

GENERALIZED INEQUALITIES FOR FUNCTIONS OF L_p SPACES VIA MONTGOMERY IDENTITY WITH PARAMETERS

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Abstract. We obtain new Montgomery identity with parameters for two variables. By using this obtained identity, we give some generalization of Ostrowski type inequality for L_p spaces with better bounds. In addition, we modify Grüss type inequality for two independent variables involving parameters.

Montgomery identity is one of the classical results that creates many important inequalities such as Ostrowski inequality, Grüss inequality and Ostrowski-Grüss inequalities. Its bivariate form has introduced some new generalization and advancement in different inequalities. These inequalities have many applications in various fields of mathematics such as numerical integration and probability theory. We can also obtain special means with the help of these inequalities. In the last 20 years rapid advancement in generalization and improvement of these types of inequalities has been observed for references see [2, 4, 12, 13, 15, 16, 21, 23, 24]. This article deals with its bivariate form in order to generate our proposed results of Ostrowski and Grüss type inequalities in terms of parameters. The idea behind the results based on parameters is to make further generalization of those results of Ostrowski and Grüss inequality which are non parametric based, as parameters extends the region of inequality more wider and provides a family of solutions and the quality of inequality will improve conclusively. If we talk about L_p spaces, this is the first ever combination of L_p space, parameters and bivariate differentiable functions, which some how connects our result with lebesgue measure.

We need the following definition to use in our results. Hölder's inequality, named after Otto Hölder, is a basic inequality and an essential tool for the study of L_p spaces. Hölder's inequality was first found by Rogers in 1888, and discovered independently by Hölder in 1889. Integral version of Hölder's Inequality [19] is stated as:

THEOREM 0.1. *Let $1 \leq p \leq \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. If $f \in L_p$ and $g \in L_q$, then $fg \in L$ and*

$$\int |f(x)g(x)|dx \leq \|f\|_p \|g\|_q,$$

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where $\|f\|_p = \left(\int |f(x)|^p dx\right)^{1/p} < \infty$.

Moreover if $f \in L_1$ and $g \in L_\infty$, then

$$\int |f(x)g(x)| dx \leq \|f\|_1 \|g\|_\infty,$$

where $\|f\|_\infty = \operatorname{ess\,sup}_{\forall x} |f(x)|$.

In the first section, we obtain Montgomery identity with parameters of two independent variables, while in the second section, we establish new inequalities of Ostrowski type for two variables in terms of parameters for L_p spaces. In the Last section, we will achieve Grüss type inequalities with its Čebyšev functional.

Throughout the paper, we have $I = [\alpha_a, \alpha_b]$ and $J = [\alpha_c, \alpha_d]$.

1. Montgomery identity for functions of two variables involving parameters

Montgomery identity is very useful to gain some interesting inequalities. Here we state the classical Montgomery identity from “Inequalities for Functions and their Integrals and Derivatives” by Mitrinović et al. in [18, p. 565].

PROPOSITION 1.1. *Let $f : I \rightarrow \mathbb{R}$ be an absolutely continuous function. Then*

$$f(x) = \frac{1}{\alpha_b - \alpha_a} \int_{\alpha_a}^{\alpha_b} f(t)dt + \frac{1}{\alpha_b - \alpha_a} \int_{\alpha_a}^{\alpha_b} P(x,t)f'(t)dt, \tag{1.1}$$

where Peano kernel $P(x,t)$ is given as

$$P(x,t) = \begin{cases} t - \alpha_a, & \text{if } t \in [\alpha_a, x], \\ t - \alpha_b, & \text{if } t \in (x, \alpha_b]. \end{cases} \tag{1.2}$$

In 2001, Dragomir et al. introduced the previous identity with parameters in [7] as recalled in the following proposition.

PROPOSITION 1.2. *Let $f : I \rightarrow \mathbb{R}$ is differentiable on $[\alpha_a, \alpha_b]$ with f' integrable on (α_a, α_b) , where $\varepsilon \in [0, 1]$. Then generalized integral identity holds*

$$(1 - \varepsilon)f(x) + \varepsilon \frac{f(\alpha_a) + f(\alpha_b)}{2} - \frac{1}{\alpha_b - \alpha_a} \int_{\alpha_a}^{\alpha_b} f(t)dt = \frac{1}{\alpha_b - \alpha_a} \int_{\alpha_a}^{\alpha_b} P_1(x,t)f'(t)dt, \tag{1.3}$$

where $\varepsilon \in [0, 1]$ and $P_1(x,t)$ is defined as

$$P_1(x,t) = \begin{cases} t - (\alpha_a + \varepsilon \frac{\alpha_b - \alpha_a}{2}), & \text{if } t \in [\alpha_a, x], \\ t - (\alpha_b - \varepsilon \frac{\alpha_b - \alpha_a}{2}), & \text{if } t \in (x, \alpha_b], \end{cases} \tag{1.4}$$

forall $x \in [\alpha_a, \alpha_b]$.

In [1], (see also [6]) authors proved the double integral Montgomery identity for two independent variables stated as follows:

PROPOSITION 1.3. *If $f : I \times J \rightarrow \mathbb{R}$ is differentiable such that $\frac{\partial^2 f(t,s)}{\partial t \partial s}$ is integrable on interior of $I \times J$, then*

$$\begin{aligned}
 f(x,y) &= \frac{1}{(\alpha_b - \alpha_a)} \int_{\alpha_a}^{\alpha_b} f(t,y)dt + \frac{1}{(\alpha_d - \alpha_c)} \int_{\alpha_c}^{\alpha_d} f(x,s)ds \\
 &\quad - \frac{1}{(\alpha_b - \alpha_a)(\alpha_d - \alpha_c)} \int_{\alpha_a}^{\alpha_b} \int_{\alpha_c}^{\alpha_d} f(t,s)dsdt \\
 &\quad + \int_{\alpha_a}^{\alpha_b} \int_{\alpha_c}^{\alpha_d} P(x,t)Q(y,s) \frac{\partial^2 f(t,s)}{\partial t \partial s} dsdt,
 \end{aligned}
 \tag{1.5}$$

where $P(x,t)$ and $Q(y,s)$ are the Peano kernels defined as

$$P(x,t) = \begin{cases} \frac{t - \alpha_a}{\alpha_b - \alpha_a}, & \text{if } t \in [\alpha_a, x], \\ \frac{t - \alpha_b}{\alpha_b - \alpha_a}, & \text{if } t \in (x, \alpha_b), \end{cases}
 \tag{1.6}$$

and

$$Q(y,s) = \begin{cases} \frac{s - \alpha_c}{\alpha_d - \alpha_c}, & \text{if } s \in [\alpha_c, y], \\ \frac{s - \alpha_d}{\alpha_d - \alpha_c}, & \text{if } s \in (y, \alpha_d). \end{cases}
 \tag{1.7}$$

Now, we are going to establish new Montgomery identity with two parameters and two independent variables, which will provide generalization of existing Montgomery identities. Here we state our first main result.

THEOREM 1.4. *If $f : I \times J \rightarrow \mathbb{R}$ be absolutely continuous such that $\frac{\partial^2 f(t,s)}{\partial t \partial s}$ is integrable on interior of $I \times J$, then*

$$\begin{aligned}
 &(1 - \varepsilon)(1 - \kappa)f(x,y) \\
 &= \frac{(1 - \kappa)}{\alpha_b - \alpha_a} \int_{\alpha_a}^{\alpha_b} f(t,y)dt + \frac{(1 - \varepsilon)}{\alpha_d - \alpha_c} \int_{\alpha_c}^{\alpha_d} f(x,s)ds \\
 &\quad - \frac{1}{(\alpha_b - \alpha_a)(\alpha_d - \alpha_c)} \int_{\alpha_a}^{\alpha_b} \int_{\alpha_c}^{\alpha_d} f(t,s)dsdt + \frac{1}{2} \psi_{\varepsilon, \kappa}(f) \\
 &\quad + \frac{1}{(\alpha_b - \alpha_a)(\alpha_d - \alpha_c)} \int_{\alpha_a}^{\alpha_b} \int_{\alpha_c}^{\alpha_d} P_1(x,t)Q_1(y,s) \frac{\partial^2 f(t,s)}{\partial t \partial s} dsdt,
 \end{aligned}
 \tag{1.8}$$

where

$$\begin{aligned} \Psi_{\varepsilon, \kappa}(f) &= \frac{\kappa}{(\alpha_b - \alpha_a)} \int_{\alpha_a}^{\alpha_b} (f(t, \alpha_c) + f(t, \alpha_d)) dt - \kappa(1 - \varepsilon) (f(x, \alpha_c) + f(x, \alpha_d)) \\ &\quad + \frac{\varepsilon}{(\alpha_d - \alpha_c)} \int_{\alpha_c}^{\alpha_d} (f(\alpha_a, s) + f(\alpha_b, s)) ds - \varepsilon(1 - \kappa) (f(\alpha_a, y) + f(\alpha_b, y)) \\ &\quad - \frac{\varepsilon \kappa}{2} (f(\alpha_a, \alpha_c) + f(\alpha_a, \alpha_d) + f(\alpha_b, \alpha_c) + f(\alpha_b, \alpha_d)), \end{aligned} \quad (1.9)$$

also $P_1(x, t)$ is defined as in (1.4) and $Q_1(y, s)$ defined as

$$Q_1(y, s) = \begin{cases} s - \left(\alpha_c + \kappa \frac{\alpha_d - \alpha_c}{2} \right), & \text{if } s \in [\alpha_c, y], \\ s - \left(\alpha_d - \kappa \frac{\alpha_d - \alpha_c}{2} \right), & \text{if } s \in (y, \alpha_d], \end{cases} \quad (1.10)$$

where $\varepsilon, \kappa \in [0, 1]$.

Proof. By using (1.4) and (1.10), we have

$$\begin{aligned} &\int_{\alpha_a}^{\alpha_b} \int_{\alpha_c}^{\alpha_d} P_1(x, t) Q_1(y, s) \frac{\partial^2 f(t, s)}{\partial t \partial s} ds dt \\ &= \int_{\alpha_a}^x \int_{\alpha_c}^y \left(t - \left(\alpha_a + \varepsilon \frac{\alpha_b - \alpha_a}{2} \right) \right) \left(s - \left(\alpha_c + \kappa \frac{\alpha_d - \alpha_c}{2} \right) \right) \frac{\partial^2 f(t, s)}{\partial t \partial s} ds dt \\ &\quad + \int_{\alpha_a}^x \int_y^{\alpha_d} \left(t - \left(\alpha_a + \varepsilon \frac{\alpha_b - \alpha_a}{2} \right) \right) \left(s - \left(\alpha_d - \kappa \frac{\alpha_d - \alpha_c}{2} \right) \right) \frac{\partial^2 f(t, s)}{\partial t \partial s} ds dt \\ &\quad + \int_x^{\alpha_b} \int_{\alpha_c}^y \left(t - \left(\alpha_b - \varepsilon \frac{\alpha_b - \alpha_a}{2} \right) \right) \left(s - \left(\alpha_c + \kappa \frac{\alpha_d - \alpha_c}{2} \right) \right) \frac{\partial^2 f(t, s)}{\partial t \partial s} ds dt \\ &\quad + \int_x^{\alpha_b} \int_y^{\alpha_d} \left(t - \left(\alpha_b - \varepsilon \frac{\alpha_b - \alpha_a}{2} \right) \right) \left(s - \left(\alpha_d - \kappa \frac{\alpha_d - \alpha_c}{2} \right) \right) \frac{\partial^2 f(t, s)}{\partial t \partial s} ds dt \\ &= I_1 + I_2 + I_3 + I_4. \end{aligned} \quad (1.11)$$

After some calculations and simplifications, we have

$$\begin{aligned} I_1 &= \left(x - \left(\alpha_a + \varepsilon \frac{\alpha_b - \alpha_a}{2} \right) \right) \left(y - \left(\alpha_c + \kappa \frac{\alpha_d - \alpha_c}{2} \right) \right) f(x, y) \\ &\quad - \left(y - \left(\alpha_c + \kappa \frac{\alpha_d - \alpha_c}{2} \right) \right) \int_{\alpha_a}^x f(t, y) dt - \left(x - \left(\alpha_a + \varepsilon \frac{\alpha_b - \alpha_a}{2} \right) \right) \int_{\alpha_c}^y f(x, s) ds \\ &\quad + \varepsilon \frac{\alpha_b - \alpha_a}{2} \left[\left(y - \left(\alpha_c + \kappa \frac{\alpha_d - \alpha_c}{2} \right) \right) f(\alpha_a, y) - \int_{\alpha_c}^y f(\alpha_a, s) ds \right] \\ &\quad + \kappa \frac{\alpha_d - \alpha_c}{2} \left[\left(x - \left(\alpha_a + \varepsilon \frac{\alpha_b - \alpha_a}{2} \right) \right) f(x, \alpha_c) - \int_{\alpha_a}^x f(t, \alpha_c) dt \right] \\ &\quad + \kappa \varepsilon \frac{(\alpha_b - \alpha_a)(\alpha_d - \alpha_c)}{4} f(\alpha_a, \alpha_c) + \int_{\alpha_a}^x \int_{\alpha_c}^y f(t, s) ds dt, \end{aligned}$$

$$\begin{aligned}
 I_2 = & - \left(x - \left(\alpha_a + \varepsilon \frac{\alpha_b - \alpha_a}{2} \right) \right) \left(y - \left(\alpha_d - \kappa \frac{\alpha_d - \alpha_c}{2} \right) \right) f(x, y) \\
 & + \left(y - \left(\alpha_d - \kappa \frac{\alpha_d - \alpha_c}{2} \right) \right) \int_{\alpha_a}^x f(t, y) dt - \left(x - \left(\alpha_a + \varepsilon \frac{\alpha_b - \alpha_a}{2} \right) \right) \int_y^{\alpha_d} f(x, s) ds \\
 & + \varepsilon \frac{\alpha_b - \alpha_a}{2} \left[- \left(y - \left(\alpha_d - \kappa \frac{\alpha_d - \alpha_c}{2} \right) \right) f(\alpha_a, y) - \int_y^{\alpha_d} f(\alpha_a, s) ds \right] \\
 & + \kappa \frac{\alpha_d - \alpha_c}{2} \left[\left(x - \left(\alpha_a + \varepsilon \frac{\alpha_b - \alpha_a}{2} \right) \right) f(x, \alpha_d) - \int_{\alpha_a}^x f(t, \alpha_d) dt \right] \\
 & + \kappa \varepsilon \frac{(\alpha_b - \alpha_a)(\alpha_d - \alpha_c)}{4} f(\alpha_a, \alpha_d) + \int_{\alpha_a}^x \int_y^{\alpha_d} f(t, s) ds dt,
 \end{aligned}$$

$$\begin{aligned}
 I_3 = & - \left(x - \left(\alpha_b - \varepsilon \frac{\alpha_b - \alpha_a}{2} \right) \right) \left(y - \left(\alpha_c + \kappa \frac{\alpha_d - \alpha_c}{2} \right) \right) f(x, y) \\
 & - \left(y - \left(\alpha_c + \kappa \frac{\alpha_d - \alpha_c}{2} \right) \right) \int_x^{\alpha_b} f(t, y) dt + \left(x - \left(\alpha_b - \varepsilon \frac{\alpha_b - \alpha_a}{2} \right) \right) \int_{\alpha_c}^y f(x, s) ds \\
 & + \varepsilon \frac{\alpha_b - \alpha_a}{2} \left[\left(y - \left(\alpha_c + \kappa \frac{\alpha_d - \alpha_c}{2} \right) \right) f(\alpha_b, y) - \int_{\alpha_c}^y f(\alpha_b, s) ds \right] \\
 & + \kappa \frac{\alpha_d - \alpha_c}{2} \left[- \left(x - \left(\alpha_b - \varepsilon \frac{\alpha_b - \alpha_a}{2} \right) \right) f(x, \alpha_c) - \int_x^{\alpha_b} f(t, \alpha_c) dt \right] \\
 & + \kappa \varepsilon \frac{(\alpha_b - \alpha_a)(\alpha_d - \alpha_c)}{4} f(\alpha_b, \alpha_c) + \int_x^{\alpha_b} \int_{\alpha_c}^y f(t, s) ds dt,
 \end{aligned}$$

and

$$\begin{aligned}
 I_4 = & \left(x - \left(\alpha_b - \varepsilon \frac{\alpha_b - \alpha_a}{2} \right) \right) \left(y - \left(\alpha_d - \kappa \frac{\alpha_d - \alpha_c}{2} \right) \right) f(x, y) \\
 & + \left(y - \left(\alpha_d - \kappa \frac{\alpha_d - \alpha_c}{2} \right) \right) \int_x^{\alpha_b} f(t, y) dt + \left(x - \left(\alpha_b - \varepsilon \frac{\alpha_b - \alpha_a}{2} \right) \right) \int_y^{\alpha_d} f(x, s) ds \\
 & + \varepsilon \frac{\alpha_b - \alpha_a}{2} \left[- \left(y - \left(\alpha_d - \kappa \frac{\alpha_d - \alpha_c}{2} \right) \right) f(\alpha_b, y) - \int_y^{\alpha_d} f(\alpha_b, s) ds \right] \\
 & + \kappa \frac{\alpha_d - \alpha_c}{2} \left[- \left(x - \left(\alpha_b - \varepsilon \frac{\alpha_b - \alpha_a}{2} \right) \right) f(x, \alpha_d) - \int_x^{\alpha_b} f(t, \alpha_d) dt \right] \\
 & + \kappa \varepsilon \frac{(\alpha_b - \alpha_a)(\alpha_d - \alpha_c)}{4} f(\alpha_b, \alpha_d) + \int_x^{\alpha_b} \int_y^{\alpha_d} f(t, s) ds dt.
 \end{aligned}$$

By substituting the values of I_1, I_2, I_3 and I_4 in (1.11), after some simplifications we get

$$\begin{aligned}
 & \int_{\alpha_a}^{\alpha_b} \int_{\alpha_c}^{\alpha_d} P_1(x, t) Q_1(y, s) \frac{\partial^2 f(t, s)}{\partial t \partial s} ds dt \\
 = & (1 - \varepsilon)(1 - \kappa)(\alpha_b - \alpha_a)(\alpha_d - \alpha_c) f(x, y) - (1 - \kappa)(\alpha_d - \alpha_c) \int_{\alpha_a}^{\alpha_b} f(t, y) dt \\
 & - (1 - \varepsilon)(\alpha_b - \alpha_a) \int_{\alpha_c}^{\alpha_d} f(x, s) ds + \int_{\alpha_a}^{\alpha_b} \int_{\alpha_c}^{\alpha_d} f(t, s) ds dt
 \end{aligned}$$

$$\begin{aligned}
& -\kappa \frac{\alpha_d - \alpha_c}{2} \int_{\alpha_a}^{\alpha_b} (f(t, \alpha_c) + f(t, \alpha_d)) dt + \frac{\kappa(1 - \varepsilon)(\alpha_b - \alpha_a)(\alpha_d - \alpha_c)}{2} \\
& \times (f(x, \alpha_c) + f(x, \alpha_d)) - \varepsilon \frac{\alpha_b - \alpha_a}{2} \int_{\alpha_c}^{\alpha_d} (f(\alpha_a, s) + f(\alpha_b, s)) ds \\
& + \frac{\varepsilon(1 - \kappa)(\alpha_b - \alpha_a)(\alpha_d - \alpha_c)}{2} (f(\alpha_a, y) + f(\alpha_b, y)) \\
& + \frac{\varepsilon\kappa(\alpha_b - \alpha_a)(\alpha_d - \alpha_c)}{4} (f(\alpha_a, \alpha_c) + f(\alpha_a, \alpha_d) + f(\alpha_b, \alpha_c) + f(\alpha_b, \alpha_d)),
\end{aligned}$$

which is our required identity. \square

REMARK 1.5.

1. If we substitute $\varepsilon, \kappa = 0$ in (1.8), then it gives (1.5) of [1] as stated in Proposition 1.3.
2. If we substitute $f(t, s) = h(t)h(s)$ and $x = y$ in (1.8), then it gives (1.3) of [7] as stated in Proposition 1.2. Hence we easily get (1.1) of [18] as stated in Proposition 1.1 by substituting $\varepsilon = 0$ in Proposition 1.2.

REMARK 1.6. If we substitute $\varepsilon = \kappa$, then we get a special type of Montgomery identity as established in [17].

$$\begin{aligned}
(1 - \varepsilon)^2 f(x, y) &= \frac{(1 - \varepsilon)}{\alpha_b - \alpha_a} \int_{\alpha_a}^{\alpha_b} f(t, y) dt + \frac{(1 - \varepsilon)}{\alpha_d - \alpha_c} \int_{\alpha_c}^{\alpha_d} f(x, s) ds \\
&- \frac{1}{(\alpha_b - \alpha_a)(\alpha_d - \alpha_c)} \int_{\alpha_a}^{\alpha_b} \int_{\alpha_c}^{\alpha_d} f(t, s) ds dt + \frac{1}{2} \psi_\varepsilon(f) \\
&+ \frac{1}{(\alpha_b - \alpha_a)(\alpha_d - \alpha_c)} \int_{\alpha_a}^{\alpha_b} \int_{\alpha_c}^{\alpha_d} P_1(x, t) Q_1'(y, s) \frac{\partial^2 f(t, s)}{\partial t \partial s} ds dt,
\end{aligned}$$

where

$$\begin{aligned}
\psi_\varepsilon(f) &= \frac{\varepsilon}{(\alpha_b - \alpha_a)} \int_{\alpha_a}^{\alpha_b} (f(t, \alpha_c) + f(t, \alpha_d)) dt + \frac{\varepsilon}{(\alpha_d - \alpha_c)} \int_{\alpha_c}^{\alpha_d} (f(\alpha_a, s) \\
&+ f(\alpha_b, s)) ds - \varepsilon(1 - \varepsilon) (f(x, \alpha_c) + f(x, \alpha_d) + f(\alpha_a, y) + f(\alpha_b, y)) \\
&- \frac{\varepsilon^2}{2} (f(\alpha_a, \alpha_c) + f(\alpha_a, \alpha_d) + f(\alpha_b, \alpha_c) + f(\alpha_b, \alpha_d)),
\end{aligned}$$

also Peano kernels $P_1(x, t)$ and $Q_1'(y, s)$ are defined as in (1.4) and (1.10) respectively.

In the next section, we are going to present Ostrowski type inequality for L_p and L_∞ spaces by using the Montgomery identity (1.8) as we obtained in the current section.

2. Generalized Ostrowski type inequalities

The classical inequality introduced by Ostrowski in 1938 named in the literature as Ostrowski inequality [20]. It is basically the absolute deviation of functional value from its integral mean. It also approximates area under the curve of a function by a rectangle. It is given in the following proposition.

PROPOSITION 2.1. *Let $f : I \rightarrow \mathbb{R}$ be a differentiable mappings on I^o such that $f \in L[\alpha_a, \alpha_b]$ where $\alpha_a < \alpha_b$ whose derivative f' is bounded on interior of I , i.e., $\|f'\|_\infty := \sup_{t \in (\alpha_a, \alpha_b)} |f'(t)| < \infty$. Then*

$$\left| f(x) - \frac{1}{\alpha_b - \alpha_a} \int_{\alpha_a}^{\alpha_b} f(t) dt \right| \leq (\alpha_b - \alpha_a) \left[\frac{1}{4} + \frac{(x - \frac{\alpha_a + \alpha_b}{2})^2}{(\alpha_b - \alpha_a)^2} \right] \|f'(x)\|_\infty. \tag{2.1}$$

The constant $\frac{1}{4}$ is the best possible constant that it cannot be replaced by smaller one.

In 1997, Dragomir and Wang established the following inequality [8] of Ostrowski type for differentiable functions where $f' \in L_p$ space.

PROPOSITION 2.2. *Let $f : I \rightarrow \mathbb{R}$ be a differentiable function on I^o where $\alpha_a < \alpha_b$ such that $f' \in L_p[\alpha_a, \alpha_b]$ where $1 \leq p \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\left| f(x) - \frac{1}{\alpha_b - \alpha_a} \int_{\alpha_a}^{\alpha_b} f(t) dt \right| \leq \frac{1}{\alpha_b - \alpha_a} \left[\frac{(x - \alpha_a)^{q+1} + (\alpha_b - x)^{q+1}}{q + 1} \right]^{\frac{1}{q}} \|f'\|_p, \tag{2.2}$$

for all $x \in [\alpha_a, \alpha_b]$.

In 2001, Ostrowski type inequality for double integrals was introduced by Barnett and Dragomir in [1].

PROPOSITION 2.3. *Let $f : I \times J \rightarrow \mathbb{R}$ is differentiable such that $\frac{\partial^2 f(t, s)}{\partial t \partial s}$ is integrable on interior of $I \times J$ and is bounded in L_∞ space. Then*

$$\begin{aligned} & \left| f(x, y) - \frac{1}{(\alpha_b - \alpha_a)} \int_{\alpha_a}^{\alpha_b} f(t, y) dt - \frac{1}{(\alpha_d - \alpha_c)} \int_{\alpha_c}^{\alpha_d} f(x, s) ds \right. \\ & \left. + \frac{1}{(\alpha_b - \alpha_a)(\alpha_d - \alpha_c)} \int_{\alpha_a}^{\alpha_b} \int_{\alpha_c}^{\alpha_d} f(t, s) ds dt \right| \\ & \leq \frac{1}{4(\alpha_b - \alpha_a)(\alpha_d - \alpha_c)} [(x - \alpha_a)^2 + (\alpha_b - x)^2] \\ & \quad \times [(y - \alpha_c)^2 + (\alpha_d - y)^2] \left\| \frac{\partial^2 f(t, s)}{\partial t \partial s} \right\|_{\infty}. \end{aligned} \tag{2.3}$$

Furthermore in 2000, Dragomir et al. in [4] generalized the results of [1] for L_p space.

PROPOSITION 2.4. *Let $f : I \times J \rightarrow \mathbb{R}$ is differentiable such that $\frac{\partial^2 f(t,s)}{\partial t \partial s}$ is integrable on interior of $I \times J$ and is bounded in L_p space where $1 \leq p \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\begin{aligned} & \left| f(x,y) - \frac{1}{(\alpha_b - \alpha_a)} \int_{\alpha_a}^{\alpha_b} f(t,y)dt - \frac{1}{(\alpha_d - \alpha_c)} \int_{\alpha_c}^{\alpha_d} f(x,s)ds \right. \\ & \left. + \frac{1}{(\alpha_b - \alpha_a)(\alpha_d - \alpha_c)} \int_{\alpha_a}^{\alpha_b} \int_{\alpha_c}^{\alpha_d} f(t,s)dsdt \right| \\ & \leq \frac{1}{(\alpha_b - \alpha_a)(\alpha_d - \alpha_c)} \left\| \frac{\partial^2 f(t,s)}{\partial t \partial s} \right\|_p \\ & \times \left[\frac{(x - \alpha_a)^{q+1} + (\alpha_b - x)^{q+1}}{q+1} \right]^{\frac{1}{q}} \left[\frac{(y - \alpha_c)^{q+1} + (\alpha_d - y)^{q+1}}{q+1} \right]^{\frac{1}{q}}. \end{aligned} \tag{2.4}$$

In 2000, Dragomir et al. in [7] generalized the classical Ostrowski inequality [20] as stated in the following proposition.

PROPOSITION 2.5. *Let $f : I \rightarrow \mathbb{R}$ be a differentiable mappings on I^o such that $f \in L[\alpha_a, \alpha_b]$, where $\alpha_a < \alpha_b$ whose derivative f' is bounded on (α_a, α_b) , i.e., $\|f'\|_\infty := \sup_{t \in (\alpha_a, \alpha_b)} |f'(t)| < \infty$. Then*

$$\begin{aligned} & \left| (1 - \varepsilon)f(x) + \varepsilon \frac{f(\alpha_a) + f(\alpha_b)}{2} - \frac{1}{\alpha_b - \alpha_a} \int_{\alpha_a}^{\alpha_b} f(t)dt \right| \\ & \leq (\alpha_b - \alpha_a) \left[\frac{1}{4} \{ \varepsilon^2 + (\varepsilon - 1)^2 \} + \frac{(x - \frac{\alpha_a + \alpha_b}{2})^2}{(\alpha_b - \alpha_a)^2} \right] \|f'(x)\|_\infty, \end{aligned} \tag{2.5}$$

where $\varepsilon \in [0, 1]$.

In 2003, Yang established Ostrowski inequality for L_p spaces in [27] that is infact a generalization of (2.5).

PROPOSITION 2.6. *Let all assumptions of Proposition 2.2 be true. Then*

$$\begin{aligned} & \left| (1 - \varepsilon)f(x) + \varepsilon \frac{f(\alpha_a) + f(\alpha_b)}{2} - \frac{1}{\alpha_b - \alpha_a} \int_{\alpha_a}^{\alpha_b} f(t)dt \right| \\ & \leq \frac{1}{(\alpha_b - \alpha_a)(q+1)^{\frac{1}{q}}} \left[\left(x - \left(\alpha_a + \varepsilon \frac{\alpha_b - \alpha_a}{2} \right) \right)^{q+1} \right. \\ & \left. + \left(\left(\alpha_b - \varepsilon \frac{\alpha_b - \alpha_a}{2} \right) - x \right)^{q+1} + 2 \left(\varepsilon \frac{\alpha_b - \alpha_a}{2} \right)^{q+1} \right]^{\frac{1}{q}} \|f'(x)\|_p, \end{aligned} \tag{2.6}$$

where $\varepsilon \in [0, 1]$.

Now, we are going to present Ostrowski inequality of double integrals for L_p space and L_∞ space with parameters.

THEOREM 2.7. *Let $f : I \times J \rightarrow \mathbb{R}$ is differentiable such that $\frac{\partial^2 f(t,s)}{\partial t \partial s}$ is integrable on interior of $I \times J$ and is bounded in L_p space where $1 \leq p \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\begin{aligned} & \left| (1-\varepsilon)(1-\kappa)f(x,y) - \frac{(1-\kappa)}{(\alpha_b-\alpha_a)} \int_{\alpha_a}^{\alpha_b} f(t,y)dt - \frac{(1-\varepsilon)}{(\alpha_d-\alpha_c)} \int_{\alpha_c}^{\alpha_d} f(x,s)ds \right. \\ & \left. + \frac{1}{(\alpha_b-\alpha_a)(\alpha_d-\alpha_c)} \int_{\alpha_a}^{\alpha_b} \int_{\alpha_c}^{\alpha_d} f(t,s)dsdt - \frac{1}{2} \Psi_{\varepsilon,\kappa}(f) \right| \\ & \leq \frac{1}{(\alpha_b-\alpha_a)(\alpha_d-\alpha_c)(q+1)^{\frac{2}{q}}} \left\| \frac{\partial^2 f(t,s)}{\partial t \partial s} \right\|_p \\ & \times \left[\left(x - \left(\alpha_a + \varepsilon \frac{\alpha_b - \alpha_a}{2} \right) \right)^{q+1} + \left(\left(\alpha_b - \varepsilon \frac{\alpha_b - \alpha_a}{2} \right) - x \right)^{q+1} \right. \\ & \left. + 2 \left(\varepsilon \frac{\alpha_b - \alpha_a}{2} \right)^{q+1} \right]^{\frac{1}{q}} \left[\left(y - \left(\alpha_c + \kappa \frac{\alpha_d - \alpha_c}{2} \right) \right)^{q+1} \right. \\ & \left. + \left(\left(\alpha_d - \kappa \frac{\alpha_d - \alpha_c}{2} \right) - y \right)^{q+1} + 2 \left(\kappa \frac{\alpha_d - \alpha_c}{2} \right)^{q+1} \right]^{\frac{1}{q}}, \end{aligned} \tag{2.7}$$

where $\varepsilon, \kappa \in [0, 1]$.

Proof. From Theorem 1.4, we have

$$\begin{aligned} & (1-\varepsilon)(1-\kappa)f(x,y) - \frac{(1-\kappa)}{(\alpha_b-\alpha_a)} \int_{\alpha_a}^{\alpha_b} f(t,y)dt - \frac{1}{2} \Psi_{\varepsilon,\kappa}(f) \\ & - \frac{(1-\varepsilon)}{(\alpha_d-\alpha_c)} \int_{\alpha_c}^{\alpha_d} f(x,s)ds + \frac{1}{(\alpha_b-\alpha_a)(\alpha_d-\alpha_c)} \int_{\alpha_a}^{\alpha_b} \int_{\alpha_c}^{\alpha_d} f(t,s)dsdt \\ & = \frac{1}{(\alpha_b-\alpha_a)(\alpha_d-\alpha_c)} \int_{\alpha_a}^{\alpha_b} \int_{\alpha_c}^{\alpha_d} P_1(x,t)Q_1(y,s) \frac{\partial^2 f(t,s)}{\partial t \partial s} dsdt. \end{aligned} \tag{2.8}$$

Applying absolute on both sides of (2.8) and using Hölder's inequality, we get

$$\begin{aligned} & \left| (1-\varepsilon)(1-\kappa)f(x,y) - \frac{(1-\kappa)}{(\alpha_b-\alpha_a)} \int_{\alpha_a}^{\alpha_b} f(t,y)dt - \frac{1}{2} \Psi_{\varepsilon,\kappa}(f) \right. \\ & \left. - \frac{(1-\varepsilon)}{(\alpha_d-\alpha_c)} \int_{\alpha_c}^{\alpha_d} f(x,s)ds + \frac{1}{(\alpha_b-\alpha_a)(\alpha_d-\alpha_c)} \int_{\alpha_a}^{\alpha_b} \int_{\alpha_c}^{\alpha_d} f(t,s)dsdt \right| \\ & = \left| \frac{1}{(\alpha_b-\alpha_a)(\alpha_d-\alpha_c)} \int_{\alpha_a}^{\alpha_b} \int_{\alpha_c}^{\alpha_d} P_1(x,t)Q_1(y,s) \frac{\partial^2 f(t,s)}{\partial t \partial s} dsdt \right| \end{aligned}$$

$$\begin{aligned}
 &\leq \frac{1}{(\alpha_b - \alpha_a)(\alpha_d - \alpha_c)} \int_{\alpha_a}^{\alpha_b} \int_{\alpha_c}^{\alpha_d} |P_1(x,t)Q_1(y,s)| \left| \frac{\partial^2 f(t,s)}{\partial t \partial s} \right| ds dt \\
 &\leq \frac{1}{(\alpha_b - \alpha_a)(\alpha_d - \alpha_c)} \left(\int_{\alpha_a}^{\alpha_b} \int_{\alpha_c}^{\alpha_d} |P_1(x,t)Q_1(y,s)|^q ds dt \right)^{\frac{1}{q}} \left(\int_{\alpha_a}^{\alpha_b} \int_{\alpha_c}^{\alpha_d} \left| \frac{\partial^2 f(t,s)}{\partial t \partial s} \right|^p ds dt \right)^{\frac{1}{p}} \\
 &= \frac{1}{(\alpha_b - \alpha_a)(\alpha_d - \alpha_c)(q+1)^{\frac{2}{q}}} \left\| \frac{\partial^2 f(t,s)}{\partial t \partial s} \right\|_p \\
 &\quad \times \left[\left(x - \left(\alpha_a + \varepsilon \frac{\alpha_b - \alpha_a}{2} \right) \right)^{q+1} + \left(\left(\alpha_b - \varepsilon \frac{\alpha_b - \alpha_a}{2} \right) - x \right)^{q+1} + 2 \left(\varepsilon \frac{\alpha_b - \alpha_a}{2} \right)^{q+1} \right]^{\frac{1}{q}} \\
 &\quad \times \left[\left(y - \left(\alpha_c + \kappa \frac{\alpha_d - \alpha_c}{2} \right) \right)^{q+1} + \left(\left(\alpha_d - \kappa \frac{\alpha_d - \alpha_c}{2} \right) - y \right)^{q+1} + 2 \left(\kappa \frac{\alpha_d - \alpha_c}{2} \right)^{q+1} \right]^{\frac{1}{q}}.
 \end{aligned}$$

□

COROLLARY 2.8. *Let all the assumptions of Proposition 2.5 be valid. Also if we select $q = 1$ and $p \rightarrow \infty$ in (2.7), then we get following result*

$$\begin{aligned}
 &\left| (1 - \varepsilon)(1 - \kappa)f(x, y) - \frac{(1 - \kappa)}{(\alpha_b - \alpha_a)} \int_{\alpha_a}^{\alpha_b} f(t, y) dt - \frac{1}{2} \Psi_{\varepsilon, \kappa}(f) \right. \\
 &\quad \left. - \frac{(1 - \varepsilon)}{(\alpha_d - \alpha_c)} \int_{\alpha_c}^{\alpha_d} f(x, s) ds + \frac{1}{(\alpha_b - \alpha_a)(\alpha_d - \alpha_c)} \int_{\alpha_a}^{\alpha_b} \int_{\alpha_c}^{\alpha_d} f(t, s) ds dt \right| \\
 &\leq \frac{1}{4(\alpha_b - \alpha_a)(\alpha_d - \alpha_c)} [(x - \alpha_a)^2 + (\alpha_b - x)^2 - \varepsilon(1 - \varepsilon)(\alpha_b - \alpha_a)^2] \\
 &\quad \times [(y - \alpha_c)^2 + (\alpha_d - y)^2 - \kappa(1 - \kappa)(\alpha_d - \alpha_c)^2] \left\| \frac{\partial^2 f(t, s)}{\partial t \partial s} \right\|_{\infty} \tag{2.9}
 \end{aligned}$$

$$\begin{aligned}
 &= (\alpha_b - \alpha_a)(\alpha_d - \alpha_c) \left[\frac{1}{4} \{ \varepsilon^2 + (\varepsilon - 1)^2 \} + \frac{(x - \frac{\alpha_a + \alpha_b}{2})^2}{(\alpha_b - \alpha_a)^2} \right] \\
 &\quad \times \left[\frac{1}{4} \{ \kappa^2 + (\kappa - 1)^2 \} + \frac{(y - \frac{\alpha_c + \alpha_d}{2})^2}{(\alpha_d - \alpha_c)^2} \right] \left\| \frac{\partial^2 f(t, s)}{\partial t \partial s} \right\|_{\infty}, \tag{2.10}
 \end{aligned}$$

where $\varepsilon, \kappa \in [0, 1]$.

REMARK 2.9. It is to be noted the constant $\frac{1}{4}$ is sharp in (2.10) in the first and second bracket in the sense that it cannot be replaced by any smaller values.

To be more specific, if we suppose the inequality (2.10) be valid for constants $H_1, H_2 > 0$, i.e.,

$$\begin{aligned}
 &\left| (1 - \varepsilon)(1 - \kappa)f(x, y) - \frac{(1 - \kappa)}{(\alpha_b - \alpha_a)} \int_{\alpha_a}^{\alpha_b} f(t, y) dt - \frac{1}{2} \Psi_{\varepsilon, \kappa}(f) \right. \\
 &\quad \left. - \frac{(1 - \varepsilon)}{(\alpha_d - \alpha_c)} \int_{\alpha_c}^{\alpha_d} f(x, s) ds + \frac{1}{(\alpha_b - \alpha_a)(\alpha_d - \alpha_c)} \int_{\alpha_a}^{\alpha_b} \int_{\alpha_c}^{\alpha_d} f(t, s) ds dt \right|
 \end{aligned}$$

$$\leq (\alpha_b - \alpha_a)(\alpha_d - \alpha_c) \left[H_1 \left\{ \varepsilon^2 + (\varepsilon - 1)^2 \right\} + \frac{\left(x - \frac{\alpha_a + \alpha_b}{2}\right)^2}{(\alpha_b - \alpha_a)^2} \right] \\ \times \left[H_2 \left\{ \kappa^2 + (\kappa - 1)^2 \right\} + \frac{\left(y - \frac{\alpha_c + \alpha_d}{2}\right)^2}{(\alpha_d - \alpha_c)^2} \right] \left\| \frac{\partial^2 f(t, s)}{\partial t \partial s} \right\|_\infty.$$

Consider $f(s, t) = st$, $x = \alpha_a$, $y = \alpha_c$, and $\varepsilon, \kappa = 0$ then above inequality reduces to

$$\frac{1}{4} \leq \left(H_1 + \frac{1}{4}\right) \left(H_2 + \frac{1}{4}\right) \\ \frac{1}{2} \times \frac{1}{2} \leq \left(H_1 + \frac{1}{4}\right) \left(H_2 + \frac{1}{4}\right),$$

which gives that $H_1 \geq \frac{1}{4}$ and $H_2 \geq \frac{1}{4}$. Hence we are true in our claim.

In the similar manner one can find out that the improved bounds will be obtained by choosing $\varepsilon, \kappa = \frac{1}{2}$.

From (2.7) and (2.10) we can get many results of Ostrowski type inequality.

REMARK 2.10.

1. If we substitute $\varepsilon = \kappa = 0$ in (2.7), then it gives (2.4) of [4] as stated in Proposition 2.4.
2. If we substitute $\varepsilon = \kappa = 0$ in (2.10), then it gives (2.3) of [1] as stated in Proposition 2.3.
3. If we substitute $f(t, s) = h(t)h(s)$, here h be absolutely continuous function, also let $\|h'\| < \infty$ and $x = y$ in (2.7), then it gives (2.6) of [27] as stated in Proposition 2.6. Further if we choose $\varepsilon = \kappa = 0$, then we get (2.2) of [8] as stated in Proposition 2.2.
4. If we substitute $f(t, s) = h(t)h(s)$, here h be absolutely continuous function, also let $\|h'\| < \infty$ and $x = y$ in (2.10), then it gives (2.5) of [7] as stated in Proposition 2.5. Further if we choose $\varepsilon = \kappa = 0$, then we get (2.1) of [20] as stated in Proposition 2.1.

COROLLARY 2.11. *If we take $\varepsilon = \kappa = 0$, $x = \frac{\alpha_a + \alpha_b}{2}$ and $y = \frac{\alpha_c + \alpha_d}{2}$ in (2.7), then we get*

$$\left| f\left(\frac{\alpha_a + \alpha_b}{2}, \frac{\alpha_c + \alpha_d}{2}\right) - \frac{1}{(\alpha_b - \alpha_a)} \int_{\alpha_a}^{\alpha_b} f\left(t, \frac{\alpha_c + \alpha_d}{2}\right) dt \right. \\ \left. - \frac{1}{(\alpha_d - \alpha_c)} \int_{\alpha_c}^{\alpha_d} f\left(\frac{\alpha_a + \alpha_b}{2}, s\right) ds + \frac{1}{(\alpha_b - \alpha_a)(\alpha_d - \alpha_c)} \int_{\alpha_a}^{\alpha_b} \int_{\alpha_c}^{\alpha_d} f(t, s) ds dt \right| \\ \leq \frac{1}{4} \left[\frac{(\alpha_b - \alpha_a)(\alpha_d - \alpha_c)}{(q + 1)^2} \right]^{\frac{1}{q}} \left\| \frac{\partial^2 f(t, s)}{\partial t \partial s} \right\|_p.$$

The above inequality is Corollary 5 of [4].

COROLLARY 2.12. *If we take $\varepsilon = \kappa = 0$, $x = \frac{\alpha_a + \alpha_b}{2}$ and $y = \frac{\alpha_c + \alpha_d}{2}$ in (2.10), then we get*

$$\begin{aligned} & \left| f\left(\frac{\alpha_a + \alpha_b}{2}, \frac{\alpha_c + \alpha_d}{2}\right) - \frac{1}{(\alpha_b - \alpha_a)} \int_{\alpha_a}^{\alpha_b} f\left(t, \frac{\alpha_c + \alpha_d}{2}\right) dt \right. \\ & - \frac{1}{(\alpha_d - \alpha_c)} \int_{\alpha_c}^{\alpha_d} f\left(\frac{\alpha_a + \alpha_b}{2}, s\right) ds \\ & \left. + \frac{1}{(\alpha_b - \alpha_a)(\alpha_d - \alpha_c)} \int_{\alpha_a}^{\alpha_b} \int_{\alpha_c}^{\alpha_d} f(t, s) ds dt \right| \\ & \leq \frac{(\alpha_b - \alpha_a)(\alpha_d - \alpha_c)}{16} \left\| \frac{\partial^2 f(t, s)}{\partial t \partial s} \right\|_{\infty}. \end{aligned}$$

The above inequality is Corollary 2.2 of [1].

REMARK 2.13. It is easy to see that in all our results, we get better bounds for substituting $x = \frac{\alpha_a + \alpha_b}{2}$, $y = \frac{\alpha_c + \alpha_d}{2}$ and $\varepsilon = \kappa = \frac{1}{2}$.

REMARK 2.14. We can also get many interesting results by varying the values of p and q in our main result (2.7). The case $p = q = 2$ is of special interest.

3. Generalized Grüss type inequalities

Čebyšev introduced the following inequality in his article [3] for two absolutely continuous functions, in the literature this inequality is named as Grüss inequality which is obtained by classical Montgomery identity defined previously in the Proposition 1.1. It gives the estimation of bounded functional for two absolutely continuous functions. Here is the inequality as given in the proposition stated below:

PROPOSITION 3.1. *Let $f, g : I \rightarrow \mathbb{R}$ be two absolutely continuous function such that $f', g' \in L_{\infty}$ spaces. Then we have*

$$|T(f, g)| \leq \frac{1}{12} (\alpha_b - \alpha_a)^2 \|f'\|_{\infty} \|g'\|_{\infty}, \tag{3.1}$$

where $T(f, g)$ is Čebyšev functional defined as

$$T(f, g) = \frac{1}{\alpha_b - \alpha_a} \int_{\alpha_a}^{\alpha_b} f(x)g(x)dx - \left(\int_{\alpha_a}^{\alpha_b} f(x)dx \right) \left(\int_{\alpha_a}^{\alpha_b} g(x)dx \right), \tag{3.2}$$

for all $x \in [\alpha_a, \alpha_b]$.

Pachpatte [23] obtained the another generalized of (3.1), which states that:

PROPOSITION 3.2. *Let $f, g : I \rightarrow \mathbb{R}$ be two absolutely continuous function such that $f', g' \in L_p[\alpha_a, \alpha_b]$ spaces where $1 \leq p \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ and $T(f, g)$ is a Čebyšev functional defined in (3.2). Then*

$$|T(f, g)| \leq \frac{1}{(\alpha_b - \alpha_a)^3} \|f'\|_p \|g'\|_p \int_{\alpha_a}^{\alpha_b} (B(x))^{\frac{2}{q}} dx \tag{3.3}$$

and

$$|T(f, g)| \leq \frac{1}{2(\alpha_b - \alpha_a)^2} \int_{\alpha_a}^{\alpha_b} [|g(x)| \|f'\|_p + |f(x)| \|g'\|_p] (B(x))^{\frac{1}{q}} dx, \tag{3.4}$$

where

$$B(x) = \frac{(x - \alpha_a)^{q+1} + (\alpha_b - x)^{q+1}}{q + 1},$$

for all $x \in [\alpha_a, \alpha_b]$.

In 2011, Gaeuzane-Lakoud and Aissaoui in [16] extended this inequality for two independent variable as can be seen in the following proposition.

PROPOSITION 3.3. *Let $f, g : I \times J \rightarrow \mathbb{R}$ be differentiable functions such that their second order partial derivatives $\frac{\partial^2 f(t, s)}{\partial t \partial s}$ and $\frac{\partial^2 g(t, s)}{\partial t \partial s}$ are integrable on $I \times J$. Then*

$$|T_*(f, g)| \leq \frac{49}{3600} (\alpha_b - \alpha_a)^2 (\alpha_d - \alpha_c)^2 \left\| \frac{\partial^2 f(t, s)}{\partial t \partial s} \right\|_{\infty} \left\| \frac{\partial^2 g(t, s)}{\partial t \partial s} \right\|_{\infty} \tag{3.5}$$

and

$$\begin{aligned} |T_*(f, g)| &\leq \frac{1}{8(\alpha_b - \alpha_a)^2 (\alpha_d - \alpha_c)^2} \\ &\times \int_{\alpha_a}^{\alpha_b} \int_{\alpha_c}^{\alpha_d} \left(|g(x, y)| \left\| \frac{\partial^2 f(t, s)}{\partial t \partial s} \right\|_{\infty} + |f(x, y)| \left\| \frac{\partial^2 g(t, s)}{\partial t \partial s} \right\|_{\infty} \right) \\ &\times \{ (x - \alpha_a)^2 + (\alpha_b - x)^2 \} \{ (y - \alpha_c)^2 + (\alpha_d - y)^2 \} dy dx, \end{aligned} \tag{3.6}$$

where

$$\begin{aligned} T_*(f, g) &= \frac{1}{(\alpha_b - \alpha_a)(\alpha_d - \alpha_c)} \int_{\alpha_a}^{\alpha_b} \int_{\alpha_c}^{\alpha_d} f(x, y) g(x, y) dy dx \\ &- \frac{1}{(\alpha_b - \alpha_a)^2 (\alpha_d - \alpha_c)} \int_{\alpha_a}^{\alpha_b} \int_{\alpha_c}^{\alpha_d} g(x, y) \int_{\alpha_a}^{\alpha_b} f(t, y) dt dy dx \\ &- \frac{1}{(\alpha_b - \alpha_a)(\alpha_d - \alpha_c)^2} \int_{\alpha_a}^{\alpha_b} \int_{\alpha_c}^{\alpha_d} g(x, y) \int_{\alpha_c}^{\alpha_d} f(x, s) ds dy dx \\ &+ \frac{1}{(\alpha_b - \alpha_a)^2 (\alpha_d - \alpha_c)^2} \int_{\alpha_a}^{\alpha_b} \int_{\alpha_c}^{\alpha_d} g(x, y) dy dx \int_{\alpha_a}^{\alpha_b} \int_{\alpha_c}^{\alpha_d} f(t, s) ds dt. \end{aligned} \tag{3.7}$$

In recent years, a number of research papers related to Grüss type inequality and its Čebyšev functional have been published, we may mention the works [2, 16, 21, 22, 24, 25]. Now we would like to generalize Grüss type inequalities of [16] for functions of L_p space and by introducing some parameters.

THEOREM 3.4. *Let $f, g : I \times J \rightarrow \mathbb{R}$ be differentiable functions such that their second order partial derivatives $\frac{\partial^2 f(t, s)}{\partial t \partial s}$ and $\frac{\partial^2 g(t, s)}{\partial t \partial s}$ are integrable on $I^o \times J^o$ and are bounded in L_p spaces where $1 \leq p \leq \infty$ and $\frac{1}{p} + \frac{1}{q}$. Then*

$$\begin{aligned}
 & |T_1(f, g; \varepsilon, \kappa)| \\
 & \leq \frac{1}{(\alpha_b - \alpha_a)^3 (\alpha_d - \alpha_c)^3 (q+1)^{\frac{4}{q}}} \left\| \frac{\partial^2 f(t, s)}{\partial t \partial s} \right\|_p \left\| \frac{\partial^2 g(t, s)}{\partial t \partial s} \right\|_q \\
 & \times \int_{\alpha_a}^{\alpha_b} \int_{\alpha_c}^{\alpha_d} \left[\left(x - \left(\alpha_a + \varepsilon \frac{\alpha_b - \alpha_a}{2} \right) \right)^{q+1} + \left(\left(\alpha_b - \varepsilon \frac{\alpha_b - \alpha_a}{2} \right) - x \right)^{q+1} \right. \\
 & \left. + 2 \left(\varepsilon \frac{\alpha_b - \alpha_a}{2} \right)^{q+1} \right]^{\frac{2}{q}} \left[\left(y - \left(\alpha_c + \kappa \frac{\alpha_d - \alpha_c}{2} \right) \right)^{q+1} \right. \\
 & \left. + \left(\left(\alpha_d - \kappa \frac{\alpha_d - \alpha_c}{2} \right) - y \right)^{q+1} + 2 \left(\kappa \frac{\alpha_d - \alpha_c}{2} \right)^{q+1} \right]^{\frac{2}{q}} dy dx, \tag{3.8}
 \end{aligned}$$

where

$$\begin{aligned}
 T_1(f, g; \varepsilon, \kappa) &= \frac{(1 - \varepsilon)^2 (1 - \kappa)^2}{(\alpha_b - \alpha_a)(\alpha_d - \alpha_c)} \int_{\alpha_a}^{\alpha_b} \int_{\alpha_c}^{\alpha_d} f(x, y) g(x, y) dy dx \\
 & - \frac{(1 - 2\varepsilon)(1 - \kappa)^2}{(\alpha_b - \alpha_a)^2 (\alpha_d - \alpha_c)} \int_{\alpha_a}^{\alpha_b} \int_{\alpha_c}^{\alpha_d} g(x, y) \int_{\alpha_a}^{\alpha_b} f(t, y) dt dy dx \\
 & - \frac{(1 - \varepsilon)^2 (1 - 2\kappa)}{(\alpha_b - \alpha_a)(\alpha_d - \alpha_c)^2} \int_{\alpha_a}^{\alpha_b} \int_{\alpha_c}^{\alpha_d} g(x, y) \int_{\alpha_c}^{\alpha_d} f(x, s) ds dy dx \\
 & + \frac{2(2\varepsilon\kappa - \varepsilon - \kappa) + 1}{(\alpha_b - \alpha_a)^2 (\alpha_d - \alpha_c)^2} \int_{\alpha_a}^{\alpha_b} \int_{\alpha_c}^{\alpha_d} g(x, y) dy dx \int_{\alpha_a}^{\alpha_b} \int_{\alpha_c}^{\alpha_d} f(t, s) ds dt \\
 & - \frac{1}{2} \int_{\alpha_a}^{\alpha_b} \int_{\alpha_c}^{\alpha_d} \left(F \psi_{\varepsilon, \kappa}(g) + G \psi_{\varepsilon, \kappa}(f) + \frac{1}{2} \psi_{\varepsilon, \kappa}(f) \psi_{\varepsilon, \kappa}(g) \right) dy dx, \tag{3.9}
 \end{aligned}$$

where $\varepsilon, \kappa \in [0, 1]$.

Proof. Let F, G, \tilde{F} and \tilde{G} be defined as follows

$$\begin{aligned}
 F &= (1 - \varepsilon)(1 - \kappa)f(x, y) - \frac{(1 - \kappa)}{(\alpha_b - \alpha_a)} \int_{\alpha_a}^{\alpha_b} f(s, y) ds - \frac{(1 - \varepsilon)}{(\alpha_d - \alpha_c)} \int_{\alpha_c}^{\alpha_d} f(x, t) dt \\
 & + \frac{1}{(\alpha_b - \alpha_a)(\alpha_d - \alpha_c)} \int_{\alpha_a}^{\alpha_b} \int_{\alpha_c}^{\alpha_d} f(t, s) ds dt - \frac{1}{2} \psi_{\varepsilon, \kappa}(f),
 \end{aligned}$$

$$\begin{aligned} \tilde{F} &= \frac{1}{(\alpha_b - \alpha_a)(\alpha_d - \alpha_c)} \int_{\alpha_a}^{\alpha_b} \int_{\alpha_c}^{\alpha_d} P_1(x,t) Q_1(y,s) \frac{\partial^2 f(t,s)}{\partial t \partial s} ds dt, \\ G &= (1 - \varepsilon)(1 - \kappa)g(x,y) - \frac{(1 - \kappa)}{(\alpha_b - \alpha_a)} \int_{\alpha_a}^{\alpha_b} g(s,y) ds - \frac{(1 - \varepsilon)}{(\alpha_d - \alpha_c)} \int_{\alpha_c}^{\alpha_d} g(x,t) dt \\ &\quad + \frac{1}{(\alpha_b - \alpha_a)(\alpha_d - \alpha_c)} \int_{\alpha_a}^{\alpha_b} \int_{\alpha_c}^{\alpha_d} g(t,s) ds dt - \frac{1}{2} \Psi_{\varepsilon, \kappa}(g), \end{aligned}$$

and

$$\tilde{G} = \frac{1}{(\alpha_b - \alpha_a)(\alpha_d - \alpha_c)} \int_{\alpha_a}^{\alpha_b} \int_{\alpha_c}^{\alpha_d} P_1(x,t) Q_1(y,s) \frac{\partial^2 g(t,s)}{\partial t \partial s} ds dt.$$

Then using the condition,

$$FG = \tilde{F}\tilde{G},$$

multiplying the resultant by $\frac{1}{(\alpha_b - \alpha_a)(\alpha_d - \alpha_c)}$ and integrate from α_a to α_b over x and integrate α_c to α_d over y , we get

$$\begin{aligned} T_1(f, g; \varepsilon, \kappa) &= \frac{(1 - \varepsilon)^2(1 - \kappa)^2}{(\alpha_b - \alpha_a)(\alpha_d - \alpha_c)} \int_{\alpha_a}^{\alpha_b} \int_{\alpha_c}^{\alpha_d} f(x,y)g(x,y)dydx \\ &\quad - \frac{(1 - 2\varepsilon)(1 - \kappa)^2}{(\alpha_b - \alpha_a)^2(\alpha_d - \alpha_c)} \int_{\alpha_a}^{\alpha_b} \int_{\alpha_c}^{\alpha_d} g(x,y) \int_{\alpha_a}^{\alpha_b} f(t,y)dt dy dx \\ &\quad - \frac{(1 - \varepsilon)^2(1 - 2\kappa)}{(\alpha_b - \alpha_a)(\alpha_d - \alpha_c)^2} \int_{\alpha_a}^{\alpha_b} \int_{\alpha_c}^{\alpha_d} g(x,y) \int_{\alpha_c}^{\alpha_d} f(x,s)ds dy dx \\ &\quad + \frac{2(2\varepsilon\kappa - \varepsilon - \kappa) + 1}{(\alpha_b - \alpha_a)^2(\alpha_d - \alpha_c)^2} \int_{\alpha_a}^{\alpha_b} \int_{\alpha_c}^{\alpha_d} g(x,y)dy dx \int_{\alpha_a}^{\alpha_b} \int_{\alpha_c}^{\alpha_d} f(t,s)ds dt \\ &\quad - \frac{1}{2} \int_{\alpha_a}^{\alpha_b} \int_{\alpha_c}^{\alpha_d} \left(F \Psi_{\varepsilon, \kappa}(g) + G \Psi_{\varepsilon, \kappa}(f) + \frac{1}{2} \Psi_{\varepsilon, \kappa}(f) \Psi_{\varepsilon, \kappa}(g) \right) dy dx \\ &= \frac{1}{(\alpha_b - \alpha_a)^3(\alpha_d - \alpha_c)^3} \int_{\alpha_a}^{\alpha_b} \int_{\alpha_c}^{\alpha_d} \left(\int_{\alpha_a}^{\alpha_b} \int_{\alpha_c}^{\alpha_d} P(x,t,y,s) \frac{\partial^2 f(t,s)}{\partial t \partial s} ds dt \right) \\ &\quad \times \left(\int_{\alpha_a}^{\alpha_b} \int_{\alpha_c}^{\alpha_d} P(x,t,y,s) \frac{\partial^2 g(t,s)}{\partial t \partial s} ds dt \right) dy dx. \end{aligned}$$

Applying absolute, we get

$$\begin{aligned} &|T_1(f, g; \varepsilon, \kappa)| \\ &\leq \frac{1}{(\alpha_b - \alpha_a)^3(\alpha_d - \alpha_c)^3} \int_{\alpha_a}^{\alpha_b} \int_{\alpha_c}^{\alpha_d} \left(\int_{\alpha_a}^{\alpha_b} \int_{\alpha_c}^{\alpha_d} \left| P_1(x,t) Q_1(y,s) \frac{\partial^2 f(t,s)}{\partial t \partial s} \right| ds dt \right) \\ &\quad \times \left(\int_{\alpha_a}^{\alpha_b} \int_{\alpha_c}^{\alpha_d} \left| P_1(x,t) Q_1(y,s) \frac{\partial^2 g(t,s)}{\partial t \partial s} \right| ds dt \right) dy dx. \end{aligned}$$

Using Hölder’s inequality, we get

$$\begin{aligned}
 & |T_1(f, g; \varepsilon, \kappa)| \\
 & \leq \frac{1}{(\alpha_b - \alpha_a)^3(\alpha_d - \alpha_c)^3} \\
 & \quad \times \int_{\alpha_a}^{\alpha_b} \int_{\alpha_c}^{\alpha_d} \left(\int_{\alpha_a}^{\alpha_b} \int_{\alpha_c}^{\alpha_d} |P_1(x, t)Q_1(y, s)|^q ds dt \right)^{\frac{1}{q}} \left\| \frac{\partial^2 f(t, s)}{\partial t \partial s} \right\|_p \\
 & \quad \times \left(\int_{\alpha_a}^{\alpha_b} \int_{\alpha_c}^{\alpha_d} |P_1(x, t)Q_1(y, s)|^q ds dt \right)^{\frac{1}{q}} \left\| \frac{\partial^2 g(t, s)}{\partial t \partial s} \right\|_p dy dx \\
 & = \frac{1}{(\alpha_b - \alpha_a)^3(\alpha_d - \alpha_c)^3} \left\| \frac{\partial^2 f(t, s)}{\partial t \partial s} \right\|_p \left\| \frac{\partial^2 g(t, s)}{\partial t \partial s} \right\|_p \\
 & \quad \times \int_{\alpha_a}^{\alpha_b} \int_{\alpha_c}^{\alpha_d} \left(\int_{\alpha_a}^{\alpha_b} \int_{\alpha_c}^{\alpha_d} |P_1(x, t)Q_1(y, s)|^q ds dt \right)^{\frac{2}{q}} dy dx \\
 & = \frac{1}{(\alpha_b - \alpha_a)^3(\alpha_d - \alpha_c)^3} \left\| \frac{\partial^2 f(t, s)}{\partial t \partial s} \right\|_p \left\| \frac{\partial^2 g(t, s)}{\partial t \partial s} \right\|_p \int_{\alpha_a}^{\alpha_b} \int_{\alpha_c}^{\alpha_d} H(x, y)^2 dy dx \\
 & \leq \frac{1}{(\alpha_b - \alpha_a)^3(\alpha_d - \alpha_c)^3(q+1)^{\frac{4}{q}}} \left\| \frac{\partial^2 f(t, s)}{\partial t \partial s} \right\|_p \left\| \frac{\partial^2 g(t, s)}{\partial t \partial s} \right\|_p \\
 & \quad \times \int_{\alpha_a}^{\alpha_b} \int_{\alpha_c}^{\alpha_d} \left[\left(x - \left(\alpha_a + \varepsilon \frac{\alpha_b - \alpha_a}{2} \right) \right)^{q+1} + \left(\left(\alpha_b - \varepsilon \frac{\alpha_b - \alpha_a}{2} \right) - x \right)^{q+1} \right. \\
 & \quad \left. + 2 \left(\varepsilon \frac{\alpha_b - \alpha_a}{2} \right)^{q+1} \right]^{\frac{2}{q}} \left[\left(y - \left(\alpha_c + \kappa \frac{\alpha_d - \alpha_c}{2} \right) \right)^{q+1} \right. \\
 & \quad \left. + \left(\left(\alpha_d - \kappa \frac{\alpha_d - \alpha_c}{2} \right) - y \right)^{q+1} + 2 \left(\kappa \frac{\alpha_d - \alpha_c}{2} \right)^{q+1} \right]^{\frac{2}{q}} dy dx. \quad \square
 \end{aligned}$$

REMARK 3.5. If we substitute $\varepsilon = \kappa = 0$ in (3.8), we get

$$\begin{aligned}
 |T_*(f, g)| & \leq \frac{1}{(\alpha_b - \alpha_a)^3(\alpha_d - \alpha_c)^3} \left\| \frac{\partial^2 f(t, s)}{\partial t \partial s} \right\|_p \left\| \frac{\partial^2 g(t, s)}{\partial t \partial s} \right\|_p \\
 & \quad \times \int_{\alpha_a}^{\alpha_b} \int_{\alpha_c}^{\alpha_d} \left[\frac{(x - \alpha_a)^{q+1} + (\alpha_b - x)^{q+1}}{q+1} \right]^{\frac{2}{q}} \\
 & \quad \times \left[\frac{(y - \alpha_c)^{q+1} + (\alpha_d - y)^{q+1}}{q+1} \right]^{\frac{2}{q}} dy dx, \tag{3.10}
 \end{aligned}$$

where $T_*(f, g)$ is defined in (3.7). The above result is generalized case for L_p spaces of (3.5) of [16].

COROLLARY 3.6. *If we substitute $q = 1$ and $p \rightarrow \infty$ in (3.8), then we get*

$$|T_1(f, g; \varepsilon, \kappa)| \leq \frac{(\alpha_b - \alpha_a)^2(\alpha_d - \alpha_c)^2}{144} \left\| \frac{\partial^2 f(t, s)}{\partial t \partial s} \right\|_\infty \left\| \frac{\partial^2 g(t, s)}{\partial t \partial s} \right\|_\infty \times \left[\frac{7}{5} + \varepsilon(1 - \varepsilon)\{3(1 - \varepsilon) - 4\} \right] \left[\frac{7}{5} + \kappa(1 - \kappa)\{3(1 - \kappa) - 4\} \right] \quad (3.11)$$

where $T_1(f, g; \varepsilon, \kappa)$ is defined as in (3.9).

REMARK 3.7. If we substitute $\varepsilon = \kappa = 0$ in (3.11), or $q = 1$ then $p \rightarrow \infty$ in (3.10), then we get (3.5) of [16] as stated in the Proposition 3.3

Now we are going to present the second main result of Čebyšev inequality.

THEOREM 3.8. *Let all assumptions of Theorems 3.4 be valid. Then*

$$|T_2(f, g, \varepsilon, \kappa)| \leq \frac{1}{2(\alpha_b - \alpha_a)^2(\alpha_d - \alpha_c)^2(q + 1)^2} \times \int_{\alpha_a}^{\alpha_b} \int_{\alpha_c}^{\alpha_d} \left(|g(x, y)| \left\| \frac{\partial^2 f(t, s)}{\partial t \partial s} \right\|_p + |f(x, y)| \left\| \frac{\partial^2 g(t, s)}{\partial t \partial s} \right\|_p \right) \times \left[\left(x - \left(\alpha_a + \varepsilon \frac{\alpha_b - \alpha_a}{2} \right) \right)^{q+1} + \left(\left(\alpha_b - \varepsilon \frac{\alpha_b - \alpha_a}{2} \right) - x \right)^{q+1} + 2 \left(\varepsilon \frac{\alpha_b - \alpha_a}{2} \right)^{q+1} \right]^{\frac{1}{q}} \left[\left(y - \left(\alpha_c + \kappa \frac{\alpha_d - \alpha_c}{2} \right) \right)^{q+1} + \left(\left(\alpha_d - \kappa \frac{\alpha_d - \alpha_c}{2} \right) - y \right)^{q+1} + 2 \left(\kappa \frac{\alpha_d - \alpha_c}{2} \right)^{q+1} \right]^{\frac{1}{q}} dy dx, \quad (3.12)$$

where

$$T_2(f, g; \varepsilon, \kappa) = \frac{(1 - \varepsilon)(1 - \kappa)}{(\alpha_b - \alpha_a)(\alpha_d - \alpha_c)} \int_{\alpha_a}^{\alpha_b} \int_{\alpha_c}^{\alpha_d} f(x, y)g(x, y)dydx - \frac{(1 - \kappa)}{(\alpha_b - \alpha_a)^2(\alpha_d - \alpha_c)} \int_{\alpha_a}^{\alpha_b} \int_{\alpha_c}^{\alpha_d} g(x, y) \int_{\alpha_a}^{\alpha_b} f(t, y)dt dy dx - \frac{(1 - \varepsilon)}{(\alpha_b - \alpha_a)(\alpha_d - \alpha_c)^2} \int_{\alpha_a}^{\alpha_b} \int_{\alpha_c}^{\alpha_d} g(x, y) \int_{\alpha_c}^{\alpha_d} f(x, s)ds dy dx + \frac{1}{(\alpha_b - \alpha_a)^2(\alpha_d - \alpha_c)^2} \int_{\alpha_a}^{\alpha_b} \int_{\alpha_c}^{\alpha_d} g(x, y)dy dx \int_{\alpha_a}^{\alpha_b} \int_{\alpha_c}^{\alpha_d} f(t, s)ds dt - \frac{1}{2(\alpha_b - \alpha_a)(\alpha_d - \alpha_c)} \int_{\alpha_a}^{\alpha_b} \int_{\alpha_c}^{\alpha_d} \left(f(x, y)\psi_{\varepsilon, \kappa}(g) + g(x, y)\psi_{\varepsilon, \kappa}(f) \right) dy dx, \quad (3.13)$$

where $\varepsilon, \kappa \in [0, 1]$.

Proof. Applying identity (2.3) to the function g , we get

$$\begin{aligned}
 & (1 - \varepsilon)(1 - \kappa)g(x, y) \\
 = & \frac{(1 - \kappa)}{\alpha_b - \alpha_a} \int_{\alpha_a}^{\alpha_b} g(t, y) dt + \frac{(1 - \varepsilon)}{\alpha_d - \alpha_c} \int_{\alpha_c}^{\alpha_d} g(x, s) ds \\
 & - \frac{1}{(\alpha_b - \alpha_a)(\alpha_d - \alpha_c)} \int_{\alpha_a}^{\alpha_b} \int_{\alpha_c}^{\alpha_d} g(t, s) ds dt + \frac{1}{2} \Psi_{\varepsilon, \kappa}(g) \\
 & + \frac{1}{(\alpha_b - \alpha_a)(\alpha_d - \alpha_c)} \int_{\alpha_a}^{\alpha_b} \int_{\alpha_c}^{\alpha_d} P_1(x, t) Q_1(y, s) \frac{\partial^2 g(t, s)}{\partial t \partial s} ds dt. \tag{3.14}
 \end{aligned}$$

Multiplying (1.8) by $\frac{1}{(\alpha_b - \alpha_a)(\alpha_d - \alpha_c)} g(x, y)$, (3.14) by $\frac{1}{(\alpha_b - \alpha_a)(\alpha_d - \alpha_c)} f(x, y)$, summing the resultant identities, then integrate from α_a to α_b over x and integrate α_c to α_d over y , we obtain

$$\begin{aligned}
 & \frac{(1 - \varepsilon)(1 - \kappa)}{(\alpha_b - \alpha_a)(\alpha_d - \alpha_c)} \int_{\alpha_a}^{\alpha_b} \int_{\alpha_c}^{\alpha_d} f(x, y) g(x, y) dy dx \\
 = & \frac{(1 - \kappa)}{(\alpha_b - \alpha_a)^2(\alpha_d - \alpha_c)} \int_{\alpha_a}^{\alpha_b} \int_{\alpha_c}^{\alpha_d} g(x, y) \int_{\alpha_a}^{\alpha_b} f(t, y) dt dy dx \\
 & + \frac{(1 - \varepsilon)}{(\alpha_b - \alpha_a)(\alpha_d - \alpha_c)^2} \int_{\alpha_a}^{\alpha_b} \int_{\alpha_c}^{\alpha_d} g(x, y) \int_{\alpha_c}^{\alpha_d} f(x, s) ds dy dx \\
 & - \frac{1}{(\alpha_b - \alpha_a)^2(\alpha_d - \alpha_c)^2} \int_{\alpha_a}^{\alpha_b} \int_{\alpha_c}^{\alpha_d} g(x, y) dy dx \int_{\alpha_a}^{\alpha_b} \int_{\alpha_c}^{\alpha_d} f(t, s) ds dt \\
 & + \frac{1}{2(\alpha_b - \alpha_a)(\alpha_d - \alpha_c)} \int_{\alpha_a}^{\alpha_b} \int_{\alpha_c}^{\alpha_d} \left(f(x, y) \Psi_{\varepsilon, \kappa}(g) + g(x, y) \Psi_{\varepsilon, \kappa}(f) \right) dy dx, \\
 & + \frac{1}{2(\alpha_b - \alpha_a)^2(\alpha_d - \alpha_c)^2} \\
 & \times \int_{\alpha_a}^{\alpha_b} \int_{\alpha_c}^{\alpha_d} \left(g(x, y) \int_{\alpha_a}^{\alpha_b} \int_{\alpha_c}^{\alpha_d} P_1(x, t) Q_1(y, s) \frac{\partial^2 f(t, s)}{\partial t \partial s} dt ds \right. \\
 & \left. + f(x, y) \int_{\alpha_a}^{\alpha_b} \int_{\alpha_c}^{\alpha_d} P_1(x, t) Q_1(y, s) \frac{\partial^2 g(t, s)}{\partial t \partial s} dt ds \right) dy dx, \tag{3.15}
 \end{aligned}$$

from that we deduce,

$$\begin{aligned}
 & T_2(f, g; \varepsilon, \kappa) \\
 = & \frac{1}{2(\alpha_b - \alpha_a)^2(\alpha_d - \alpha_c)^2} \\
 & \times \int_{\alpha_a}^{\alpha_b} \int_{\alpha_c}^{\alpha_d} \left(g(x, y) \int_{\alpha_a}^{\alpha_b} \int_{\alpha_c}^{\alpha_d} P_1(x, t) Q_1(y, s) \frac{\partial^2 f(t, s)}{\partial t \partial s} dt ds \right. \\
 & \left. + f(x, y) \int_{\alpha_a}^{\alpha_b} \int_{\alpha_c}^{\alpha_d} P_1(x, t) Q_1(y, s) \frac{\partial^2 g(t, s)}{\partial t \partial s} dt ds \right) dy dx. \tag{3.16}
 \end{aligned}$$

Consequently taking absolute value on it and then applying Hölder’s Inequality, we have

$$\begin{aligned}
 & |T_2(f, g; \varepsilon, \kappa)| \\
 & \leq \frac{1}{2(\alpha_b - \alpha_a)^2(\alpha_d - \alpha_c)^2(q + 1)^2} \\
 & \quad \times \int_{\alpha_a}^{\alpha_b} \int_{\alpha_c}^{\alpha_d} \left(|g(x, y)| \left\| \frac{\partial^2 f(t, s)}{\partial t \partial s} \right\|_p + |f(x, y)| \left\| \frac{\partial^2 g(t, s)}{\partial t \partial s} \right\|_p \right) \\
 & \quad \times \left[\left(x - \left(\alpha_a + \varepsilon \frac{\alpha_b - \alpha_a}{2} \right) \right)^{q+1} + \left(\left(\alpha_b - \varepsilon \frac{\alpha_b - \alpha_a}{2} \right) - x \right)^{q+1} \right. \\
 & \quad \left. + 2 \left(\varepsilon \frac{\alpha_b - \alpha_a}{2} \right)^{q+1} \right]^{\frac{1}{q}} \left[\left(y - \left(\alpha_c + \kappa \frac{\alpha_d - \alpha_c}{2} \right) \right)^{q+1} \right. \\
 & \quad \left. + \left(\left(\alpha_d - \kappa \frac{\alpha_d - \alpha_c}{2} \right) - y \right)^{q+1} + 2 \left(\kappa \frac{\alpha_d - \alpha_c}{2} \right)^{q+1} \right]^{\frac{1}{q}} dy dx. \quad \square \quad (3.17)
 \end{aligned}$$

REMARK 3.9. If we substitute $\varepsilon = \kappa = 0$ in (3.12), then we get

$$\begin{aligned}
 & |T_*(f, g)| \\
 & \leq \frac{1}{2(\alpha_b - \alpha_a)^2(\alpha_d - \alpha_c)^2(q + 1)^2} \\
 & \quad \times \int_{\alpha_a}^{\alpha_b} \int_{\alpha_c}^{\alpha_d} \left(|g(x, y)| \left\| \frac{\partial^2 f(t, s)}{\partial t \partial s} \right\|_p + |f(x, y)| \left\| \frac{\partial^2 g(t, s)}{\partial t \partial s} \right\|_p \right) \\
 & \quad \times \left[(x - \alpha_a)^{q+1} + (\alpha_b - x)^{q+1} \right]^{\frac{1}{q}} \left[(y - \alpha_c)^{q+1} + (\alpha_d - y)^{q+1} \right]^{\frac{1}{q}} dy dx, \quad (3.18)
 \end{aligned}$$

where $T_*(f, g)$ is defined as in (3.7).

REMARK 3.10. If we substitute $q = 1$ and $p \rightarrow \infty$ in (3.12), then we get

$$\begin{aligned}
 & |T_2(f, g, \varepsilon, \kappa)| \\
 & \leq \frac{1}{8(\alpha_b - \alpha_a)^2(\alpha_d - \alpha_c)^2} \\
 & \quad \times \int_{\alpha_a}^{\alpha_b} \int_{\alpha_c}^{\alpha_d} \left(|g(x, y)| \left\| \frac{\partial^2 f(t, s)}{\partial t \partial s} \right\|_\infty + |f(x, y)| \left\| \frac{\partial^2 g(t, s)}{\partial t \partial s} \right\|_\infty \right) \\
 & \quad \times \left[(x - \alpha_a)^2 + (\alpha_b - x)^2 - \varepsilon(1 - \varepsilon)(\alpha_b - \alpha_a)^2 \right] \\
 & \quad \times \left[(y - \alpha_c)^2 + (\alpha_d - y)^2 - \kappa(1 - \kappa)(\alpha_d - \alpha_c)^2 \right] dy dx, \quad (3.19)
 \end{aligned}$$

where $T_2(f, g; \varepsilon, \kappa)$ is defined as in (3.13).

REMARK 3.11. If we substitute $\varepsilon = \kappa = 0$ in (3.19), then we get inequality (3.6) of [16] as stated in Proposition 3.3.

REMARK 3.12. We can get many interesting inequalities by varying the values of ε and κ . It is to be noted that the better bound for (3.8) and (3.12) is derived from $\varepsilon, \kappa = \frac{1}{2}$.

4. Conclusion

In this article, we acquired some new results of Montgomery identity, Ostrowski inequality and Grüss inequality with parameters for L_p spaces. We have obtained various inequalities with better bounds. Our proposed results capture number of results stated in [1, 7, 8, 16, 17, 18, 20, 27] as special cases.

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