

HERMITE–HADAMARD TYPE INTEGRAL INEQUALITIES FOR THE CLASS OF STRONGLY CONVEX FUNCTIONS ON TIME SCALES

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Abstract. In this paper, we introduce the notion of a strongly convex function with respect to two non-negative auxiliary functions on time scales. We establish several new dynamic inequalities for these classes of strongly convex functions. The results obtained in this paper are the generalization of the results of Rashid *et al.* (Mathematics, 7 (10), 956, 2019). Further, we discuss some special cases which may be deduced from our main results. Moreover, some examples of our main results are mentioned.

1. Introduction

In 1988, Hilger [1] introduced the theory of time scale which is a unification of the discrete theory with the continuous theory. Recently, much attention has been given to the time scales calculus by many researchers, see for instance [3, 17, 19, 29]. Consequently, the concept of time scale theory has been extended and generalized. Time scales calculus has applications in various fields such as Economics, Engineering, Physics, Signal processing, Aerospace, Dynamic programming, Recurrent neural networks, and Control theory, see references [2, 5, 6, 7, 8, 9].

Bohner and Peterson [3, 4] provided foundational results for the analysis of dynamic equations defined on generalized domains which can be a mixture of continuous and discrete. Analysis of dynamic equations is now unrestricted by domains. Dinu [10] introduced the notion of convex functions on time scales and defined the subdifferential of convex functions on time scales. Many researchers investigated time scales versions of several dynamic inequalities that essentially depend on integral inequalities, see references [11, 13, 14, 16, 18, 19]. Dinu [12] investigated Ostrowski type inequalities on time scales. Donchev *et al.* [15] obtained Hardy type inequalities with general kernels to arbitrary time scales using multivariable convex functions.

The concept of strongly convex functions was introduced by Karamardian [21] and showed that every differentiable function is strongly convex if and only if its gradient is strongly monotone. Karamardian also established a relationship between the strongly convex function and the Hessian matrix. For more details, one can refer to [22, 23,

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27]. Hermite [24] and Hadamard [25] discovered an interesting inequality for a convex function which provides a lower and an upper estimate for the integral average of any convex function defined on a compact interval. It states that if $f : X = [u_1, u_2] \rightarrow \mathbb{R}$ is a convex function with $u_1 \leq u_2$. Then the following double inequality holds:

$$f\left(\frac{u_1 + u_2}{2}\right) \leq \frac{1}{u_2 - u_1} \int_{u_1}^{u_2} f(\omega) d\omega \leq \frac{f(u_1) + f(u_2)}{2},$$

for all $u_1, u_2 \in [u_1, u_2]$, which is known as Hermite-Hadamard inequality in the literature. Further, Dinu [17] obtained Hermite-Hadamard inequality for convex functions on time scales. In recent year, there have been many extensions and generalizations of Hermite-Hadamard inequalities studied in [26, 27, 28, 29, 30, 31, 32, 33, 34]. In 2019, Tahir *et al.* [19] established some new Hermite-Hadamard type integral inequalities using the concept of time scales. Recently, Rashid *et al.* [20] investigated the time scales version of two non-negative auxiliary functions for the class of convex functions and obtained several dynamical variants that are essentially based on Hermite-Hadamard inequality.

Inspired by the work and ideas of above mentioned research, we introduce the notion of a strongly convex function with respect to two auxiliary functions ϕ_1 and ϕ_2 on Time scales \mathbb{T} . We derive some new dynamic inequalities for these strongly convex functions. Further, our main results include several new results in particular cases. Some examples are also mentioned in the support of our theory.

2. Preliminaries

Time scale is a nonempty closed subset of the set of real numbers \mathbb{R} . The set of integers \mathbb{Z} , the set of real numbers \mathbb{R} , finite unions of disjoint intervals, limit sets such as $\{0\} \cup \{\frac{1}{n}\} : n = 1, 2, \dots$, Cantor sets etc are the examples of time scales. Throughout this paper, we denote time scale by \mathbb{T} , time-scaled interval by $[u_1, u_2]_{\mathbb{T}}$, and the interior of X by X^0 . There are two types of the operator: The forward jump operator $\sigma(\alpha) = \inf\{\omega \in \mathbb{T} : \omega > \alpha\}$ and the backward jump operator $\zeta(\alpha) = \sup\{\omega \in \mathbb{T} : \omega < \alpha\}$ for all $\alpha \in \mathbb{T}$. The forward jump operator represents the next element and the backward jump operator represents the previous element in the domain. If \mathbb{T} has a maximum α , then $\sigma(\alpha) = \alpha$, and if \mathbb{T} has a minimum α , then $\zeta(\alpha) = \alpha$.

If $\sigma(\alpha) > \alpha$, then α is called right-scattered and if $\zeta(\alpha) < \alpha$, then α is called left-scattered. The point α is said to be isolated if it is both right-scattered and left-scattered: $\sigma(\alpha) > \alpha > \zeta(\alpha)$ for $\alpha \in \mathbb{T}$. It is a characteristic of discrete domains that all points within them are isolated. α is said to be right-dense if $\sigma(\alpha) = \alpha$ and α is said to be left-dense if $\zeta(\alpha) = \alpha$. The point α is said to be dense if it is both left-dense and right-dense: $\sigma(\alpha) = \alpha = \zeta(\alpha)$ for $\alpha \in \mathbb{T}$.

The mappings $\eta, \zeta : \mathbb{T} \rightarrow [0, \infty)$ defined by

$$\eta(\alpha) := \sigma(\alpha) - \alpha, \quad \zeta(\alpha) := \alpha - \zeta(\alpha)$$

are said to be forward and backward graininess functions, respectively. The graininess function measures the step size between two consecutive points in \mathbb{T} . The set \mathbb{T}^k which

is derived from time scale \mathbb{T} is defined as follows: If \mathbb{T} has a left-scattered maximum m , then $\mathbb{T}^k = \mathbb{T} - m$; otherwise $\mathbb{T}^k = \mathbb{T}$.

The delta derivative is a basic time scale derivative and is denoted by $f^\Delta(\alpha)$. Let $f : \mathbb{T} \rightarrow \mathbb{R}$ be a function. Then the delta derivative $f^\Delta(\alpha)$ of f at a point $\alpha \in \mathbb{T}^k$ is defined to be the number such that given $\varepsilon > 0$, there exists a neighborhood $N = (\alpha - \delta, \alpha + \delta)$ for some $\delta > 0$, such that

$$|f(\sigma(\alpha)) - f(s) - f^\Delta(\alpha)(\sigma(\alpha) - s)| \leq \varepsilon |\sigma(\alpha) - s|,$$

for all $s \in N$.

If $\mathbb{T} = \mathbb{R}$, then the delta derivative $f^\Delta = f'$ where f' is the derivative from continuous calculus.

If $\mathbb{T} = \mathbb{Z}$, then the delta derivative $f^\Delta = \Delta f$ where Δf is the forward difference operator from discrete calculus.

DEFINITION 2.1. [3] A function $f : \mathbb{T} \rightarrow \mathbb{R}$ is called *rd*-continuous if it is continuous at every right-dense point of \mathbb{T} and if its left-sided limit is finite at any left dense point of \mathbb{T} . All *rd*-continuous functions are denoted by C_{rd} .

DEFINITION 2.2. [3] A function $F : \mathbb{T} \rightarrow \mathbb{R}$ is called an antiderivative of $f : \mathbb{T} \rightarrow \mathbb{R}$ if $F^\Delta(\alpha) = f(\alpha)$, for all $\alpha \in \mathbb{T}^k$. Then, delta integral is defined by

$$\int_{u_1}^s f(\alpha)\Delta\alpha = F(s) - F(u_1),$$

where $s, u_1 \in \mathbb{T}$.

THEOREM 2.1. [3] If $f \in C_{rd}$ and $\alpha \in \mathbb{T}^k$ then

$$\int_{\alpha}^{\sigma(\alpha)} f(\omega)\Delta(\omega) = \eta(\alpha)f(\alpha).$$

THEOREM 2.2. [3] Let $f_1, f_2 \in C_{rd}, \lambda \in \mathbb{R}$ and $u_1, u_2, u_3 \in \mathbb{T}$ then

- (i) $\int_{u_1}^{u_2} (f_1(\omega) + f_2(\omega))\Delta\omega = \int_{u_1}^{u_2} f_1(\omega)\Delta\omega + \int_{u_1}^{u_2} f_2(\omega)\Delta\omega;$
- (ii) $\int_{u_1}^{u_2} \lambda f(\omega)\Delta\omega = \lambda \int_{u_1}^{u_2} f(\omega)\Delta\omega;$
- (iii) $\int_{u_1}^{u_2} f(\omega)\Delta\omega = - \int_{u_2}^{u_1} f(\omega)\Delta\omega;$
- (iv) $\int_{u_1}^{u_2} f(\omega)\Delta\omega = \int_{u_1}^{u_3} f(\omega)\Delta\omega + \int_{u_3}^{u_2} f(\omega)\Delta\omega;$
- (v) $\int_{u_1}^{u_2} f_1^\sigma(\omega)f_2^\Delta(\omega)\Delta\omega = (f_1f_2)(u_2) - (f_1f_2)(u_1) - \int_{u_1}^{u_2} f_1^\Delta(\omega)f_2(\omega)\Delta\omega;$
- (vi) $\int_{u_1}^{u_2} f_1(\omega)f_2^\Delta(\omega)\Delta\omega = (f_1f_2)(u_2) - (f_1f_2)(u_1) - \int_{u_1}^{u_2} f_1^\Delta(\omega)f_2^\sigma(\omega)\Delta\omega;$
- (vii) $\int_{u_1}^{u_1} f(\omega)\Delta\omega = 0;$

(viii) If $f(\omega) \geq 0$ for all ω , then $\int_{u_1}^{u_2} f(\omega)\Delta\omega \geq 0$;

(ix) If $|f_1(\omega)| \leq f_2(\omega)$ on $[u_1, u_2]$, then $|\int_{u_1}^{u_2} f_1(\omega)\Delta\omega| \leq \int_{u_1}^{u_2} f_2(\omega)\Delta\omega$.

From assertion (ix) of Theorem 2.2 for $f_2(\omega) = |f_1(\omega)|$ on $[u_1, u_2]$, we have

$$\left| \int_{u_1}^{u_2} f_1(\omega)\Delta\omega \right| \leq \int_{u_1}^{u_2} |f_1(\omega)|\Delta\omega.$$

DEFINITION 2.3. [20] Consider a time scale \mathbb{T} and let $\phi_1, \phi_2 : (0, 1) \rightarrow \mathbb{R}$ be two nonnegative functions. A function $f : X = [u_1, u_2]_{\mathbb{T}} \rightarrow \mathbb{R}$ is said to be a (ϕ_1, ϕ_2) -convex function with respect to two nonnegative functions ϕ_1 and ϕ_2 if

$$f((1 - \alpha)u_1 + \alpha u_2) \leq \phi_1(1 - \alpha)\phi_2(\alpha)f(u_1) + \phi_2(1 - \alpha)\phi_1(\alpha)f(u_2),$$

$$\forall u_1, u_2 \in X, \alpha \in [0, 1].$$

DEFINITION 2.4. [21] A function $f : X \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is said to be strongly convex on a convex set $X \subseteq \mathbb{R}$ if there exists a constant $c > 0$ such that

$$f((1 - \alpha)u_1 + \alpha u_2) \leq (1 - \alpha)f(u_1) + \alpha f(u_2) - c\alpha(1 - \alpha)(u_2 - u_1)^2,$$

$$\forall u_1, u_2 \in X, \alpha \in [0, 1].$$

DEFINITION 2.5. [11] Let $\gamma_k : \mathbb{T}^2 \rightarrow \mathbb{R}$, $k \in \mathbb{N}_0$ be defined by

$$\gamma_0(\alpha, \beta) = 1, \forall \alpha, \beta \in \mathbb{T}$$

and then recursively by

$$\gamma_{k+1}(\alpha, \beta) = \int_{\beta}^{\alpha} \gamma_k(\vartheta, \beta)\Delta\vartheta, \forall \alpha, \beta \in \mathbb{T}.$$

LEMMA 2.1. [19] Let $f : \mathbb{T} \rightarrow \mathbb{R}$ be a delta differentiable mapping and $u_1, u_2 \in \mathbb{T}$ with $u_1 < u_2$. If $f^{\Delta} \in C_{rd}$, then the following equality holds:

$$f(u_1)\{1 - \gamma_2(1, 0)\} + f(u_2)\gamma_2(1, 0) - \frac{1}{u_2 - u_1} \int_{u_1}^{u_2} f^{\sigma}(\omega)\Delta\omega$$

$$= \frac{u_2 - u_1}{2} \int_0^1 \int_0^1 [f^{\Delta}(\alpha u_1 + (1 - \alpha)u_2) - f^{\Delta}(\beta u_1 + (1 - \beta)u_2)](\alpha - \beta)\Delta\alpha\Delta\beta.$$

LEMMA 2.2. [19] Let $f : [u_1, u_2]_{\mathbb{T}} \rightarrow \mathbb{R}$ be a delta differentiable mapping on \mathbb{T}^0 and $u_1, u_2 \in \mathbb{T}$ with $u_1 < u_2$. If $f^{\Delta} \in C_{rd}$, then the following equality holds:

$$f\left(\frac{u_1 + u_2}{2}\right) - \frac{1}{u_2 - u_1} \int_{u_1}^{u_2} f^{\sigma}(\omega)\Delta\omega$$

$$= \frac{u_2 - u_1}{2} \int_0^1 \int_0^1 [f^{\Delta}(\alpha u_1 + (1 - \alpha)u_2) - f^{\Delta}(\beta u_1 + (1 - \beta)u_2)](\psi(\beta) - \psi(\alpha))\Delta\alpha\Delta\beta,$$

where

$$\psi(\omega) = \begin{cases} \omega, & \omega \in [0, \frac{1}{2}] \\ \omega - 1, & \omega \in (\frac{1}{2}, 1]. \end{cases}$$

COROLLARY 2.1. [20] Consider a time scale \mathbb{T} and $X = [u_1, u_2]_{\mathbb{T}}$ such that $u_1 < u_2$ and $u_1, u_2 \in \mathbb{T}$. Suppose that there is a delta differentiable function $f : X \rightarrow \mathbb{R}$ on X^0 . If $f^\Delta \in C_{rd}$, then

$$\begin{aligned} & \frac{f(u_1) + f(u_2)}{2} - \frac{1}{u_2 - u_1} \int_{u_1}^{u_2} f^\sigma(\omega) \Delta \omega \\ &= \frac{u_2 - u_1}{2} \left[\int_0^1 \alpha f^\Delta(\alpha u_2 + (1 - \alpha)u_1) \Delta \alpha - \int_0^1 \alpha f^\Delta(\alpha u_1 + (1 - \alpha)u_2) \Delta \alpha \right]. \end{aligned}$$

COROLLARY 2.2. [20] Consider a time scale \mathbb{T} and $X = [u_1, u_2]_{\mathbb{T}}$ such that $u_1 < u_2$ and $u_1, u_2 \in \mathbb{T}$. Suppose that there is a delta differentiable function $f : X \rightarrow \mathbb{R}$ on X^0 . If $f^\Delta \in C_{rd}$, then

$$\begin{aligned} & f\left(\frac{u_1 + u_2}{2}\right) - \frac{1}{u_2 - u_1} \int_{u_1}^{u_2} f^\sigma(\omega) \Delta \omega \\ &= (u_2 - u_1) \left[\int_0^{1/2} \alpha f^\Delta(\alpha u_2 + (1 - \alpha)u_1) \Delta \alpha + \int_{1/2}^1 (\alpha - 1) f^\Delta(\alpha u_2 + (1 - \alpha)u_1) \Delta \alpha \right]. \end{aligned}$$

3. Main results

Now, we define a class of strongly convex function with respect to two auxiliary functions ϕ_1 and ϕ_2 on time scales \mathbb{T} .

DEFINITION 3.1. Consider a time scale \mathbb{T} and let $\phi_1, \phi_2 : (0, 1) \rightarrow \mathbb{R}$ be two non-negative functions. A function $f : X = [u_1, u_2]_{\mathbb{T}} \rightarrow \mathbb{R}$ is said to be a (ϕ_1, ϕ_2) -strongly convex function with respect to two nonnegative functions ϕ_1, ϕ_2 and modulus c if

$$\begin{aligned} f((1 - \alpha)u_1 + \alpha u_2) &\leq \phi_1(1 - \alpha)\phi_2(\alpha)f(u_1) + \phi_2(1 - \alpha)\phi_1(\alpha)f(u_2) \\ &\quad - c\alpha(1 - \alpha)(u_2 - u_1)^2, \quad \forall u_1, u_2 \in X, \alpha \in [0, 1]. \end{aligned}$$

Now, we discuss some new special cases of Definition 3.1.

- (I). If $\phi_1(\alpha) = \phi_2(\alpha) = \alpha^s$ in Definition 3.1, then we get Breckner type of s -strongly convex functions.

DEFINITION 3.2. Consider a time scale \mathbb{T} and $s \in [0, 1]$ be a real number. A function $f : X = [u_1, u_2]_{\mathbb{T}} \rightarrow \mathbb{R}$ is a Breckner type s -strongly convex function, if

$$\begin{aligned} f((1 - \alpha)u_1 + \alpha u_2) &\leq (1 - \alpha)^s \alpha^s [f(u_1) + f(u_2)] - c\alpha(1 - \alpha)(u_2 - u_1)^2, \\ &\quad \forall u_1, u_2 \in X, \alpha \in [0, 1]. \end{aligned}$$

- (II). If $\phi_1(\alpha) = \phi_2(\alpha) = 1$ in Definition 3.1, then we get P -strongly convex functions.

DEFINITION 3.3. Consider a time scale \mathbb{T} , then $f : X = [u_1, u_2]_{\mathbb{T}} \rightarrow \mathbb{R}$ is a P -strongly convex function, if

$$f((1 - \alpha)u_1 + \alpha u_2) \leq f(u_1) + f(u_2) - c\alpha(1 - \alpha)(u_2 - u_1)^2, \quad \forall u_1, u_2 \in X, \alpha \in [0, 1].$$

THEOREM 3.1. Consider a time scale \mathbb{T} and $X = [u_1, u_2]_{\mathbb{T}}$ such that $u_1 < u_2$ and $u_1, u_2 \in \mathbb{T}$. Suppose that there is a delta differentiable function $f : X \rightarrow \mathbb{R}$ on X^0 . If $|f^\Delta|$ is (ϕ_1, ϕ_2) -strongly convex function with respect to two nonnegative functions ϕ_1, ϕ_2 and modulus c , then

$$\begin{aligned} & \left| \frac{f(u_1) + f(u_2)}{2} - \frac{1}{u_2 - u_1} \int_{u_1}^{u_2} f^\sigma(\omega) \Delta\omega \right| \\ & \leq \frac{u_2 - u_1}{2} [(A^*(\alpha) + B^*(\alpha))(|f^\Delta(u_1)| + |f^\Delta(u_2)|) - 2c(u_2 - u_1)^2 C^*(\alpha)], \end{aligned}$$

where

$$A^*(\alpha) = \int_0^1 \alpha \phi_1(\alpha) \phi_2(1 - \alpha) \Delta\alpha,$$

$$B^*(\alpha) = \int_0^1 \alpha \phi_2(\alpha) \phi_1(1 - \alpha) \Delta\alpha$$

and

$$C^*(\alpha) = \int_0^1 \alpha^2(1 - \alpha) \Delta\alpha.$$

Proof. Using Corollary 2.1, modulus property and (ϕ_1, ϕ_2) -strong convexity of $|f^\Delta|$, we obtain

$$\begin{aligned} & \left| \frac{f(u_1) + f(u_2)}{2} - \frac{1}{u_2 - u_1} \int_{u_1}^{u_2} f^\sigma(\omega) \Delta\omega \right| \\ & \leq \frac{u_2 - u_1}{2} \left[\int_0^1 \alpha |f^\Delta(\alpha u_2 + (1 - \alpha)u_1)| \Delta\alpha + \int_0^1 \alpha |f^\Delta(\alpha u_1 + (1 - \alpha)u_2)| \Delta\alpha \right] \\ & \leq \frac{u_2 - u_1}{2} \left[\int_0^1 \alpha \{ \phi_1(\alpha) \phi_2(1 - \alpha) |f^\Delta(u_2)| + \phi_2(\alpha) \phi_1(1 - \alpha) |f^\Delta(u_1)| \right. \\ & \quad \left. - c\alpha(1 - \alpha)(u_2 - u_1)^2 \} \Delta\alpha + \int_0^1 \alpha \{ \phi_1(\alpha) \phi_2(1 - \alpha) |f^\Delta(u_1)| \right. \\ & \quad \left. + \phi_2(\alpha) \phi_1(1 - \alpha) |f^\Delta(u_2)| - c\alpha(1 - \alpha)(u_2 - u_1)^2 \} \Delta\alpha \right] \\ & = \frac{u_2 - u_1}{2} \left[\int_0^1 \alpha \{ \phi_1(\alpha) \phi_2(1 - \alpha) (|f^\Delta(u_1)| + |f^\Delta(u_2)|) + \phi_2(\alpha) \phi_1(1 - \alpha) (|f^\Delta(u_1)| \right. \\ & \quad \left. + |f^\Delta(u_2)|) \} \Delta\alpha - 2c(u_2 - u_1)^2 \int_0^1 \alpha^2(1 - \alpha) \Delta\alpha \right] \\ & = \frac{u_2 - u_1}{2} [(A^*(\alpha) + B^*(\alpha))(|f^\Delta(u_1)| + |f^\Delta(u_2)|) - 2c(u_2 - u_1)^2 C^*(\alpha)]. \end{aligned}$$

This completes the proof. \square

COROLLARY 3.1. *In Theorem 3.1, if $|f^\Delta|$ is a Breckner type s -strongly convex function, then*

$$\begin{aligned} & \left| \frac{f(u_1) + f(u_2)}{2} - \frac{1}{u_2 - u_1} \int_{u_1}^{u_2} f^\sigma(\omega) \Delta\omega \right| \\ & \leq (u_2 - u_1) [H_1(\alpha) (|f^\Delta(u_1)| + |f^\Delta(u_2)|) - 2c(u_2 - u_1)^2 \int_0^1 \alpha^2 (1 - \alpha) \Delta\alpha], \end{aligned}$$

where

$$H_1(\alpha) = \int_0^1 \alpha^{s+1} (1 - \alpha)^s \Delta\alpha.$$

REMARK 3.1. If $\mathbb{T} = \mathbb{R}$, then delta integral reduces to the usual Riemann integral from calculus. Hence, Theorem 3.1 becomes

$$\begin{aligned} & \left| \frac{f(u_1) + f(u_2)}{2} - \frac{1}{u_2 - u_1} \int_{u_1}^{u_2} f(\omega) d\omega \right| \\ & \leq \frac{u_2 - u_1}{2} \left[(A^*(\alpha) + B^*(\alpha)) (|f'(u_1)| + |f'(u_2)|) - 2c(u_2 - u_1)^2 C^*(\alpha) \right], \end{aligned}$$

where

$$A^*(\alpha) = \int_0^1 \alpha \phi_1(\alpha) \phi_2(1 - \alpha) d\alpha,$$

$$B^*(\alpha) = \int_0^1 \alpha \phi_2(\alpha) \phi_1(1 - \alpha) d\alpha$$

and

$$C^*(\alpha) = \int_0^1 \alpha^2 (1 - \alpha) d\alpha.$$

THEOREM 3.2. *Consider a time scale \mathbb{T} and $X = [u_1, u_2]_{\mathbb{T}}$ such that $u_1 < u_2$ and $u_1, u_2 \in \mathbb{T}$. Suppose that there is a delta differentiable function $f : X \rightarrow \mathbb{R}$ on X^0 . If $|f^\Delta|^b$ is (ϕ_1, ϕ_2) -strongly convex function with respect to two nonnegative functions ϕ_1, ϕ_2 and modulus c , where $\frac{1}{a} + \frac{1}{b} = 1$ with $b > 1$. Then, we have*

$$\begin{aligned} & \left| \frac{f(u_1) + f(u_2)}{2} - \frac{1}{u_2 - u_1} \int_{u_1}^{u_2} f^\sigma(\omega) \Delta\omega \right| \\ & \leq \frac{u_2 - u_1}{2} \left(\int_0^1 \alpha^a \Delta\alpha \right)^{\frac{1}{a}} \left[\left(\int_0^1 \{ \phi_1(\alpha) \phi_2(1 - \alpha) |f^\Delta(u_2)|^b + \phi_2(\alpha) \phi_1(1 - \alpha) |f^\Delta(u_1)|^b \right. \right. \\ & \quad \left. \left. - c\alpha(1 - \alpha)(u_2 - u_1)^2 \} \Delta\alpha \right)^{\frac{1}{b}} + \left(\int_0^1 \alpha \{ \phi_1(\alpha) \phi_2(1 - \alpha) |f^\Delta(u_1)|^b \right. \right. \\ & \quad \left. \left. + \phi_2(\alpha) \phi_1(1 - \alpha) |f^\Delta(u_2)|^b - c\alpha(1 - \alpha)(u_2 - u_1)^2 \} \Delta\alpha \right)^{\frac{1}{b}} \right]. \end{aligned}$$

Proof. Using Corollary 2.1, modulus property, Hölder’s integral inequality and (ϕ_1, ϕ_2) -strong convexity of $|f^\Delta|$, we get

$$\begin{aligned} & \left| \frac{f(u_1) + f(u_2)}{2} - \frac{1}{u_2 - u_1} \int_{u_1}^{u_2} f^\sigma(\omega) \Delta\omega \right| \\ & \leq \frac{u_2 - u_1}{2} \left[\left| \int_0^1 \alpha f^\Delta(\alpha u_2 + (1 - \alpha)u_1) \Delta\alpha \right| + \left| \int_0^1 \alpha f^\Delta(\alpha u_1 + (1 - \alpha)u_2) \Delta\alpha \right| \right] \\ & \leq \frac{u_2 - u_1}{2} \left(\int_0^1 \alpha^a \Delta\alpha \right)^{\frac{1}{a}} \left[\left(\int_0^1 |f^\Delta(\alpha u_2 + (1 - \alpha)u_1)|^b \Delta\alpha \right)^{\frac{1}{b}} \right. \\ & \quad \left. + \left(\int_0^1 |f^\Delta(\alpha u_1 + (1 - \alpha)u_2)|^b \Delta\alpha \right)^{\frac{1}{b}} \right] \\ & \leq \frac{u_2 - u_1}{2} \left(\int_0^1 \alpha^a \Delta\alpha \right)^{\frac{1}{a}} \left[\left(\int_0^1 \{ \phi_1(\alpha)\phi_2(1 - \alpha) |f^\Delta(u_2)|^b + \phi_2(\alpha)\phi_1(1 - \alpha) |f^\Delta(u_1)|^b \right. \right. \\ & \quad \left. \left. - c\alpha(1 - \alpha)(u_2 - u_1)^2 \} \Delta\alpha \right)^{\frac{1}{b}} + \left(\int_0^1 \{ (\phi_1(\alpha)\phi_2(1 - \alpha)) |f^\Delta(u_1)|^b \right. \right. \\ & \quad \left. \left. + (\phi_2(\alpha)\phi_1(1 - \alpha)) |f^\Delta(u_2)|^b - c\alpha(1 - \alpha)(u_2 - u_1)^2 \} \Delta\alpha \right)^{\frac{1}{b}} \right]. \end{aligned}$$

This completes the proof. \square

COROLLARY 3.2. *In Theorem 3.2, if $|f^\Delta|^b$ is a Breckner type s -strongly convex function, then*

$$\begin{aligned} & \left| \frac{f(u_1) + f(u_2)}{2} - \frac{1}{u_2 - u_1} \int_{u_1}^{u_2} f^\sigma(\omega) \Delta\omega \right| \\ & \leq \frac{u_2 - u_1}{2} \left(\int_0^1 \alpha^a \Delta\alpha \right)^{\frac{1}{a}} \left[\left(\int_0^1 \{ \alpha^s(1 - \alpha)^s (|f^\Delta(u_1)|^b + |f^\Delta(u_2)|^b) \right. \right. \\ & \quad \left. \left. - c\alpha(1 - \alpha)(u_2 - u_1)^2 \} \Delta\alpha \right)^{\frac{1}{b}} \right]. \end{aligned}$$

REMARK 3.2. If $\mathbb{T} = \mathbb{R}$, then delta integral reduces to the usual Riemann integral from calculus. Hence, Theorem 3.2 becomes

$$\begin{aligned} & \left| \frac{f(u_1) + f(u_2)}{2} - \frac{1}{u_2 - u_1} \int_{u_1}^{u_2} f(\omega) d\omega \right| \\ & \leq \frac{u_2 - u_1}{2} \left(\int_0^1 \alpha^a d\alpha \right)^{\frac{1}{a}} \left[\left(\int_0^1 \{ \phi_1(\alpha)\phi_2(1 - \alpha) |f'(u_2)|^b + \phi_2(\alpha)\phi_1(1 - \alpha) |f'(u_1)|^b \right. \right. \\ & \quad \left. \left. - c\alpha(1 - \alpha)(u_2 - u_1)^2 \} d\alpha \right)^{\frac{1}{b}} + \left(\int_0^1 \{ (\phi_1(\alpha)\phi_2(1 - \alpha)) |f'(u_1)|^b \right. \right. \\ & \quad \left. \left. + \phi_2(\alpha)\phi_1(1 - \alpha) |f'(u_2)|^b - c\alpha(1 - \alpha)(u_2 - u_1)^2 \} d\alpha \right)^{\frac{1}{b}} \right]. \end{aligned}$$

THEOREM 3.3. Consider a time scale \mathbb{T} and $X = [u_1, u_2]_{\mathbb{T}}$ such that $u_1 < u_2$ and $u_1, u_2 \in \mathbb{T}$. Suppose that there is a delta differentiable function $f : X \rightarrow \mathbb{R}$ on X^0 . If $|f^\Delta|$ is (ϕ_1, ϕ_2) -strongly convex function with respect to two nonnegative functions ϕ_1, ϕ_2 and modulus c , then

$$\left| f\left(\frac{u_1+u_2}{2}\right) - \frac{1}{u_2-u_1} \int_{u_1}^{u_2} f^\sigma(\omega) \Delta\omega \right| \\ \leq (u_2-u_1) \left[A^{**}(\alpha) |f^\Delta(u_2)| + B^{**}(\alpha) |f^\Delta(u_1)| - c(u_2-u_1)^2 C^{**}(\alpha) \right],$$

where

$$A^{**}(\alpha) = \int_0^{1/2} \alpha \phi_1(\alpha) \phi_2(1-\alpha) \Delta\alpha + \int_{1/2}^1 (1-\alpha) \phi_1(\alpha) \phi_2(1-\alpha) \Delta\alpha,$$

$$B^{**}(\alpha) = \int_0^{1/2} \alpha \phi_2(\alpha) \phi_1(1-\alpha) \Delta\alpha + \int_{1/2}^1 (1-\alpha) \phi_2(\alpha) \phi_1(1-\alpha) \Delta\alpha$$

and

$$C^{**}(\alpha) = \int_0^{1/2} \alpha^2(1-\alpha) \Delta\alpha + \int_{1/2}^1 \alpha(1-\alpha)^2 \Delta\alpha.$$

Proof. Using Corollary 2.2, modulus property and (ϕ_1, ϕ_2) -strong convexity of $|f^\Delta|$, we get

$$\left| f\left(\frac{u_1+u_2}{2}\right) - \frac{1}{u_2-u_1} \int_{u_1}^{u_2} f^\sigma(\omega) \Delta\omega \right| \\ \leq (u_2-u_1) \left[\int_0^{1/2} \alpha |f^\Delta(\alpha u_2 + (1-\alpha)u_1)| \Delta\alpha + \int_{1/2}^1 |\alpha-1| |f^\Delta(\alpha u_2 + (1-\alpha)u_1)| \Delta\alpha \right] \\ \leq (u_2-u_1) \left[\int_0^{1/2} \alpha \{ \phi_1(\alpha) \phi_2(1-\alpha) |f^\Delta(u_2)| + \phi_2(\alpha) \phi_1(1-\alpha) |f^\Delta(u_1)| \right. \\ \left. - c\alpha(1-\alpha)(u_2-u_1)^2 \} \Delta\alpha + \int_{1/2}^1 (1-\alpha) \{ \phi_1(\alpha) \phi_2(1-\alpha) |f^\Delta(u_2)| \right. \\ \left. + \phi_2(\alpha) \phi_1(1-\alpha) |f^\Delta(u_1)| - c\alpha(1-\alpha)(u_2-u_1)^2 \} \Delta\alpha \right] \\ = (u_2-u_1) \left[\left\{ \int_0^{1/2} \alpha \phi_1(\alpha) \phi_2(1-\alpha) \Delta\alpha + \int_{1/2}^1 (1-\alpha) \phi_1(\alpha) \phi_2(1-\alpha) \Delta\alpha \right\} |f^\Delta(u_2)| \right. \\ \left. + \left\{ \int_0^{1/2} \alpha \phi_2(\alpha) \phi_1(1-\alpha) \Delta\alpha + \int_{1/2}^1 (1-\alpha) \phi_2(\alpha) \phi_1(1-\alpha) \Delta\alpha \right\} |f^\Delta(u_1)| \right. \\ \left. - c(u_2-u_1)^2 \left\{ \int_0^{1/2} \alpha^2(1-\alpha) \Delta\alpha + \int_{1/2}^1 \alpha(1-\alpha)^2 \Delta\alpha \right\} \right] \\ = (u_2-u_1) \left[A^{**}(\alpha) |f^\Delta(u_2)| + B^{**}(\alpha) |f^\Delta(u_1)| - c(u_2-u_1)^2 C^{**}(\alpha) \right].$$

This completes the proof. \square

REMARK 3.3. If $\mathbb{T} = \mathbb{R}$, then our delta integral reduces to the usual Riemann integral from calculus. Hence, Theorem 3.3 becomes

$$\left| f\left(\frac{u_1 + u_2}{2}\right) - \frac{1}{u_2 - u_1} \int_{u_1}^{u_2} f(\omega) d\omega \right| \leq (u_2 - u_1) [A^{**}(\alpha)|f'(u_1)| + B^{**}(\alpha)|f'(u_2)| - c(u_2 - u_1)^2 C^{**}(\alpha)],$$

where

$$A^{**}(\alpha) = \int_0^{1/2} \alpha \phi_1(\alpha) \phi_2(1 - \alpha) d\alpha + \int_{1/2}^1 (1 - \alpha) \phi_1(\alpha) \phi_2(1 - \alpha) d\alpha,$$

$$B^{**}(\alpha) = \int_0^{1/2} \alpha \phi_2(\alpha) \phi_1(1 - \alpha) d\alpha + \int_{1/2}^1 (1 - \alpha) \phi_2(\alpha) \phi_1(1 - \alpha) d\alpha$$

and

$$C^{**}(\alpha) = \int_0^{1/2} \alpha^2(1 - \alpha) d\alpha + \int_{1/2}^1 \alpha(1 - \alpha)^2 d\alpha.$$

THEOREM 3.4. Consider a time scale \mathbb{T} and $X = [u_1, u_2]_{\mathbb{T}}$ such that $u_1 < u_2$ and $u_1, u_2 \in \mathbb{T}$. Suppose that there is a delta differentiable function $f : X \rightarrow \mathbb{R}$ on X^0 . If $|f^\Delta|^b$ is (ϕ_1, ϕ_2) -strongly convex function with respect to two nonnegative functions ϕ_1, ϕ_2 and modulus c , where $\frac{1}{a} + \frac{1}{b} = 1$ with $b > 1$. Then, we have

$$\begin{aligned} & \left| f\left(\frac{u_1 + u_2}{2}\right) - \frac{1}{u_2 - u_1} \int_{u_1}^{u_2} f^\sigma(\omega) \Delta\omega \right| \\ & \leq (u_2 - u_1) \left[\left(\int_0^{1/2} \alpha^a \Delta\alpha\right)^{\frac{1}{a}} \left(\int_0^{1/2} \{\phi_1(\alpha)\phi_2(1 - \alpha)|f^\Delta(u_2)|^b \right. \right. \\ & \quad \left. \left. + \phi_2(\alpha)\phi_1(1 - \alpha)|f^\Delta(u_1)|^b - c\alpha(1 - \alpha)(u_2 - u_1)^2\} \Delta\alpha\right)^{\frac{1}{b}} \right. \\ & \quad \left. + \left(\int_{1/2}^1 |1 - \alpha|^a \Delta\alpha\right)^{\frac{1}{a}} \left(\int_{1/2}^1 \{\phi_1(\alpha)\phi_2(1 - \alpha)|f^\Delta(u_2)|^b \right. \right. \\ & \quad \left. \left. + \phi_2(\alpha)\phi_1(1 - \alpha)|f^\Delta(u_1)|^b - c\alpha(1 - \alpha)(u_2 - u_1)^2\} \Delta\alpha\right)^{\frac{1}{b}} \right]. \end{aligned}$$

Proof. Using Corollary 2.2, modulus property, Hölder’s integral inequality and (ϕ_1, ϕ_2) -strong convexity of $|f^\Delta|$, we get

$$\left| f\left(\frac{u_1 + u_2}{2}\right) - \frac{1}{u_2 - u_1} \int_{u_1}^{u_2} f^\sigma(\omega) \Delta\omega \right| \leq (u_2 - u_1) \left[\left| \int_0^{1/2} \alpha f^\Delta(\alpha u_2 + (1 - \alpha)u_1) \Delta\alpha \right| + \left| \int_{1/2}^1 (\alpha - 1) f^\Delta(\alpha u_2 + (1 - \alpha)u_1) \Delta\alpha \right| \right]$$

$$\begin{aligned}
&\leq (u_2 - u_1) \left[\left(\int_0^{1/2} \alpha^a \Delta \alpha \right)^{\frac{1}{a}} \left(\int_0^{1/2} |f^\Delta(\alpha u_2 + (1 - \alpha)u_1|^b \Delta \alpha \right)^{\frac{1}{b}} \right. \\
&\quad \left. + \left(\int_{1/2}^1 |1 - \alpha|^a \Delta \alpha \right)^{\frac{1}{a}} \left(\int_{1/2}^1 |f^\Delta(\alpha u_2 + (1 - \alpha)u_1|^b \Delta \alpha \right)^{\frac{1}{b}} \right] \\
&\leq (u_2 - u_1) \left[\left(\int_0^{1/2} \alpha^a \Delta \alpha \right)^{\frac{1}{a}} \left(\int_0^{1/2} \{ \phi_1(\alpha) \phi_2(1 - \alpha) |f^\Delta(u_2)|^b \right. \right. \\
&\quad \left. \left. + \phi_2(\alpha) \phi_1(1 - \alpha) |f^\Delta(u_1)|^b - c \alpha(1 - \alpha)(u_2 - u_1)^2 \} \Delta \alpha \right)^{\frac{1}{b}} \right. \\
&\quad \left. + \left(\int_{1/2}^1 |1 - \alpha|^a \Delta \alpha \right)^{\frac{1}{a}} \left(\int_{1/2}^1 \{ \phi_1(\alpha) \phi_2(1 - \alpha) |f^\Delta(u_2)|^b \right. \right. \\
&\quad \left. \left. + \phi_2(\alpha) \phi_1(1 - \alpha) |f^\Delta(u_1)|^b - c \alpha(1 - \alpha)(u_2 - u_1)^2 \} \Delta \alpha \right)^{\frac{1}{b}} \right].
\end{aligned}$$

This completes the proof. \square

REMARK 3.4. If $\mathbb{T} = \mathbb{R}$, then our delta integral reduces to the usual Riemann integral from calculus. Hence, Theorem 3.4 becomes

$$\begin{aligned}
&\left| f\left(\frac{u_1 + u_2}{2}\right) - \frac{1}{u_2 - u_1} \int_{u_1}^{u_2} f(\omega) d\omega \right| \\
&\leq (u_2 - u_1) \left[\left(\int_0^{1/2} \alpha^a d\alpha \right)^{\frac{1}{a}} \left(\int_0^{1/2} \{ \phi_1(\alpha) \phi_2(1 - \alpha) |f'(u_2)|^b \right. \right. \\
&\quad \left. \left. + \phi_2(\alpha) \phi_1(1 - \alpha) |f'(u_1)|^b - c \alpha(1 - \alpha)(u_2 - u_1)^2 \} d\alpha \right)^{\frac{1}{b}} \right. \\
&\quad \left. + \left(\int_{1/2}^1 |1 - \alpha|^a d\alpha \right)^{\frac{1}{a}} \left(\int_{1/2}^1 \{ \phi_1(\alpha) \phi_2(1 - \alpha) |f'(u_2)|^b \right. \right. \\
&\quad \left. \left. + \phi_2(\alpha) \phi_1(1 - \alpha) |f'(u_1)|^b - c \alpha(1 - \alpha)(u_2 - u_1)^2 \} d\alpha \right)^{\frac{1}{b}} \right].
\end{aligned}$$

THEOREM 3.5. Let $f : \mathbb{T} \rightarrow \mathbb{R}$ be a differentiable mapping and $u_1, u_2 \in \mathbb{T}$ with $u_1 < u_2$. Let $|f^\Delta|$ be (ϕ_1, ϕ_2) -strongly convex function with respect to two nonnegative functions ϕ_1, ϕ_2 and modulus c , then

$$\begin{aligned}
&\left| f(u_1) \{1 - \gamma_2(1, 0)\} + f(u_2) \gamma_2(1, 0) - \frac{1}{u_2 - u_1} \int_{u_1}^{u_2} f^\sigma(\omega) \Delta \omega \right| \\
&\leq \frac{u_2 - u_1}{2} [A^{***}(\alpha, \beta) |f^\Delta(u_1)| + B^{***}(\alpha, \beta) |f^\Delta(u_2)| - c(u_2 - u_1)^2 C^{***}(\alpha, \beta)],
\end{aligned}$$

where

$$A^{***}(\alpha, \beta) = \int_0^1 \int_0^1 \{ \phi_1(\alpha) \phi_2(1 - \alpha) + \phi_1(\beta) \phi_2(1 - \beta) \} (\alpha + \beta) \Delta \alpha \Delta \beta,$$

$$B^{***}(\alpha, \beta) = \int_0^1 \int_0^1 \{\phi_2(\alpha)\phi_1(1-\alpha) + \phi_2(\beta)\phi_1(1-\beta)\}(\alpha + \beta)\Delta\alpha\Delta\beta,$$

and

$$C^{***}(\alpha, \beta) = \int_0^1 \int_0^1 \{\alpha(1-\alpha) + \beta(1-\beta)\}(\alpha + \beta)\Delta\alpha\Delta\beta.$$

Proof. Using Lemma 2.1, property of modulus and (ϕ_1, ϕ_2) -strong convexity of $|f^\Delta|$, we obtain

$$\begin{aligned} & \left| f(u_1)\{1 - \gamma_2(1, 0)\} + f(u_2)\gamma_2(1, 0) - \frac{1}{u_2 - u_1} \int_{u_1}^{u_2} f^\sigma(\omega)\Delta\omega \right| \\ & \leq \frac{u_2 - u_1}{2} \int_0^1 \int_0^1 |f^\Delta(\alpha u_1 + (1 - \alpha)u_2) - f^\Delta(\beta u_1 + (1 - \beta)u_2)| |\alpha - \beta| \Delta\alpha\Delta\beta \\ & \leq \frac{u_2 - u_1}{2} \int_0^1 \int_0^1 \{|f^\Delta(\alpha u_1 + (1 - \alpha)u_2)| + |f^\Delta(\beta u_1 + (1 - \beta)u_2)|\} (\alpha + \beta) \Delta\alpha\Delta\beta \\ & \leq \frac{u_2 - u_1}{2} \int_0^1 \int_0^1 \{\phi_1(\alpha)\phi_2(1 - \alpha)|f^\Delta(u_1)| + \phi_2(\alpha)\phi_1(1 - \alpha)|f^\Delta(u_2)| \\ & \quad - c\alpha(1 - \alpha)(u_2 - u_1)^2 + \phi_1(\beta)\phi_2(1 - \beta)|f^\Delta(u_1)| + \phi_2(\beta)\phi_1(1 - \beta)|f^\Delta(u_2)| \\ & \quad - c\beta(1 - \beta)(u_2 - u_1)^2\} (\alpha + \beta) \Delta\alpha\Delta\beta \\ & = \frac{u_2 - u_1}{2} \int_0^1 \int_0^1 [\{\phi_1(\alpha)\phi_2(1 - \alpha) + \phi_1(\beta)\phi_2(1 - \beta)\}|f^\Delta(u_1)| + \{\phi_2(\alpha)\phi_1(1 - \alpha) \\ & \quad + \phi_2(\beta)\phi_1(1 - \beta)\}|f^\Delta(u_2)| - c(u_2 - u_1)^2\{\alpha(1 - \alpha) + \beta(1 - \beta)\}] (\alpha + \beta) \Delta\alpha\Delta\beta \\ & = \frac{u_2 - u_1}{2} [A^{***}(\alpha, \beta)|f^\Delta(u_1)| + B^{***}(\alpha, \beta)|f^\Delta(u_2)| - c(u_2 - u_1)^2 C^{***}(\alpha, \beta)]. \end{aligned}$$

This completes the proof. \square

REMARK 3.5. If $\mathbb{T} = \mathbb{R}$, then our delta integral reduces to the usual Riemann integral from calculus. Hence, Theorem 3.5 becomes

$$\begin{aligned} & \left| \frac{f(u_1) + f(u_2)}{2} - \frac{1}{u_2 - u_1} \int_{u_1}^{u_2} f(\omega)d\omega \right| \\ & \leq \frac{u_2 - u_1}{2} [A^{***}(\alpha, \beta)|f'(u_1)| + B^{***}(\alpha, \beta)|f'(u_2)| - c(u_2 - u_1)^2 C^{***}(\alpha, \beta)], \end{aligned}$$

where

$$A^{***}(\alpha, \beta) = \int_0^1 \int_0^1 \{\phi_1(\alpha)\phi_2(1 - \alpha) + \phi_1(\beta)\phi_2(1 - \beta)\}(\alpha + \beta)d\alpha d\beta,$$

$$B^{***}(\alpha, \beta) = \int_0^1 \int_0^1 \{\phi_2(\alpha)\phi_1(1 - \alpha) + \phi_2(\beta)\phi_1(1 - \beta)\}(\alpha + \beta)d\alpha d\beta,$$

$$C^{***}(\alpha, \beta) = \int_0^1 \int_0^1 \{\alpha(1 - \alpha) + \beta(1 - \beta)\}(\alpha + \beta)d\alpha d\beta$$

and

$$\gamma_2(1,0) = \int_0^1 (1 - \alpha)d\alpha = 1/2.$$

THEOREM 3.6. *Let $f : [u_1, u_2]_{\mathbb{T}} \rightarrow \mathbb{R}$ be a delta differentiable mapping on \mathbb{T}^0 such that $u_1 < u_2$. If $|f^\Delta|$ is (ϕ_1, ϕ_2) -strongly convex function with respect to two nonnegative functions ϕ_1, ϕ_2 and modulus c , then*

$$\begin{aligned} & \left| f\left(\frac{u_1 + u_2}{2}\right) - \frac{1}{u_2 - u_1} \int_{u_1}^{u_2} f^\sigma(\omega)\Delta\omega \right| \\ & \leq \frac{u_2 - u_1}{2} [A^{****}(\alpha, \beta)|f^\Delta(u_1)| + B^{****}(\alpha, \beta)|f^\Delta(u_2)| - c(u_2 - u_1)^2 C^{****}(\alpha, \beta)], \end{aligned}$$

where

$$A^{****}(\alpha, \beta) = \int_0^1 \int_0^1 |\psi(\beta) - \psi(\alpha)| [\phi_1(\alpha)\phi_2(1 - \alpha) + \phi_1(\beta)\phi_2(1 - \beta)] \Delta\alpha\Delta\beta,$$

$$B^{****}(\alpha, \beta) = \int_0^1 \int_0^1 |\psi(\beta) - \psi(\alpha)| [\phi_1(1 - \alpha)\phi_2(\alpha) + \phi_1(1 - \beta)\phi_2(\beta)] \Delta\alpha\Delta\beta$$

and

$$C^{****}(\alpha, \beta) = \int_0^1 \int_0^1 |\psi(\beta) - \psi(\alpha)| [\alpha(1 - \alpha) + \beta(1 - \beta)] \Delta\alpha\Delta\beta.$$

Proof. Using Lemma 2.2, property of modulus and (ϕ_1, ϕ_2) -strong convexity of $|f^\Delta|$, we obtain

$$\begin{aligned} & \left| f\left(\frac{u_1 + u_2}{2}\right) - \frac{1}{u_2 - u_1} \int_{u_1}^{u_2} f^\sigma(\omega)\Delta\omega \right| \\ & \leq \frac{u_2 - u_1}{2} \int_0^1 \int_0^1 |f^\Delta(\alpha u_1 + (1 - \alpha)u_2) - f^\Delta(\beta u_1 + (1 - \beta)u_2)| |\psi(\beta) - \psi(\alpha)| \Delta\alpha\Delta\beta \\ & \leq \frac{u_2 - u_1}{2} \int_0^1 \int_0^1 \{|f^\Delta(\alpha u_1 + (1 - \alpha)u_2)| + |f^\Delta(\beta u_1 + (1 - \beta)u_2)|\} |\psi(\beta) - \psi(\alpha)| \Delta\alpha\Delta\beta \\ & \leq \frac{u_2 - u_1}{2} \int_0^1 \int_0^1 \{\phi_1(\alpha)\phi_2(1 - \alpha)|f^\Delta(u_1)| + \phi_2(\alpha)\phi_1(1 - \alpha)|f^\Delta(u_2)| \\ & \quad - c\alpha(1 - \alpha)(u_2 - u_1)^2 + \phi_1(\beta)\phi_2(1 - \beta)|f^\Delta(u_1)| + \phi_2(\beta)\phi_1(1 - \beta)|f^\Delta(u_2)| \\ & \quad - c\beta(1 - \beta)|u_2 - u_1|^2\} |\psi(\beta) - \psi(\alpha)| \Delta\alpha\Delta\beta \\ & = \frac{u_2 - u_1}{2} \int_0^1 \int_0^1 [\{\phi_1(\alpha)\phi_2(1 - \alpha) + \phi_1(\beta)\phi_2(1 - \beta)\}|f^\Delta(u_1)| + \{\phi_2(\alpha)\phi_1(1 - \alpha) \\ & \quad + \{\phi_2(\beta)\phi_1(1 - \beta)\}|f^\Delta(u_2)| - c(u_2 - u_1)^2\{\alpha(1 - \alpha) + \beta(1 - \beta)\}] |\psi(\beta) - \psi(\alpha)| \Delta\alpha\Delta\beta \\ & = \frac{u_2 - u_1}{2} [A^{****}(\alpha, \beta)|f^\Delta(u_1)| + B^{****}(\alpha, \beta)|f^\Delta(u_2)| - c(u_2 - u_1)^2 C^{****}(\alpha, \beta)]. \end{aligned}$$

This completes the proof. \square

REMARK 3.6. If $\mathbb{T} = \mathbb{R}$, then our delta integral reduces to the usual Riemann integral from calculus. Hence, Theorem 3.6 becomes

$$\begin{aligned} & \left| f\left(\frac{u_1 + u_2}{2}\right) - \frac{1}{u_2 - u_1} \int_{u_1}^{u_2} f(\omega) d\omega \right| \\ & \leq \frac{u_2 - u_1}{2} [A^{****}(\alpha, \beta) |f'(u_1)| + B^{****}(\alpha, \beta) |f'(u_2)| - c(u_2 - u_1)^2 C^{****}(\alpha, \beta)], \end{aligned}$$

where

$$A^{****}(\alpha, \beta) = \int_0^1 \int_0^1 |\psi(\beta) - \psi(\alpha)| [\phi_1(\alpha)\phi_2(1 - \alpha) + \phi_1(\beta)\phi_2(1 - \beta)] d\alpha d\beta,$$

$$B^{****}(\alpha, \beta) = \int_0^1 \int_0^1 |\psi(\beta) - \psi(\alpha)| [\phi_1(1 - \alpha)\phi_2(\alpha) + \phi_1(1 - \beta)\phi_2(\beta)] d\alpha d\beta$$

and

$$C^{****}(\alpha, \beta) = \int_0^1 \int_0^1 |\psi(\beta) - \psi(\alpha)| [\alpha(1 - \alpha) + \beta(1 - \beta)] d\alpha d\beta.$$

EXAMPLE 3.1. Let $\mathbb{T} = \mathbb{R}$. Obviously, $f(\alpha) = \alpha$ is a strongly convex function with $\phi_1(\alpha) = 2 - \alpha$, $\phi_2(\alpha) = 1$, $c = 1$, and continuous on $(0, \infty)$, so we may apply Theorem 3.1 with $u_1 = 1/2$ and $u_2 = 1$. Clearly

$$\begin{aligned} & \left| \frac{f(u_1) + f(u_2)}{2} - \frac{1}{u_2 - u_1} \int_{u_1}^{u_2} f(\omega) d\omega \right| \\ & = \frac{3}{4} - 2 \int_{1/2}^1 \omega d\omega = 0. \end{aligned} \tag{1}$$

On the other hand

$$\begin{aligned} & \frac{u_2 - u_1}{2} [(A^*(\alpha) + B^*(\alpha))(|f'(u_1)| + |f'(u_2)|) - 2c(u_2 - u_1)^2 C^*(\alpha)] \\ & = \frac{1}{4} \left[\frac{9}{6} \times 2 - 2 \times 1 \times \frac{1}{4} \times \frac{1}{12} \right] \approx 0.7395, \end{aligned} \tag{2}$$

where

$$A^*(\alpha) = \int_0^1 \alpha(2 - \alpha) d\alpha = \frac{2}{3},$$

$$B^*(\alpha) = \int_0^1 \alpha(1 + \alpha) d\alpha = \frac{5}{6}$$

and

$$C^*(\alpha) = \int_0^1 \alpha^2(1 - \alpha) d\alpha = \frac{1}{12}.$$

From (1) and (2), we see that $0 < 0.7395$.

EXAMPLE 3.2. Let $\mathbb{T} = \mathbb{R}$. Obviously, $f(\alpha) = \alpha + 1$ is strongly convex with $\phi_1(\alpha) = 2$, $\phi_2(\alpha) = 4$, $c = 1/4$ and continuous on $(0, \infty)$, so we may apply Theorem 3.3 with $u_1 = 0$ and $u_2 = 1/4$. Clearly

$$\begin{aligned} & \left| f\left(\frac{u_1+u_2}{2}\right) - \frac{1}{u_2-u_1} \int_{u_1}^{u_2} f(\omega) d\omega \right| \\ &= \left| \frac{9}{8} - 4 \int_0^{1/2} (\omega+1) d\omega \right| = 0. \end{aligned} \quad (3)$$

On the other hand

$$\begin{aligned} & (u_2 - u_1) [A^{**}(\alpha)|f'(u_1)| + B^{**}(\alpha)|f'(u_2)| - c(u_2 - u_1)^2 C^{**}] \\ &= \frac{1}{4} \left[(2 \times 1 + 2 \times 1) - \frac{1}{4} \times \frac{1}{16} \times \frac{5}{96} \right] \approx 0.9997, \end{aligned} \quad (4)$$

where

$$\begin{aligned} A^{**}(\alpha) &= B^{**}(\alpha) = 8 \int_0^{1/2} \alpha d\alpha + 8 \int_{1/2}^1 (1-\alpha) d\alpha = 2, \\ C^{**}(\alpha) &= \int_0^{1/2} \alpha^2 (1-\alpha) d\alpha + \int_{1/2}^1 \alpha (1-\alpha)^2 d\alpha = \frac{5}{96}. \end{aligned}$$

From (3) and (4), we see that $0 < 0.9997$.

EXAMPLE 3.3. Let $\mathbb{T} = \mathbb{R}$. Obviously, $f(\alpha) = \sqrt{\alpha}$ is strongly convex with $\phi_1(\alpha) = 4$, $\phi_2(\alpha) = \alpha$, $c = 1/4$, and continuous on $(0, \infty)$, so we may apply Theorem 3.5 with $u_1 = 2$ and $u_2 = 4$. Clearly

$$\begin{aligned} & \left| f\left(\frac{u_1+u_2}{2}\right) - \frac{1}{u_2-u_1} \int_{u_1}^{u_2} f(\omega) d\omega \right| \\ &= \left| \sqrt{\frac{2+4}{2}} - \frac{1}{2} \int_2^4 \sqrt{\omega} d\omega \right| \approx 0.0081. \end{aligned} \quad (5)$$

On the other hand

$$\begin{aligned} & \frac{u_2-u_1}{2} [A^{***}(\alpha, \beta)|f^\Delta(u_1)| + B^{***}(\alpha, \beta)|f^\Delta(u_2)| - c(u_2 - u_1)^2 C^{***}(\alpha, \beta)] \\ &= \frac{4-2}{2} \left[\frac{10}{3} \times \frac{1}{2\sqrt{2}} + \frac{14}{3} \times \frac{1}{4} - \frac{1}{4} \times 4 \times \frac{1}{3} \right] \approx 2.0118, \end{aligned} \quad (6)$$

where

$$\begin{aligned} A^{***}(\alpha, \beta) &= \int_0^1 \int_0^1 \{4(1-\alpha) + 4(1-\beta)\}(\alpha + \beta) d\alpha d\beta = \frac{10}{3}, \\ B^{***}(\alpha, \beta) &= \int_0^1 \int_0^1 (4\alpha + 4\beta)(\alpha + \beta) d\alpha d\beta = \frac{14}{3} \end{aligned}$$

and

$$C^{***}(\alpha, \beta) = \int_0^1 \int_0^1 \{\alpha(1-\alpha) + \beta(1-\beta)\}(\alpha + \beta) d\alpha d\beta = \frac{1}{3}.$$

From (5) and (6), we see that $0.0081 < 2.0118$.

4. Conclusions

In this paper, we have introduced the concept of strongly convex functions on time scales by selecting the appropriate values of functions ϕ_1 and ϕ_2 . Further, we have established Hermite-Hadamard type inequalities for (ϕ_1, ϕ_2) -strongly convex functions with respect to two nonnegative functions ϕ_1, ϕ_2 on time scales. We have also discussed several particular cases when $\mathbb{T} = \mathbb{R}$. The results obtained in this paper are the generalization of the previously known results. The idea of (ϕ_1, ϕ_2) -strongly convex function and obtained results in this paper may have further applications in future research work.

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