rightarrow C(S^2)$  associated to the operator given by (4.1) is defined as

$$\begin{aligned} & \mathbb{G}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}(f(s, t); x_1, x_2) \\ &= \mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}(f(x_1, t) + f(s, x_2) - f(s, t); x_1, x_2) \\ &= \sum_{\ell_1=1}^{\infty} \sum_{\ell_2=0}^{\infty} \mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}(x_1, x_2) \left( f\left(x_1, \frac{\ell_2}{\sigma_2 + \ell_2}\right) + f\left(\frac{\ell_1}{\sigma_1 + \ell_1}, x_2\right) - f\left(\frac{\ell_1}{\sigma_1 + \ell_1}, \frac{\ell_2}{\sigma_2 + \ell_2}\right) \right), \end{aligned} \tag{5.2}$$

for all  $(x_1, x_2) \in [0, 1)^2$  and  $\mathbb{G}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}(f; x_1, x_2) = f(x_1, x_2)$  for all  $(x_1, x_2) \in S^2 \setminus [0, 1)^2$ .

Now we estimate the rate of convergence of (5.2) in terms of  $\omega_{mixed}$ .

**THEOREM 10.** *For every  $f \in C_b(S^2)$ , we have*

$$\|\mathbb{G}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}(f) - f\|_{C(S^2)} \leq 4 \omega_{mixed}(f; \sqrt{\delta_{\sigma_1, 2}}, \sqrt{\delta_{\sigma_2, 2}}).$$

*Proof.* From the definition of  $\omega_{mixed}$  and the inequality (5.1),

$$\begin{aligned} |\Delta_{(s,t)}f(x_1, x_2)| &\leq \omega_{mixed}(f; |s - x_1|, |t - x_2|) \\ &\leq \left(1 + \frac{|s - x_1|}{\delta_1}\right) \left(1 + \frac{|t - x_2|}{\delta_2}\right) \omega_{mixed}(f; \delta_1, \delta_2), \end{aligned} \tag{5.3}$$

for every  $(s, t) \in S^2$ ,  $(x_1, x_2) \in S^2$  and for any  $\delta_1, \delta_2 > 0$ . From the definition of  $\Delta_{(s,t)}f(x_1, x_2)$ , we get

$$f(x_1, t) + f(s, x_2) - f(s, t) = f(x_1, x_2) - \Delta_{(s,t)}f(x_1, x_2).$$

Applying  $\mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}(\cdot; x_1, x_2)$  on the above equation, in view of (5.3) we have

$$\mathbb{G}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}(f; x_1, x_2) = f(x_1, x_2) \mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}((1; x_1, x_2) - \mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}(\Delta_{(s,t)}f(x_1, x_2); x_1, x_2)) \tag{5.4}$$

Hence using (5.2), we obtain

$$\begin{aligned} & |\mathbb{G}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}(f; x_1, x_2) - f(x_1, x_2)| \\ &\leq \mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}(|\Delta_{(s,t)}f(x_1, x_2)|; x_1, x_2) \\ &\leq \left( \mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}(1; x_1, x_2) + \frac{1}{\delta_1} \mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}(|s - x_1|; x_1, x_2) \right. \\ &\quad \left. + \frac{1}{\delta_2} \mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}(|t - x_2|; x_1, x_2) + \frac{1}{\delta_1 \delta_2} \mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}(|s - x_1|; x_1, x_2) \right. \\ &\quad \left. \times \mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}(|t - x_2|; x_2) \right) \omega_{mixed}(f; \delta_1, \delta_2). \end{aligned}$$

Applying Cauchy-Schwarz inequality and choosing  $\delta_1 = \sqrt{\delta_{\sigma_1, 2}}$  and  $\delta_2 = \sqrt{\delta_{\sigma_2, 2}}$ , we get the desired result.  $\square$

For  $f \in C_b(S^2)$ , the Lipschitz class  $Lip_B^M(v, \eta)$  with  $v, \eta \in (0, 1]$  is defined by

$$Lip_B^M(v, \eta) = \{f \in C_b(S^2) : |\Delta_{(s,t)} f(x_1, x_2)| \leq M |s - x_1|^v |t - x_2|^\eta, \\ \text{for } (s, t), (x_1, x_2) \in S^2\}.$$

In the following theorem, we obtain an error estimate of  $f$  by  $\mathbb{G}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}(f)$  for  $f \in Lip_B^M(v, \eta)$ .

**THEOREM 11.** *For  $f \in Lip_B^M(v, \eta)$ , we have*

$$\|\mathbb{G}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}((f) - f)\|_{C(S^2)} \leq M(\delta_{\sigma_1, 2})^{\frac{v}{2}}(\delta_{\sigma_2, 2})^{\frac{\eta}{2}},$$

for  $M > 0$ .

*Proof.* From (5.4) and our hypothesis, we get

$$\begin{aligned} \left| \mathbb{G}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}(f; x_1, x_2) - f(x_1, x_2) \right| &\leq \mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}(|\Delta_{(s,t)} f(x_1, x_2)|; x_1, x_2) \\ &\leq M \mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}(|s - x_1|^v |t - x_2|^\eta; x_1, x_2) \\ &= M \mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}(|s - x_1|^v; x_1, x_2) \mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}(|s - x_2|^\eta; x_1, x_2). \end{aligned}$$

Now, we apply the Hölder's inequality by taking  $l_1 = 2/v, m_1 = 2/(2 - v)$  and  $l_2 = 2/\eta, m_2 = 2/(2 - \eta)$ , to get the assertion.  $\square$

Our next result provides the rate of approximation for B-differentiable functions by  $\mathbb{G}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}$ .

**THEOREM 12.** *Let  $f \in D_b(S^2)$  with  $D_B f \in C_b(S^2) \cap B(S^2)$ , then there holds the following inequality:*

$$\|\mathbb{G}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}(f) - f\|_{C(S^2)} \leq \frac{M}{\sqrt{\sigma_1} \sqrt{\sigma_2}} \left( \|D_B f\|_\infty + \omega_{\text{mixed}}(D_B f; \sigma_1^{-1/2}, \sigma_2^{-1/2}) \right),$$

for some constant  $M > 0$ .

*Proof.* Since  $f \in D_b(S^2)$ , using mean value theorem

$$\Delta_{(s,t)} f(x_1, x_2) = (s - x_1)(t - x_2) D_B f(v, \eta), \text{ with } x < v < s; y < \eta < t. \quad (5.5)$$

Taking into account

$$D_B f(v, \eta) = \Delta_{(v,\eta)} D_B f(x_1, x_2) + D_B f(v, x_2) + D_B f(x_1, \eta) - D_B f(x_1, x_2),$$

and  $D_B f \in C_b(S^2) \cap B(S^2)$ , in view of (5.5) we may write

$$\begin{aligned} &|\mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}(\Delta_{(s,t)} f(x_1, x_2); x_1, x_2)| \\ &= |\mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}((s - x_1)(t - x_2) D_B f(v, \eta); x_1, x_2)| \\ &\leq \mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}(|s - x_1| |t - x_2| |\Delta_{(v,\eta)} D_B f(x_1, x_2)|; x_1, x_2) \\ &\quad + \mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}(|s - x_1| |t - x_2| |D_B f(v, y)| \\ &\quad + |D_B f(x, \eta)| + |D_B f(x_1, x_2)|); x_1, x_2) \end{aligned}$$

$$\begin{aligned} &\leq \mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}(|s - x_1||t - x_2| \omega_{mixed}(D_B f; |v - x_1|, |\eta - x_2|); x_1, x_2) \\ &\quad + 3 \|D_B f\|_{\infty} \mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}(|s - x_1||t - x_2|; x_1, x_2). \end{aligned} \tag{5.6}$$

Hence from (5.4), (5.6) and the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} &|\mathbb{G}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}(f; x_1, x_2) - f(x_1, x_2)| \\ &= |\mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2} \Delta_{(s,t)}(x_1, x_2); x_1, x_2| \\ &\leq 3 \|D_B f\|_{\infty} \mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}(|s - x_1||t - x_2|; x_1, x_2) \\ &\quad + \left( \mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}(|s - x_1||t - x_2|; x_1, x_2) \right. \\ &\quad + \delta_1^{-1} \mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}((s - x_1)^2 |t - x_2|; x_1, x_2) \\ &\quad + \delta_2^{-1} \mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}(|s - x_1|(t - x_2)^2; x_1, x_2) \\ &\quad \left. + \delta_1^{-1} \delta_2^{-1} \mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}((s - x_1)^2 (t - x_2)^2; x_1, x_2) \right) \omega_B(D_B f; \delta_1, \delta_2) \\ &\leq 3 \|D_B f\|_{\infty} \sqrt{\mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}((s - x_1)^2 (t - x_2)^2; x_1, x_2)} \\ &\quad + \left( \sqrt{\mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2} (s - x_1)^2 (t - x_2)^2; x_1, x_2} \right. \\ &\quad + \delta_1^{-1} \sqrt{\mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2} ((s - x_1)^4 (t - x_2)^2; x_1, x_2)} \\ &\quad + \delta_2^{-1} \sqrt{\mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2} ((s - x_1)^2 (t - x_2)^4; x_1, x_2)} \\ &\quad \left. + \delta_1^{-1} \delta_2^{-1} \mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2} ((s - x_1)^2 (t - x_2)^2; x_1, x_2) \right) \omega_{mixed}(D_B f; \delta_1, \delta_2), \end{aligned} \tag{5.7}$$

for any  $\delta_1, \delta_2 > 0$ . In view of Remark 1, for  $(s, t) \in S^2$ ,  $(x_1, x_2) \in S^2$  and  $i, j = 1, 2$

$$\begin{aligned} \mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}((s - x_1)^{2i} (t - x_2)^{2j}; x_1, x_2) &= \mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}((s - x_1)^{2i}; x_1, x_2) \mathfrak{R}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}((t - x_2)^{2j}; x_1, x_2). \\ &\leq \frac{M_1 M_2}{\sigma_1^i \sigma_2^j}, \end{aligned} \tag{5.8}$$

where  $M_1, M_2$  are some positive constants.

Let  $\delta_1 = \frac{1}{\sqrt{\sigma_1}}$ , and  $\delta_2 = \frac{1}{\sqrt{\sigma_2}}$ .

Then, by combining (5.7)–(5.8), we have

$$\begin{aligned} &|\mathbb{G}_{\sigma_1, \sigma_2}^{\ell_1, \ell_2}(f; x_1, x_2) - f(x_1, x_2)| \\ &= 3 \|D_B f\|_{\infty} O\left(\frac{1}{\sqrt{\sigma_1}}\right) O\left(\frac{1}{\sqrt{\sigma_2}}\right) \\ &\quad + O\left(\frac{1}{\sqrt{\sigma_1}}\right) O\left(\frac{1}{\sqrt{\sigma_2}}\right) \omega_{mixed}(D_B f; \sigma_1^{-1/2}, \sigma_2^{-1/2}), \text{ as } \sigma_1, \sigma_2 \rightarrow \infty, \end{aligned}$$

for all  $(x_1, x_2) \in S^2$ .  $\square$

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