

ON BOUNDEDNESS AND COMPACTNESS OF DISCRETE HARDY OPERATOR IN DISCRETE WEIGHTED VARIABLE LEBESGUE SPACES

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Abstract. In this paper, the authors establish a two-weight boundedness criterion of discrete Hardy operator and its dual operator in the scale of discrete weighted variable Lebesgue spaces. Moreover, we study the problem of compactness of the discrete Hardy operator in discrete weighted variable Lebesgue spaces. We also study a similar problem for the dual operator of discrete Hardy operator.

1. Introduction

In the literature many authors including G. H. Hardy, J. E. Littlewood and G. Polya [17] consider the following standard form of discrete Hardy's inequality in discrete Lebesgue space with constant exponent. Let $p > 1$, $p' = \frac{p}{p-1}$ and let $\{x_k\}_{k=1}^{\infty}$ be an arbitrary sequence of non-negative real numbers. Then

$$\left(\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^n x_k \right)^p \right)^{\frac{1}{p}} \leq p' \left(\sum_{n=1}^{\infty} x_n^p \right)^{\frac{1}{p}}. \quad (1)$$

The constant p' in (1) is sharp. The first result towards a weight characterization of (1) was proved by K. F. Andersen and H. P. Heinig in [1]. Moreover, a sufficient condition for the weight estimate to hold was proved by H. P. Heinig in [18]. A full weight characterization of discrete Hardy inequality was proved by G. Bennett in [6]–[8] and M.S. Braverman and V. D. Stepanov in [9]. It is well known that an essential development for Hardy-type inequalities in the discrete case is given by C. A. Okpoti, L.-E. Persson, and A. Wedestig in [30] and [31]. There has been a similar development for Hardy-type inequalities in the discrete case given by A. A. Kalybay, R. Oinarov and A. M. Temirkhanova [19] and A. A. Kalybay, L.-E. Persson and A. M. Temirkhanova [20]. For a history of Hardy type inequalities on the cones of monotone functions and sequences and for references to related results we refer to the monograph of A. Kufner,

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L. Maligranda and L.-E. Persson [22], and to the papers of M. L. Gol’dman [15] and A. Gogatishvili and V. D. Stepanov [14].

It is well known that the variable discrete Lebesgue space was first studied by W. Orlicz [32] in 1931. In [32], Hölder’s inequality for variable discrete Lebesgue space was proved. W. Orlicz also considered the variable Lebesgue space on the real line, and proved the Hölder inequality in this setting. However, this paper is essentially the only contribution of Orlicz to the study of the variable Lebesgue spaces (see also [25] and [26]). The next step in the development of the variable Lebesgue spaces came two decades later in the work of Nakano [27] and [28]. In particular, the variable Lebesgue spaces were objects of interest during the last two decades (see, [10] and [12]). The study of these spaces has been stimulated by problems of elasticity, fluid dynamics, calculus of variations and differential equations with non-standard growth conditions (see, [10] and [12]). Different characterization of the mapping properties such as boundedness and compactness of Hardy operator in the variable Lebesgue spaces were studied in [2]–[5], [11], [13], [21], [23], [24], etc.

In this paper a criterion for boundedness and compactness of discrete Hardy operator and its dual operator defined on discrete weighted variable Lebesgue spaces are established.

The remainder of the paper is structured as follows. Section 2 contains some preliminaries along with the standard ingredients used in the proofs. We also recall standard results from the theory of discrete variable Lebesgue spaces. Our principal assertions are formulated and proven in Section 3. We establish necessary and sufficient conditions on weight functions for the boundedness and compactness of discrete Hardy operator in discrete weighted variable Lebesgue spaces in Section 3.

2. Preliminaries

Let \mathbb{N} be the set of natural number and let $p = \{p_n\}_{n=1}^\infty$ be a sequence of real numbers such that $1 \leq \underline{p} \leq p_n \leq \bar{p} < \infty$, where $\underline{p} = \inf_{n \geq 1} p_n$ and $\bar{p} = \sup_{n \geq 1} p_n$. The conjugate exponent function of p_n is defined as $\frac{1}{p_n} + \frac{1}{p'_n} = 1$ for all $n \in \mathbb{N}$. Denote by χ_A the characteristic function of $A \subset \mathbb{N}$. Throughout this paper $\underline{p}' = \frac{\underline{p}}{\underline{p}-1}$. Let $w = \{w_n\}_{n=1}^\infty$ be a sequence of positive numbers, i.e., w is a weight function defined on \mathbb{N} .

DEFINITION 1. The discrete weighted Lebesgue space with variable exponents $\ell_{w_n}^{p_n}(\mathbb{N})$ is the set of sequences $x = \{x_n\}_{n=1}^\infty$ such that for some $\lambda_0 > 0$

$$\sum_{k=1}^\infty \left(\frac{|x_n|}{\lambda_0} w_k \right)^{p_k} < \infty.$$

We observe that the expression

$$\|x\|_{\ell_{w_n}^{p_n}(\mathbb{N})} = \|xw\|_{\ell^{p_n}(\mathbb{N})} = \inf \left\{ \lambda > 0 : \sum_{k=1}^\infty \left(\frac{|x_n|}{\lambda} w_k \right)^{p_k} \leq 1 \right\}$$

defines the Luxemburg norm in $\ell_{w_n}^{p_n}(\mathbb{N})$.

The space $\ell_{w_n}^{p_n}(\mathbb{N})$ is a Banach space (see, [10], [12] and [29]).

For constant exponent sequence the space $\ell_{w_n}^{p_n}(\mathbb{N})$ coincides with classical discrete weighted Lebesgue space.

LEMMA 1. [30] *Let $A_k = \sum_{n=1}^k v_n^{-p'}$, $A_0 = 0$ and let $v_n > 0$ for all $n \in \mathbb{N}$. Then the following statements holds:*

(a) *If $0 < b < 1$, then*

$$bA_k^{b-1} v_k^{-p'} \leq A_k^b - A_{k-1}^b \leq bA_{k-1}^{b-1} v_k^{-p'} \quad \text{for all } k \in \mathbb{N}.$$

(b) *If $b < 0$ or $b \geq 1$, then*

$$bA_{k-1}^{b-1} v_k^{-p'} \leq A_k^b - A_{k-1}^b \leq bA_k^{b-1} v_k^{-p'} \quad \text{for all } k \in \mathbb{N}.$$

For further convenience, we give proofs of the Lemma.

Proof. Let $f(x) = x^b$ and let $0 < b < 1$. Then by mean value theorem, we have

$$bx^{b-1}(x-y) \leq x^b - y^b \leq by^{b-1}(x-y), \quad 0 < y < x. \tag{2}$$

Applying inequality (2), one has

$$A_k^{b-1} (A_k - A_{k-1}) \leq \frac{1}{b} (A_k^b - A_{k-1}^b) \leq A_{k-1}^{b-1} (A_k - A_{k-1}).$$

So, one has

$$A_k^{b-1} v_k^{-p'} \leq \frac{1}{b} (A_k^b - A_{k-1}^b) \leq A_{k-1}^{b-1} v_k^{-p'}.$$

It is obvious that

$$\sum_{k=1}^n A_k^{b-1} v_k^{-p'} \leq \frac{1}{b} A_n^b. \tag{3}$$

In the similar way we can prove statements (b). Therefore we omit the proof. \square

We need the following Lemma.

LEMMA 2. *Let $1 \leq p_n \leq q_n \leq \bar{q} < \infty$. Suppose $v = \{v_n\}_{n=1}^\infty$ is a sequence of positive numbers.*

Then $\ell_{v_n}^{p_n}(\mathbb{N}) \hookrightarrow \ell_{v_n}^{q_n}(\mathbb{N})$ and

$$\|x\|_{\ell_{v_n}^{q_n}(\mathbb{N})} \leq \|x\|_{\ell_{v_n}^{p_n}(\mathbb{N})}.$$

Proof. Let $x = \{x_n\}_{n=1}^\infty \in \ell_{v_n}^{p_n}(\mathbb{N})$. It is obvious that

$$\left(\frac{|x_n|}{\lambda} v_n\right)^{p_n} \leq \sum_{n=1}^\infty \left(\frac{|x_n|}{\lambda} v_n\right)^{p_n} \leq 1 \text{ for all } n \in \mathbb{N}.$$

It is well known that an exponential function $f(t) = a^t$ is decreasing function for $0 < a < 1$. Then $\left(\frac{|x_n|}{\lambda} v_n\right)^{q_n} \leq \left(\frac{|x_n|}{\lambda} v_n\right)^{p_n}$. Therefore, we have

$$\sum_{n=1}^\infty \left(\frac{|x_n|}{\lambda} v_n\right)^{q_n} \leq \sum_{n=1}^\infty \left(\frac{|x_n|}{\lambda} v_n\right)^{p_n} \leq 1.$$

So, $\|x\|_{\ell_{v_n}^{q_n}(\mathbb{N})} \leq \lambda$. Choosing $\lambda = \|x\|_{\ell_{v_n}^{p_n}(\mathbb{N})}$, one has

$$\|x\|_{\ell_{v_n}^{q_n}(\mathbb{N})} \leq \|x\|_{\ell_{v_n}^{p_n}(\mathbb{N})}.$$

This completes the proof of lemma. \square

We need the following Theorem.

THEOREM 1. [29] *Let $1 \leq p_n \leq q_n \leq \bar{q} < \infty$, $\frac{1}{r_n} = \frac{1}{p_n} - \frac{1}{q_n}$, $\Omega_1 = \{n \in \mathbb{N} : p_n < q_n\}$ and $\Omega_2 = \{n \in \mathbb{N} : p_n = q_n\}$. Suppose $v = \{v_n\}_{n=1}^\infty$ and $w = \{w_n\}_{n=1}^\infty$ are the weight sequences satisfying condition*

$$\left\| \frac{v}{w} \right\|_{\ell^{r_n}(\mathbb{N})} < \infty.$$

Then $\ell_{w_n}^{q_n}(\mathbb{N}) \hookrightarrow \ell_{v_n}^{p_n}(\mathbb{N})$ and

$$\|x\|_{\ell_{v_n}^{p_n}(\mathbb{N})} \leq \left(A + B + \|\chi_{\Omega_2}\|_{\ell^\infty(\mathbb{N})}\right)^{\frac{1}{2}} \left\| \frac{v}{w} \right\|_{\ell^{r_n}(\mathbb{N})} \|x\|_{\ell_{w_n}^{q_n}(\mathbb{N})}.$$

Here $A = \sup_{n \in \Omega_1} \frac{p_n}{q_n}$, $B = \sup_{n \in \Omega_1} \frac{q_n - p_n}{q_n}$ and

$$\left\| \frac{v}{w} \right\|_{\ell^{r_n}(\mathbb{N})} = \left\| \frac{v}{w} \right\|_{\ell^{r_n}(\Omega_1)} + \left\| \frac{v}{w} \right\|_{\ell^\infty(\Omega_2)}.$$

We give the characterization of relatively compact sets in $\ell_{p_n}(\mathbb{N})$.

THEOREM 2. [16] *Let $1 \leq p_n \leq \bar{p} < \infty$ and let $\mathcal{A} \subset \ell^{p_n}(\mathbb{N})$. Then, the set $\mathcal{A} = \{a^i\}_{i \in I}$ is precompact in $\ell^{p_n}(\mathbb{N})$ if and only if the following conditions are satisfied*

- (i) \mathcal{A} is bounded, i.e. $\exists M > 0 \forall a^i \in \mathcal{A} \quad \|a^i\|_{\ell^{p_n}(\mathbb{N})} \leq M$;
- (ii) $\forall \varepsilon > 0 \exists K \in \mathbb{N} \forall a^i \in \mathcal{A} \quad \|a^i\|_{\ell^{p_n}(n > K+1)} < \varepsilon$.

3. Main results

Now we give the main results of this paper.

Let $\{x_n\}_{n=1}^\infty \in \ell_{v_n}^{p_n}(\mathbb{N})$ be an arbitrary sequence of real numbers. Suppose that $H_n = \frac{1}{n} \sum_{k=1}^n x_k$ and $H_n^* = \sum_{k=n}^\infty \frac{x_k}{k}$.

THEOREM 3. *Let $\{p_n\}_{n=1}^\infty$ and $\{q_n\}_{n=1}^\infty$ be sequences of real numbers such that $1 < \underline{p} \leq q_n \leq \bar{q} < \infty$ and let $\frac{1}{r_n} = \frac{1}{\underline{p}} - \frac{1}{p_n}$ for all $n \in \mathbb{N}$ and $\|1\|_{\ell^{r_n}(\mathbb{N})} < \infty$. Suppose $\{\omega_n\}_{n=1}^\infty$ and $\{v_n\}_{n=1}^\infty$ are sequences of positive numbers. Then the inequality*

$$\|H_n\|_{\ell_{\omega_n}^{q_n}(\mathbb{N})} \leq C \|x\|_{\ell_{v_n}^{p_n}(\mathbb{N})} \tag{4}$$

holds if and only if

$$D = \sup_{k \geq 1} \left(\sum_{n=1}^k v_n^{-p'} \right)^{\frac{1}{q p'}} \left\| \frac{\left(\sum_{m=1}^n v_m^{-p'} \right)^{\frac{1}{p' q'}}}{n} \right\|_{\ell_{\omega_n}^{q_n}(n \geq k)} < \infty. \tag{5}$$

Moreover, if $C > 0$ is the best possible constant in (4), then

$$\left(\frac{\underline{p}-1}{\underline{p}+\underline{q}-1} \right)^{\frac{1}{\underline{p}}} D \leq C \leq (\bar{q})^{\frac{1}{\underline{p}'}} \left(1 + \frac{\bar{p}-\underline{p}}{\bar{p}} + \|\chi_{\Omega_2}\|_{\ell^\infty(\mathbb{N})} \right)^{\frac{1}{\underline{p}}} D.$$

Proof. Sufficiency. Let $h_n = \left(\sum_{k=1}^n v_k^{-p'} \right)^{\frac{1}{q p'}}$ and let $h_0 = 0$. Applying Hölder inequality and Minkowski inequality, we have

$$\begin{aligned} \|H_n\|_{\ell_{\omega_n}^{q_n}(\mathbb{N})} &= \left\| \frac{1}{n} \sum_{k=1}^n x_k \right\|_{\ell_{\omega_n}^{q_n}(\mathbb{N})} = \left\| \frac{1}{n} \sum_{k=1}^n x_k v_k h_k (v_k h_k)^{-1} \right\|_{\ell_{\omega_n}^{q_n}(\mathbb{N})} \\ &\leq \left\| \frac{1}{n} \left(\sum_{k=1}^n (|x_k| v_k)^p h_k^p \right)^{\frac{1}{p}} \left(\sum_{k=1}^n (v_k h_k)^{-p'} \right)^{\frac{1}{p'}} \right\|_{\ell_{\omega_n}^{q_n}(\mathbb{N})} \\ &= \left\| \frac{1}{n} \left(\sum_{k=1}^n (|x_k| v_k)^p h_k^p \right)^{\frac{1}{p}} \left(\sum_{k=1}^n h_k^{-p'} v_k^{-p'} \right)^{\frac{1}{p'}} \right\|_{\ell_{\omega_n}^{q_n}(\mathbb{N})} \end{aligned}$$

$$\begin{aligned}
&= \left\| \left(\sum_{k=1}^n \frac{1}{n^{\underline{p}}} (|x_k| v_k)^{\underline{p}} h_k^{\underline{p}} \left(\sum_{k=1}^n h_k^{-\underline{p}'} v_k^{-\underline{p}'} \right)^{\frac{\underline{p}}{\underline{p}'}} \right)^{\frac{1}{\underline{p}}} \right\|_{\ell_{\omega_n}^{q_n(\mathbb{N})}} \\
&= \left\| \sum_{k=1}^{\infty} \frac{1}{n^{\underline{p}}} (|x_k| v_k)^{\underline{p}} h_k^{\underline{p}} \chi_{E_n}(k) \left(\sum_{k=1}^n h_k^{-\underline{p}'} v_k^{-\underline{p}'} \right)^{\frac{\underline{p}}{\underline{p}'}} \right\|_{\ell_{\omega_n}^{\frac{q_n}{\underline{p}}(\mathbb{N})}}^{\frac{1}{\underline{p}}} \\
&\leq \left(\sum_{k=1}^{\infty} \left\| \frac{1}{n^{\underline{p}}} (|x_k| v_k)^{\underline{p}} h_k^{\underline{p}} \chi_{E_n}(k) \left(\sum_{k=1}^n h_k^{-\underline{p}'} v_k^{-\underline{p}'} \right)^{\frac{\underline{p}}{\underline{p}'}} \right\|_{\ell_{\omega_n}^{\frac{q_n}{\underline{p}}(\mathbb{N})}} \right)^{\frac{1}{\underline{p}}} \\
&= \left(\sum_{k=1}^{\infty} (|x_k| v_k)^{\underline{p}} h_k^{\underline{p}} \left\| \frac{1}{n^{\underline{p}}} \left(\sum_{k=1}^n h_k^{-\underline{p}'} v_k^{-\underline{p}'} \right)^{\frac{\underline{p}}{\underline{p}'}} \right\|_{\ell_{\omega_n}^{\frac{q_n}{\underline{p}}(n \geq k)}} \right)^{\frac{1}{\underline{p}}} \\
&= \left(\sum_{k=1}^{\infty} (|x_k| v_k)^{\underline{p}} h_k^{\underline{p}} \left\| \frac{1}{n} \left(\sum_{k=1}^n h_k^{-\underline{p}'} v_k^{-\underline{p}'} \right)^{\frac{1}{\underline{p}'}} \right\|_{\ell_{\omega_n}^{q_n(n \geq k)}} \right)^{\frac{1}{\underline{p}}}.
\end{aligned}$$

Let $b = 1 - \frac{1}{\underline{q}}$. It follows from (3) that

$$\begin{aligned}
&\left(\sum_{k=1}^{\infty} (|x_k| v_k)^{\underline{p}} h_k^{\underline{p}} \left\| \frac{1}{n} \left(\sum_{k=1}^n h_k^{-\underline{p}'} v_k^{-\underline{p}'} \right)^{\frac{1}{\underline{p}'}} \right\|_{\ell_{\omega_n}^{q_n(n \geq k)}} \right)^{\frac{1}{\underline{p}}} \\
&= \left(\sum_{k=1}^{\infty} (|x_k| v_k)^{\underline{p}} h_k^{\underline{p}} \left\| \frac{1}{n} \left(\sum_{k=1}^n \left(\sum_{l=1}^k v_l^{-\underline{p}'} \right)^{-\frac{1}{\underline{q}}} v_k^{-\underline{p}'} \right)^{\frac{1}{\underline{p}'}} \right\|_{\ell_{\omega_n}^{q_n(n \geq k)}} \right)^{\frac{1}{\underline{p}}} \\
&= \left(\sum_{k=1}^{\infty} (|x_k| v_k)^{\underline{p}} h_k^{\underline{p}} \left\| \frac{1}{n} \left(\sum_{k=1}^n \left(\sum_{l=1}^k v_l^{-\underline{p}'} \right)^{(1-\frac{1}{\underline{q}})-1} v_k^{-\underline{p}'} \right)^{\frac{1}{\underline{p}'}} \right\|_{\ell_{\omega_n}^{q_n(n \geq k)}} \right)^{\frac{1}{\underline{p}}}
\end{aligned}$$

$$\begin{aligned} &\leq \frac{1}{\left(1 - \frac{1}{q}\right)^{\frac{1}{p'}}} \left(\sum_{k=1}^{\infty} (|x_k| v_k)^p h_k^p \left\| \frac{1}{n} \left(\sum_{k=1}^n v_k^{-p'} \right)^{\frac{1-\frac{1}{q}}{p'}} \right\|_{\ell_{\omega_n}^{qn}(n \geq k)}^p \right)^{\frac{1}{p}} \\ &= (\bar{q}')^{\frac{1}{p'}} \left(\sum_{k=1}^{\infty} (|x_k| v_k)^p h_k^p \left\| \frac{1}{n} (h_n)^{q-1} \right\|_{\ell_{\omega_n}^{qn}(n \geq k)}^p \right)^{\frac{1}{p}}. \end{aligned} \tag{6}$$

By condition (5) and Theorem 1, we have

$$\begin{aligned} &\left(\sum_{k=1}^{\infty} (|x_k| v_k)^p h_k^p \left\| \frac{1}{n} (h_n)^{q-1} \right\|_{\ell_{\omega_n}^{qn}(n \geq k)}^p \right)^{\frac{1}{p}} \\ &\leq D \|x\|_{\ell_{v_n}^p(\mathbb{N})} \leq D \left(1 + \frac{\bar{p}-p}{\bar{p}} + \|\chi_{\Omega_2}\|_{\ell_{\infty}(\mathbb{N})} \right)^{\frac{1}{p}} \|x\|_{\ell_{v_n}^{pn}(\mathbb{N})}. \end{aligned} \tag{7}$$

So, combining inequalities (6) and (7), we have

$$\left\| \frac{1}{n} \sum_{k=1}^n x_k \right\|_{\ell_{\omega_n}^{qn}(\mathbb{N})} \leq (\bar{q}')^{\frac{1}{p'}} D \left(1 + \frac{\bar{p}-p}{\bar{p}} + \|\chi_{\Omega_2}\|_{\ell_{\infty}(\mathbb{N})} \right)^{\frac{1}{p}} \|x\|_{\ell_{v_n}^{pn}(\mathbb{N})}.$$

So, (4) holds and $C \leq (\bar{q}')^{\frac{1}{p'}} D \left(1 + \frac{\bar{p}-p}{\bar{p}} + \|\chi_{\Omega_2}\|_{\ell_{\infty}(\mathbb{N})} \right)^{\frac{1}{p}}$.

Necessity. Let (4) be provided and for fixed natural number N we choose the following test sequence as

$$x_k = \begin{cases} h_N^{-1-\frac{q}{p-1}} v_k^{-p'} & k = 1, \dots, N \\ h_k^{-1-\frac{q}{p-1}} v_k^{-p'} & k = N + 1, \dots \end{cases}$$

For the left hand side of (4) we have that

$$\begin{aligned} \|H_n\|_{\ell_{\omega_n}^{qn}(\mathbb{N})} &= \left\| \frac{1}{n} \sum_{k=1}^n x_k \right\|_{\ell_{\omega_n}^{qn}(\mathbb{N})} \geq \left\| \frac{1}{n} \sum_{k=1}^n x_k \right\|_{\ell_{\omega_n}^{qn}(n \geq N)} \\ &= \left\| \frac{1}{n} \left(\sum_{k=1}^N h_N^{-1-\frac{q}{p-1}} v_k^{-p'} + \sum_{k=N+1}^n h_k^{-1-\frac{q}{p-1}} v_k^{-p'} \right) \right\|_{\ell_{\omega_n}^{qn}(n \geq N)} \\ &\geq \left\| \frac{1}{n} \left(h_N^{q-1} + h_n^{-1-\frac{q}{p-1}} \sum_{k=N+1}^n v_k^{-p'} \right) \right\|_{\ell_{\omega_n}^{qn}(n \geq N)} \end{aligned}$$

$$\begin{aligned}
 &= \left\| \frac{1}{n} \left(h_N^{q-1} + h_n^{-1-\frac{q}{p-1}} \left(h_n^{\frac{pq}{p-1}} - h_N^{\frac{pq}{p-1}} \right) \right) \right\|_{\ell_{\omega_n}^{qn} (n \geq N)} \\
 &= \left\| \frac{1}{n} \left(h_N^{q-1} + h_n^{q-1} - h_n^{-1-\frac{pq}{p-1}} h_N^{\frac{pq}{p-1}} \right) \right\|_{\ell_{\omega_n}^{qn} (n \geq N)} \\
 &\geq \left\| \frac{1}{n} \left(h_N^{q-1} + h_n^{q-1} - h_N^{-1-\frac{pq}{p-1}} h_N^{\frac{pq}{p-1}} \right) \right\|_{\ell_{\omega_n}^{qn} (n \geq N)} \\
 &= \left\| \frac{1}{n} \left(h_N^{q-1} + h_n^{q-1} - h_N^{q-1} \right) \right\|_{\ell_{\omega_n}^{qn} (n \geq N)} = \left\| \frac{h_n^{q-1}}{n} \right\|_{\ell_{\omega_n}^{qn} (n \geq N)}.
 \end{aligned}$$

For the right-hand side of (4), by applying Lemma 1 we have

$$\begin{aligned}
 \|x\|_{\ell_{v_n}^{pn}(\mathbb{N})} &\leq \|x\|_{\ell_{v_n}^p(\mathbb{N})} = \left(\sum_{k=1}^{\infty} (|x_k| v_k)^p \right)^{\frac{1}{p}} = \left(\sum_{k=1}^N (|x_k| v_k)^p + \sum_{k=N+1}^{\infty} (|x_k| v_k)^p \right)^{\frac{1}{p}} \\
 &= \left(\sum_{k=1}^N h_N^{-p-qp'} v_k^{-p'} + \sum_{k=N+1}^{\infty} h_k^{-p-qp'} v_k^{-p'} \right)^{\frac{1}{p}} = \left(h_N^{-p} + \sum_{k=N+1}^{\infty} h_k^{-p-qp'} v_k^{-p'} \right)^{\frac{1}{p}} \\
 &= \left(h_N^{-p} + \sum_{k=N+1}^{\infty} \left(\sum_{m=1}^k v_m^{-p'} \right)^{-\frac{p-1}{q}-1} v_k^{-p'} \right)^{\frac{1}{p}} \\
 &\leq \left(h_N^{-p} + \frac{q}{p-1} \left(\sum_{k=N+1}^{\infty} [h_{k-1}^{-p} - h_k^{-p}] \right) \right)^{\frac{1}{p}} \\
 &\leq \left(h_N^{-p} + \frac{q}{p-1} h_N^{-p} \right)^{\frac{1}{p}} = \left(\frac{p+q-1}{p-1} \right)^{\frac{1}{p}} h_N^{-1}.
 \end{aligned}$$

So (4) implies that

$$\left\| \frac{h_n^{q-1}}{n} \right\|_{\ell_{\omega_n}^{qn} (n \geq N)} \leq C \left(\frac{p+q-1}{p-1} \right)^{\frac{1}{p}} h_N^{-1}.$$

Thus, we have

$$\left(\frac{p-1}{p+q-1} \right)^{\frac{1}{p}} h_N \left\| \frac{h_n^{q-1}}{n} \right\|_{\ell_{\omega_n}^{qn} (n \geq N)} \leq C.$$

By (4) we find that (1) holds and

$$\left(\frac{\underline{p} - 1}{\underline{p} + \underline{q} - 1} \right)^{\frac{1}{\underline{p}}} D \leq C.$$

This completes the proof of theorem. \square

For the adjoint discrete Hardy operator the following theorem holds.

THEOREM 4. *Let $\{p_n\}_{n=1}^\infty$ and $\{q_n\}_{n=1}^\infty$ be sequences of real numbers such that $1 < \underline{p} \leq q_n \leq \bar{q} < \infty$ and let $\frac{1}{r_n} = \frac{1}{\underline{p}} - \frac{1}{p_n}$ for all $n \in \mathbb{N}$ and $\|1\|_{\ell^{r_n}(\mathbb{N})} < \infty$. Suppose $\{\omega_n\}_{n=1}^\infty$ and $\{v_n\}_{n=1}^\infty$ are sequences of positive real numbers. Let $\{x_n\}_{n=1}^\infty \in \ell_{v_n}^{p_n}(\mathbb{N})$ be an arbitrary sequence of real numbers. Then the inequality*

$$\|H_n^*\|_{\ell_{\omega_n}^{q_n}(\mathbb{N})} \leq C \|x\|_{\ell_{v_n}^{p_n}(\mathbb{N})} \tag{8}$$

holds if and only if

$$D^* = \sup_{k \geq 1} \left(\sum_{n=k}^\infty \frac{v_n^{-p'}}{n^{p'}} \right)^{\frac{1}{q'}} \left\| \left(\sum_{m=n}^\infty \frac{v_m^{-p'}}{m^{p'}} \right)^{\frac{1}{p'}} \right\|_{\ell_{\omega_n}^{q_n}(n \leq k)} < \infty.$$

Moreover, if $C > 0$ is the best possible constant in (8), then

$$\left(\frac{\underline{p} - 1}{\underline{p} + \underline{q} - 1} \right)^{\frac{1}{\underline{p}}} D^* \leq C \leq (\bar{q}')^{\frac{1}{p'}} \left(1 + \frac{\bar{p} - \underline{p}}{\bar{p}} + \|\chi_{\Omega_2}\|_{\ell^\infty(\mathbb{N})} \right)^{\frac{1}{\underline{p}}} D^*.$$

COROLLARY 1. *Let $p_n = p = \text{const}$, $q_n = q = \text{const}$ for all $n \in \mathbb{N}$ and $1 < p \leq q < \infty$. Suppose $\{\omega_n\}_{n=1}^\infty$ and $\{v_n\}_{n=1}^\infty$ are sequences of nonnegative numbers. Let $\{x_n\}_{n=1}^\infty \in \ell_{v_n}^p(\mathbb{N})$ be an arbitrary sequence of real numbers. Then condition (5) is equivalent to condition*

$$M = \sup_{k \geq 1} \left(\sum_{n=1}^k v_n^{-p'} \right)^{\frac{1}{p'}} \left(\sum_{n=k}^\infty \left(\frac{\omega_n}{n} \right)^q \right)^{\frac{1}{q}} < \infty.$$

Moreover,

$$M \leq D \leq q^{\frac{1}{q}} M.$$

COROLLARY 2. *Let $p_n = p = \text{const}$, $q_n = q = \text{const}$ for all $n \in \mathbb{N}$ and $1 < p \leq q < \infty$. Suppose $\{\omega_n\}_{n=1}^\infty$ and $\{v_n\}_{n=1}^\infty$ are sequences of nonnegative numbers. Let*

$\{x_n\}_{n=1}^\infty \in \ell_{v_n}^p(\mathbb{N})$ be an arbitrary sequence of real numbers. Then condition (8) is equivalent to condition

$$M^* = \sup_{k \geq 1} \left(\sum_{n=k}^\infty \frac{v_n^{-p'}}{n^{p'}} \right)^{\frac{1}{p'}} \left(\sum_{n=1}^k \omega_n \right)^{\frac{1}{q}} < \infty.$$

Moreover,

$$M^* \leq D^* \leq q^{\frac{1}{q}} M^*.$$

COROLLARY 3. Let $\{p_n\}_{n=1}^\infty$ and $\{q_n\}_{n=1}^\infty$ be sequences of real numbers such that $1 < \underline{p} \leq q_n \leq \bar{q} < \infty$ and let $\frac{1}{r_n} = \frac{1}{\underline{p}} - \frac{1}{p_n}$ for all $n \in \mathbb{N}$ and $\|1\|_{\ell^{r_n}(\mathbb{N})} < \infty$. Suppose $\omega_n = n^\alpha$ and $v_n = 1$ for all $n \in \mathbb{N}$ and $\alpha < \frac{1}{\underline{p}} - \frac{1}{\underline{q}}$. Let $\{x_n\}_{n=1}^\infty \in \ell^p(\mathbb{N})$ be an arbitrary sequence of real numbers.

Then the inequality (4) holds.

Moreover, if $C > 0$ is the best possible constant in (4), then

$$\begin{aligned} & \left(\frac{\underline{p}-1}{\underline{p}+\underline{q}-1} \right)^{\frac{1}{\underline{p}}} \sup_{k \geq 1} k^{\frac{1}{\underline{p} \underline{q}}} \left\| n^{\frac{q-1}{q\underline{p}}+\alpha-1} \right\|_{\ell^{q_n}(n \geq k)} \leq C \\ & \leq (\bar{q}')^{\frac{1}{\underline{p}'}} \left(1 + \frac{\bar{p}-\underline{p}}{\bar{p}} + \|\chi_{\Omega_2}\|_{\ell^\infty(\mathbb{N})} \right)^{\frac{1}{\underline{p}'}} \left(1 + \frac{\underline{p}}{\underline{q}-\alpha \underline{p} \underline{q}-1} \right)^{\frac{1}{\underline{q}}}. \end{aligned}$$

Now we give a compactness result of discrete Hardy operator from $\ell_{v_n}^{p_n}(\mathbb{N})$ into $\ell_{w_n}^{q_n}(\mathbb{N})$.

THEOREM 5. Let $\{p_n\}_{n=1}^\infty$ and $\{q_n\}_{n=1}^\infty$ be sequences of real numbers such that $1 < \underline{p} \leq q_n \leq \bar{q} < \infty$ and let $\frac{1}{r_n} = \frac{1}{\underline{p}} - \frac{1}{p_n}$ for all $n \in \mathbb{N}$ and $\|1\|_{\ell^{r_n}(\mathbb{N})} < \infty$. Suppose $\{\omega_n\}_{n=1}^\infty$ and $\{v_n\}_{n=1}^\infty$ are sequences of positive real numbers. Then H_n is compact from $\ell_{v_n}^{p_n}(\mathbb{N})$ into $\ell_{w_n}^{q_n}(\mathbb{N})$ if and only if

$$\lim_{k \rightarrow \infty} \left(\sum_{n=1}^k v_n^{-p'} \right)^{\frac{1}{q p'}} \left\| \frac{\left(\sum_{m=1}^n v_m^{-p'} \right)^{\frac{q-1}{q p'}}}{n} \right\|_{\ell_{\omega_n}^{q_n}(n \geq k)} = 0. \tag{9}$$

Proof. Sufficiency. Let the condition (9) be provided. Then the condition (5) of Theorem 1 is valid. Therefore, by Theorem 1, the operator H_n is bounded from $\ell_{v_n}^{p_n}(\mathbb{N})$

into $\ell_{w_n}^{q_n}(\mathbb{N})$. Assume that $\mathcal{A} = \{f_m\}_{m \in I} \subset \ell_{v_n}^{p_n}(\mathbb{N})$ and $M = \sup_{m \geq 1} \|f_m\|_{\ell_{v_n}^{p_n}(\mathbb{N})}$. Let us show that the set $\left\{w_n^{\frac{1}{q_n}} H_n f_m\right\}$ is precompact in $\ell^{q_n}(\mathbb{N})$. By Theorem 1,

$$\left\|w_n^{\frac{1}{q_n}} H_n f_m\right\|_{\ell^{q_n}(\mathbb{N})} = \|H_n f_m\|_{\ell_{\omega_n}^{q_n}(\mathbb{N})} \leq C \|f_m\|_{\ell_{v_n}^{p_n}(\mathbb{N})} \leq CM.$$

So the set $\left\{w_n^{\frac{1}{q_n}} H_n f_m\right\}$ is uniformly bounded in $\ell^{q_n}(\mathbb{N})$.

For $r > 1$ we set $\bar{w}_r = \{\bar{w}_{r,i}\}_{i=1}^\infty$, $\bar{w}_{r,i} = \begin{cases} 0, & \text{if } 1 \leq i \leq r-1, \\ w_i, & \text{if } i \geq r. \end{cases}$

Let

$$A_r = \sup_{k \geq r} \left(\sum_{n=1}^k v_n^{-p'} \right)^{\frac{1}{q p'}} \left\| \frac{1}{\bar{w}_n^{q_n}} \frac{\left(\sum_{m=1}^n v_m^{-p'} \right)^{\frac{1}{q' p'}}}{n} \right\|_{\ell^{q_n}(n \geq k)} = \sup_{k \geq r} F_k.$$

Then, by Theorem 1, we have

$$\begin{aligned} \sup_{\|f_m\|_{\ell_{v_n}^{p_n}(\mathbb{N})} \leq M} \left\|w_n^{\frac{1}{q_n}} H_n f_m\right\|_{\ell^{q_n}(n \geq r)} &= \sup_{\|f_m\|_{\ell_{v_n}^{p_n}(\mathbb{N})} \leq M} \left\|\bar{w}_{r,n}^{\frac{1}{q_n}} H_n f_m\right\|_{\ell^{q_n}(n \geq 1)} \\ &\leq M (\bar{q}')^{\frac{1}{p'}} \left(1 + \frac{\bar{p} - p}{\bar{p}} + \|\chi_{\Omega_2}\|_{\ell^\infty(\mathbb{N})} \right)^{\frac{1}{2}} A_r. \end{aligned} \tag{10}$$

So, by (10), we get

$$\begin{aligned} &\lim_{r \rightarrow \infty} \left(\sup_{\|f_m\|_{\ell_{v_n}^{p_n}(\mathbb{N})} \leq M} \left\|w_n^{\frac{1}{q_n}} H_n f_m\right\|_{\ell^{q_n}(n \geq r)} \right) \\ &\leq M (\bar{q}')^{\frac{1}{p'}} \left(1 + \frac{\bar{p} - p}{\bar{p}} + \|\chi_{\Omega_2}\|_{\ell^\infty(\mathbb{N})} \right)^{\frac{1}{2}} \lim_{r \rightarrow \infty} A_r \\ &= M (\bar{q}')^{\frac{1}{p'}} \left(1 + \frac{\bar{p} - p}{\bar{p}} + \|\chi_{\Omega_2}\|_{\ell^\infty(\mathbb{N})} \right)^{\frac{1}{2}} \lim_{r \rightarrow \infty} \sup_{k \geq r} F_k \\ &= M (\bar{q}')^{\frac{1}{p'}} \left(1 + \frac{\bar{p} - p}{\bar{p}} + \|\chi_{\Omega_2}\|_{\ell^\infty(\mathbb{N})} \right)^{\frac{1}{2}} \overline{\lim}_{r \rightarrow \infty} F_r \\ &= M (\bar{q}')^{\frac{1}{p'}} \left(1 + \frac{\bar{p} - p}{\bar{p}} + \|\chi_{\Omega_2}\|_{\ell^\infty(\mathbb{N})} \right)^{\frac{1}{2}} \lim_{r \rightarrow \infty} F_r = 0. \end{aligned}$$

Necessity. Let the operator H_n be compact from $\ell_{v_n}^{p_n}(\mathbb{N})$ into $\ell_{w_n}^{q_n}(\mathbb{N})$. For $r \geq 1$ we introduce the following sequence $f_r = \{f_{r,j}\}_{j=1}^\infty$,

$$f_{r,j} = \begin{cases} v_j^{-p'}, & \text{if } 1 \leq j \leq r, \\ 0, & \text{if } j > r. \end{cases}$$

Let $g_r = \left\{ \frac{f_{r,j}}{\|f_r\|_{\ell_{v_j}^{p_j}(\mathbb{N})}} \right\}_{j=1}^\infty$. It is obvious that $\|g_r\|_{\ell_{v_j}^{p_j}(\mathbb{N})} = 1$. Since the operator H_n is

compact from $\ell_{v_n}^{p_n}(\mathbb{N})$ into $\ell_{w_n}^{q_n}(\mathbb{N})$, it implies that the set $\left\{ w_n H_n \varphi, \|\varphi\|_{\ell_{v_n}^{p_n}(\mathbb{N})} = 1 \right\}$ is precompact in $\ell^{q_n}(\mathbb{N})$. Thus, by the criteria on precompactness of the sets in $\ell^{q_n}(\mathbb{N})$, we have

$$\lim_{r \rightarrow \infty} \left(\sup_{\|\varphi\|_{\ell_{v_n}^{p_n}(\mathbb{N})} = 1} \|w_n H_n \varphi\|_{\ell_{q_n}(n \geq r)} \right) = 0. \tag{11}$$

By Lemma 2, we have

$$\begin{aligned} \sup_{\|\varphi\|_{\ell_{v_n}^{p_n}(\mathbb{N})} = 1} \|w_n H_n \varphi\|_{\ell_{q_n}(n \geq r)} &\geq \|w_n H_n g_r\|_{\ell_{q_n}(n \geq r)} = \left\| \frac{w_n \sum_{j=1}^n f_{r,j}}{n \|f_r\|_{\ell_{v_j}^{p_j}(\mathbb{N})}} \right\|_{\ell_{q_n}(n \geq r)} \\ &\geq \left\| \frac{w_n \sum_{j=1}^n f_{r,j}}{n \|f_r\|_{\ell_{v_n}^{p_n}(\mathbb{N})}} \right\|_{\ell_{q_n}(n \geq r)} = \left\| \frac{w_n}{n} \left(\frac{\left(\sum_{j=1}^n v_j^{-p'} \right)^{\frac{1}{p}}}{\left(\sum_{j=1}^r v_j^{-p'} \right)^{\frac{1}{p}}} \right)^{\frac{1}{q}} \left(\frac{\left(\sum_{j=1}^n v_j^{-p'} \right)^{\frac{q-1}{q}}}{\left(\sum_{j=1}^r v_j^{-p'} \right)^{\frac{1}{p}}} \right)^{\frac{q-1}{q}} \right\|_{\ell_{q_n}(n \geq r)} \\ &\geq \left(\frac{\left(\sum_{j=1}^r v_j^{-p'} \right)^{\frac{1}{q}}}{\left(\sum_{j=1}^r v_j^{-p'} \right)^{\frac{1}{p}}} \right)^{\frac{1}{q}} \left\| \frac{w_n}{n} \left(\frac{\left(\sum_{j=1}^n v_j^{-p'} \right)^{\frac{1}{q}}}{\left(\sum_{j=1}^n v_j^{-p'} \right)^{\frac{1}{p}}} \right)^{\frac{1}{q'}} \right\|_{\ell_{q_n}(n \geq r)} \\ &= \left(\sum_{j=1}^r v_j^{-p'} \right)^{\frac{1}{q p'}} \left\| \frac{\left(\sum_{j=1}^n v_j^{-p'} \right)^{\frac{1}{q p'}}}{n} \right\|_{\ell_{w_n}^{q_n}(n \geq r)}. \end{aligned} \tag{12}$$

So, by (11) and (12) we have the condition (9).

This completes the proof. \square

THEOREM 6. *Let $\{p_n\}_{n=1}^{\infty}$ and $\{q_n\}_{n=1}^{\infty}$ be sequences of real numbers such that $1 < \underline{p} \leq q_n \leq \bar{q} < \infty$ and let $\frac{1}{r_n} = \frac{1}{\underline{p}} - \frac{1}{p_n}$ for all $n \in \mathbb{N}$ and $\|1\|_{\ell^{r_n}(\mathbb{N})} < \infty$. Suppose $\{\omega_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ are sequences of positive real numbers. Then H_n^* is compact from $\ell_{v_n}^{p_n}(\mathbb{N})$ into $\ell_{\omega_n}^{q_n}(\mathbb{N})$ if and only if*

$$\lim_{k \rightarrow \infty} \left(\sum_{n=k}^{\infty} \frac{v_n^{-p'}}{n^{p'}} \right)^{\frac{1}{q'}} \left\| \left(\sum_{m=n}^{\infty} \frac{v_m^{-p'}}{m^{p'}} \right)^{\frac{1}{p'}} \right\|_{\ell_{\omega_n}^{q_n} (n \leq k)} = 0. \quad (13)$$

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