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MAJORIZATION PROBLEMS FOR A CLASS OF ANALYTIC FUNCTIONS DEFINED BY SUBORDINATION

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Abstract. In the present paper we study a majorization problem for a class of analytic functions of α-convex functions defined by subordination with respect to a given function φ. In a special case we obtain the majorization problem for the class $\mathscr{M}(\alpha)$ of α-convex functions. Further, coefficient bounds for some majorized functions are estimated.

1. Introduction

Let $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$ denote the open unit disk of the complex plane $\mathbb{C}$, and let $\mathscr{A}$ be the class of functions f analytic in $\mathbb{U}$ that have the following representation

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,\ z \in \mathbb{U}.$$

Also, let $\mathscr{S}$ be the subclass of $\mathscr{A}$ consisting of all univalent functions in $\mathbb{U}$. Further, let Φ represent the category of all analytic functions ϖ in $\mathbb{U}$ that are satisfying the requirements $\varpi(0) = 0$, and $|\varpi(z)| < 1$ for all $z \in \mathbb{U}$.

DEFINITION 1. [25, 26] For two analytic functions in the unit disk θ and Θ, we state that θ is quasi-subordinate to Θ if there is a function ν analytic in $\mathbb{U}$, such that $\theta(z)/\nu(z)$ is analytic in $\mathbb{U}$,

$$\frac{\theta(z)}{\nu(z)} \prec \Theta(z),$$

and $|\nu(z)| \leqslant 1$, $z \in \mathbb{U}$, where "$\prec$" stands for the usual subordination for analytic functions in $\mathbb{U}$. We denote the above quasi-subordination by

$$\theta(z) \prec_q \Theta(z). \tag{1}$$

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We remark that the relation (1) can be rewritten as

$$\theta(z) = \nu(z)\Theta(\varpi(z)),\ z \in \mathbb{U},$$

where $|\nu(z)| \leqslant 1$, $z \in \mathbb{U}$, and $\varpi \in \Phi$. Therefore, for $\nu(z) \equiv 1$ the quasi-subordination reduces to the subordination [9], i.e.

$$\theta(z) = \Theta(\varpi(z)),\ z \in \mathbb{U},$$

written as $\theta(z) \prec \Theta(z)$, while for $\varpi(z) \equiv z$ the quasi-subordination reduces to the majorization [16], i.e.

$$\theta(z) = \nu(z)\Theta(z),\ z \in \mathbb{U},$$

written as $\theta(z) \ll \Theta(z)$, and in this case we say that θ is majorized by Θ.

Using the concept of subordination, Ma and Minda [17] introduced the classes $\mathscr{S}^*(\varphi)$ and $\mathscr{C}(\varphi)$, where φ is an analytic function with positive real part in $\mathbb{U}$, starlike with respect to $\varphi(0) = 1$, with $\varphi(\mathbb{U})$ symmetric with respect to the real axis, and $\varphi'(0) > 0$. Ma and Minda [17] defined the above mentioned classes as follows:

$$\mathscr{S}^*(\varphi) := \left\{ f \in \mathscr{A} : \frac{zf'(z)}{f(z)} \prec \varphi(z) \right\}$$

and

$$\mathscr{C}(\varphi) := \left\{ f \in \mathscr{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z) \right\}.$$

As remarkable special cases, taking $\varphi(z) = (1 + Az)/(1 + Bz)$ where $A \in \mathbb{C}$, $-1 \leqslant B \leqslant 0$ and $A \neq B$, we get the classes $\mathscr{S}^*[A, B]$ and $\mathscr{C}[A, B]$, respectively, and these classes with the restriction $-1 \leqslant B < A \leqslant 1$ reduce to the well-known *Janowski starlike* and *Janowski convex functions*, respectively. By replacing $A = 1 - 2\alpha$ and $B = -1$, where $0 \leqslant \alpha < 1$, we obtain the classes $\mathscr{S}^*(\alpha)$ and $\mathscr{C}(\alpha)$ of *starlike functions of order* α and *convex functions of order* α, respectively. In particular, $\mathscr{S}^* := \mathscr{S}^*(0)$ and $\mathscr{C} := \mathscr{C}(0)$ are the class of starlike functions and of convex functions in the unit disk $\mathbb{U}$, respectively. In this regards, see also the articles [3, 4, 6, 28, 29].

In 1969, Mocanu [21] introduced the class of α-convex functions, denoted by $\mathscr{M}(\alpha)$, as follows: for $\alpha \in \mathbb{R}$ a $f \in \mathscr{A}$ with $\dfrac{f(z)f'(z)}{z} \neq 0$, $z \in \mathbb{U}$, is said to be α-convex in $\mathbb{U}$ if

$$\operatorname{Re}\left\{ (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) \right\} > 0,\ z \in \mathbb{U}.$$

Clearly, $\mathscr{S}^* := \mathscr{M}(0)$ and $\mathscr{C} := \mathscr{M}(1)$ are the class of starlike functions and of convex functions in the unit disk $\mathbb{U}$, respectively. In 1973, Miller et al. [19] proved that functions belonging to $\mathscr{M}(\alpha)$ are starlike for all real α, and convex for $\alpha \geqslant 1$.

The unified treatment of various subclasses of starlike and convex functions by Ma and Minda [17] motivates one to consider similar classes defined by subordination. In this regards, the class $\mathscr{M}(\alpha)$ was further generalized by Ali et al. [2] by introducing the class $\mathcal{M}(\alpha,\varphi)$ of α-convex functions with respect to φ, that satisfies the above Ma and Minda conditions, by

$$\mathcal{M}(\alpha,\varphi) := \left\{ f \in \mathcal{A} : (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1+\frac{zf''(z)}{f'(z)}\right) \prec \varphi(z) \right\}. \tag{2}$$

As special remark, we see that [24]

$$\mathcal{M}\left(\alpha, \frac{1+(1-2\beta)z}{1-z}\right) =: \mathcal{M}(\alpha,\beta),\ 0 \leqslant \beta < 1,$$
$$\mathcal{M}(0,\varphi) =: \mathcal{S}^*(\varphi), \quad \text{and} \quad \mathcal{M}(1,\varphi) =: \mathcal{C}(\varphi).$$

Silverman [27] considered the class $\mathcal{G}_\beta$ defined by using various combination of $zf'(z)/f(z)$ and $1+zf''(z)/f'(z)$ and investigated properties of the functions of this class (see also [23, 32]). Thus, a function $f \in \mathcal{A}$ belongs to the class $\mathcal{G}_\beta$ if it satisfies the subordination condition

$$\frac{1+zf''(z)/f'(z)}{zf'(z)/f(z)} \prec 1+\beta z,$$

where $0 < \beta \leqslant 1$.

The majorization problem for the normalized classes of starlike functions has recently been investigated by MacGregor [16] and Altintas et al. [7]. Moreover, some authors have investigated majorization problems for the families of meromorphic and multivalent meromorphic or univalent functions including various linear and nonlinear operators (see [8, 12, 13, 30, 31]). Lately, Cho et al. [10] studied majorization problem for the general class of $\mathcal{S}^*(\varphi)$ and presented many consequences of the main result, while the given results would generalize many of the previous ones. Moreover, they investigated coefficient bounds for majorized functions associated with the mentioned class.

Motivated essentially by the recent works [16, 10], in the present paper we study a majorization problem for the general category $\mathcal{M}(\alpha,\varphi)$ and $\mathcal{G}_\beta$ with various special consequences of the main result. Also, some suitable relations of the outcomes are presented with those reported in the earlier results. Moreover, coefficient estimates for majorized functions related to the class $\mathcal{M}(\alpha)$ are obtained.

2. Main results

Throughout this paper we assume only that φ is a univalent function with positive real part in $\mathbb{U}$, with $\varphi(\mathbb{U})$ symmetric with respect to the real axis, and normalized with $\varphi(0)=1$, in other words we will use weaker conditions on φ mentioned in the previous section.

First, we study a majorization problem for the general category $\mathcal{M}(\alpha,\varphi)$. For this goal, the following outcomes will be employed in the proofs of our main results.

LEMMA 1. [20, Theorem 1] *Let β and γ be complex numbers with $\beta \neq 0$, and let $h(z) = c + h_1 z + \cdots$, be regular in $\mathbb{U}$. If $\operatorname{Re}[\beta h(z) + \gamma] > 0$ then the solution of*

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} = h(z),$$

with $q(0) = c$, is regular in $\mathbb{U}$. The solution satisfies $\operatorname{Re}[\beta q(z) + \gamma] > 0$ and is given by

$$q(z) = \begin{cases} H^{\gamma}(z)\left(\beta \displaystyle\int_0^z H^{\gamma}(t) t^{-1}\,\mathrm{d}t\right)^{-1} - \gamma/\beta, & \text{if } c = 0, \\ z^{\gamma}[H(z)]^{\beta c}\left(\beta \displaystyle\int_0^z [H(t)]^{\beta c} t^{\gamma-1}\,\mathrm{d}t\right)^{-1} - \gamma/\beta, & \text{if } c \neq 0, \end{cases}$$

where

$$H(z) = \begin{cases} z\exp\left(\dfrac{\beta}{\gamma}\displaystyle\int_0^z \frac{h(t)}{t}\,\mathrm{d}t\right), & \text{if } c = 0, \\ z\exp\left(\displaystyle\int_0^z \frac{h(t)-c}{ct}\,\mathrm{d}t\right), & \text{if } c \neq 0. \end{cases}$$

LEMMA 2. [15, Theorem 2 (see also page 210)] *Let $f(z) = z + \sum_{\nu=2}^{\infty} a_\nu z^\nu \in \mathscr{M}(\alpha)$, $\alpha > 0$. Let $S(n)$ be the set of all n-tuples $(x_l, x_2, \cdots, x_n)$ of non-negative integers for which $\sum_{i=1}^{n} i x_i = n$, and for each such n-tuple define q by $\sum_{i=1}^{n} x_i = q$. If $\Upsilon(\alpha, q) = \alpha(\alpha-1)(\alpha-2)\ldots(\alpha-q)$ with $\Upsilon(\alpha, 0) = \alpha$, then for $n = 1, 2, \cdots$,*

$$|a_{n+1}| \leqslant \sum \frac{\Upsilon(\alpha, q-1) c_1^{x_1} c_2^{x_2} \cdots c_n^{x_n}}{x_1! x_2! \cdots x_n!},$$

where summation is taken over all n-tuples in $S(n)$, and c_n are given by

$$c_n = \frac{1}{n!\alpha^n(1+n\alpha)} \prod_{k=0}^{n-1} (2 + k\alpha).$$

The bounds are sharp and for $\alpha > 0$ are attained by the function f_* defined by

$$f_*(z) = \left[\frac{1}{\alpha}\int_0^z \zeta^{\frac{1}{\alpha}-1}(1-\zeta)^{-2/\alpha}\,\mathrm{d}\zeta\right]^{\alpha}.$$

LEMMA 3. [32, Corollary 2.6] *Let A and B be such that $-1 \leqslant B < A \leqslant 1$. Then $\mathscr{G}_b \subseteq \mathscr{S}^*[A,B]$ where $b = (A-B)/(1+|A|)^2$. For $A \neq 0$ the inclusion is sharp.*

The next lemma that represents Theorem 2 of [10] holds also if we assume the weaker conditions for the function φ mentioned at the beginning of this section. Following the same proof as in [10] we saw that the assumptions that φ should be starlike with respect to $\varphi(0) = 1$, and the condition $\varphi'(0) > 0$, both assumed in [10], are not necessary.

LEMMA 4. [10, Theorem 2] *Let* $\theta \in \mathscr{A}$, $\Theta \in \mathscr{S}^*(\varphi)$ *with* $\theta(z) \ll \Theta(z)$, *then* $|\theta'(z)| \leqslant |\Theta'(z)|$ *for all* z *in the disk* $|z| \leqslant r_1$, *where* r_1 *is the smallest positive root of the equation*

$$\min_{|z|=r} |\varphi(z)| \left(1 - r^2\right) - 2r = 0,\ r \in (0,1).$$

Our first main result is the next one:

THEOREM 1. *For* $\alpha > 0$ *let* $\theta \in \mathscr{A}$, $\Theta \in \mathscr{M}(\alpha, \varphi)$ *with* $\theta(z) \ll \Theta(z)$. *If* φ *is convex in* $\mathbb{U}$, *then* $|\theta'(z)| \leqslant |\Theta'(z)|$ *for all* z *in the disk* $|z| \leqslant r'$, *where* r' *is the smallest positive root of the equation*

$$\min_{|z|=r} |\psi(z)| \left(1 - r^2\right) - 2r = 0,\ r \in (0,1),$$

with

$$\psi(z) = [H(z)]^{1/\alpha} \left(\frac{1}{\alpha} \int_0^z [H(t)]^{1/\alpha} t^{-1} \,\mathrm{d}t \right)^{-1}, \tag{3}$$

and

$$H(z) = z \exp\left(\int_0^z \frac{\varphi(t) - 1}{t} \,\mathrm{d}t \right). \tag{4}$$

Moreover, the function ψ *is univalent with real positive part in* $\mathbb{U}$.

Proof. For $\Theta \in \mathscr{M}(\alpha, \varphi)$ let define the function $p(z) := z\Theta'(z)/\Theta(z)$, $z \in \mathbb{U}$. Then, the subordination relation of the definition formula (2) for $\Theta \in \mathscr{M}(\alpha, \varphi)$ is equivalent to

$$p(z) + \alpha \frac{zp'(z)}{p(z)} \prec \varphi(z). \tag{5}$$

Supposing that the function ψ satisfies the differential equation

$$\psi(z) + \alpha \frac{z\psi'(z)}{\psi(z)} = \varphi(z),\ \psi(0) = 1, \tag{6}$$

since the function ϑ has positive real part in $\mathbb{U}$, using Lemma 1 for $\beta = 1/\alpha$, $\gamma = 0$, and $c = 1$, it follows that the solution ψ of the differential equation (6) is analytic in $\mathbb{U}$, with $\operatorname{Re} \psi(z) > 0$, $z \in \mathbb{U}$, and is given by (3), where H is defined by (4).

On other the hand, since φ is convex and ψ is analytic in $\mathbb{U}$ with $\operatorname{Re} \psi(z) > 0$, $z \in \mathbb{U}$, using [18, Theorem 3.4.g] for $\phi(w) = \alpha/w$ it follows that ψ is univalent in $\mathbb{U}$, and since p satisfies the differential subordination (5), from [18, Theorem 3.4.g] it follows that $p(z) \prec \psi(z) \prec \varphi(z)$, that is

$$\frac{z\Theta'(z)}{\Theta(z)} \prec \psi(z),$$

and ψ is the best dominant of the subordination (5). Thus, we proved that $\Theta \in \mathscr{M}(\alpha, \varphi)$ implies $\Theta \in \mathscr{S}^*(\psi)$, or

$$\mathscr{M}(\alpha, \varphi) \subset \mathscr{S}^*(\psi).$$

Therefore, by the assumptions $\theta \in \mathscr{A}$, $\Theta \in \mathscr{S}^*(\psi)$ with $\theta(z) \ll \Theta(z)$ and the above inclusion, from Lemma 4 we conclude that $|\theta'(z)| \leqslant |\Theta'(z)|$ for all z in the disk $|z| \leqslant r'$, where r' is the smallest positive root of the equation

$$\min_{|z|=r} |\psi(z)| \left(1-r^2\right) - 2r = 0,\ r \in (0,1),$$

and the proof is completed. □

For $\alpha = 1$ Theorem 1 reduces to the reported result Theorem 2.1 of [11] when θ be majorized by $\Theta \in \mathscr{C}(\varphi)$:

COROLLARY 1. *Let $\theta \in \mathscr{A}$, $\Theta \in \mathscr{C}(\varphi)$ with $\theta(z) \ll \Theta(z)$. If φ is convex in $\mathbb{U}$, then $|\theta'(z)| \leqslant |\Theta'(z)|$ for all z in the disk $|z| \leqslant r'$, where r' is the smallest positive root of the equation*

$$\min_{|z|=r} |\psi(z)| \left(1-r^2\right) - 2r = 0,\ r \in (0,1),$$

with

$$\psi(z) = H(z) \left(\int_0^z H(t) t^{-1}\, \mathrm{d}t \right)^{-1},$$

and

$$H(z) = z \exp\left(\int_0^z \frac{\varphi(t)-1}{t}\, \mathrm{d}t \right).$$

Moreover, the function ψ is univalent with real positive part in $\mathbb{U}$.

Taking $\varphi(z) = \dfrac{1}{\sqrt{1-z}}$ in Corollary 1 we get the next result for θ be majorized by Θ, where $\Theta \in \mathscr{C}\left(\dfrac{1}{\sqrt{1-z}}\right) = \mathscr{C}\mathscr{V}_{hpl}(1/2)$ (see [14]):

COROLLARY 2. *Let $\theta \in \mathscr{A}$, $\Theta \in \mathscr{C}\mathscr{V}_{hpl}(1/2)$ with $\theta(z) \ll \Theta(z)$. Then, $|\theta'(z)| \leqslant |\Theta'(z)|$ for all z in the disk $|z| \leqslant r_1'$, where r_1' is the smallest positive root of the equation*

$$\psi(-r)\left(1-r^2\right) - 2r = 0,\ r \in (0,1),$$

with

$$\psi(z) = \frac{4z}{\left(1+\sqrt{1-z}\right)^2} \cdot \frac{1}{\dfrac{-8}{1+\sqrt{1-z}} - 8\log\left(1+\sqrt{1-z}\right) + 4 + 8\ln 2}.$$

Proof. If we take $\varphi(z) = \dfrac{1}{\sqrt{1-z}}$ in Corollary 1, then φ is convex (univalent) in $\mathbb{U}$, with real positive part, $\varphi(\mathbb{U})$ is symmetric with respect to the real axes, and

$\varphi(0) = 1$. Then we obtain that the function ψ is convex (see the proof of Theorem 8 in [5], pages 5–6, relations (45)–(48)) and

$$\psi(z) = H(z)\left(\int_0^z H(t)t^{-1}\,\mathrm{d}t\right)^{-1}$$

$$= \frac{4z}{\left(1+\sqrt{1-z}\right)^2} \cdot \frac{1}{\dfrac{-8}{1+\sqrt{1-z}} - 8\log\left(1+\sqrt{1-z}\right) + 4 + 8\ln 2} = 1 + \frac{1}{4}z + \cdots,\ z \in \mathbb{U},$$

where

$$H(z) = z\exp\left(\int_0^z \frac{\vartheta(t)-1}{t}\,\mathrm{d}t\right) = \frac{4z}{\left(1+\sqrt{1-z}\right)^2},$$

and all powers are considered at the principal branch, that is $\log 1 = 0$. Further, since $\psi(\overline{z}) = \overline{\psi(z)}$ for all $z \in \mathbb{U}$ it follows that $\psi(\mathbb{U})$ is symmetric with respect to the real axis. Thus, combining with the fact that ψ is convex in $\mathbb{U}$, and because $\operatorname{Re}\psi(z) > 0$, $z \in \mathbb{U}$, it follows that

$$\min\left\{\operatorname{Re}\psi(z) : |z| \leqslant r\right\} = \psi(-r)$$

or

$$\min\left\{\operatorname{Re}\psi(z) : |z| \leqslant r\right\} = \psi(r),\ r \in (0,1).$$

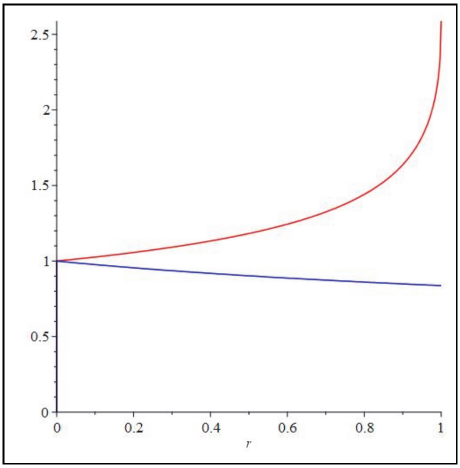

Figure 1: *The plot of* $\psi(r)$ – *red, and* $\psi(-r)$ – *blue, for* $r \in (0,1)$

From the Fig. 1 made with MAPLE™ software we get

$$\psi(-r) < \psi(r),\ r \in (0,1).$$

Therefore

$$\min\left\{\operatorname{Re}\psi(z) : |z| \leqslant r\right\} = \psi(-r) = \min_{|z|=r}|\psi(z)|,\ r \in (0,1),$$

and this completes the proof. $\square$

For $\varphi(z) = 1+\rho z$, with $\rho \in (0,1]$, Corollary 1 leads to the next result for the functions θ majorized by $\Theta \in \mathscr{C}(1+\rho z)$ (see [5]):

COROLLARY 3. *Let $\theta \in \mathscr{A}$, $\Theta \in \mathscr{C}(1+\rho z)$, $\rho \in (0,1]$, with $\theta(z) \ll \Theta(z)$. Then, $|\theta'(z)| \leqslant |\Theta'(z)|$ for all z in the disk $|z| \leqslant r_2'$, where r_2' is the smallest positive root of the equation*

$$\psi(-r)\left(1-r^2\right) - 2r = 0,\ r \in (0,1),$$

where

$$\psi(z) = \frac{\rho z \exp(\rho z)}{\exp(\rho z) - 1}.$$

Proof. We will take $\varphi(z) = 1+\rho z$, $\rho \in (0,1]$, in Corollary 1, which is convex (univalent) with real positive part in $\mathbb{U}$, $\varphi(\mathbb{U})$ is symmetric with respect to the real axis, and $\varphi(0) = 1$. Also, the function ψ is convex (see the proof of Theorem 6 in [5]) and

$$\psi(z) = H(z)\left(\int_0^z \frac{H(t)}{t}\,\mathrm{d}t\right)^{-1} = \frac{\rho z \exp(\rho z)}{\exp(\rho z) - 1} = 1 + \frac{\rho}{2}z + \cdots,\ z \in \mathbb{U},$$

with

$$H(z) = z\exp\left(\int_0^z \frac{\varphi(t)-1}{t}\,\mathrm{d}t\right) = z\exp(\rho z).$$

From the same reasons like those of the proof of Corollary 2 we have

$$\min\{\operatorname{Re}\psi(z) : |z| \leqslant r\} = \psi(-r)$$

or

$$\min\{\operatorname{Re}\psi(z) : |z| \leqslant r\} = \psi(r),\ r \in (0,1).$$

If we define the function $\Psi : [-r,r] \setminus \{0\} \to \mathbb{U}$, with $r \in (0,1)$, by

$$\Psi(x) := \begin{cases} \dfrac{\rho x \exp(\rho x)}{\exp(\rho x) - 1}, & \text{if } x \in [-r,r] \setminus \{0\}, \\ 1, & \text{if } x = 0, \end{cases}$$

then

$$\psi'(x) = \frac{\rho \exp(\rho x)\left(\exp(\rho x) - 1 - \rho x\right)}{\left(\exp(\rho x) - 1\right)^2},\ x \in [-r,r] \setminus \{0\}.$$

It is easy to check that $\ln(1+y) \leqslant y$ for $y \in (-1,+\infty)$, which implies $\exp(\rho x) \geqslant 1+\rho x$ for all $x \in [-r,r] \setminus \{0\}$, therefore $\psi'(x) \geqslant 0$, $x \in [-r,r] \setminus \{0\}$. A simple computation shows that Ψ is continuous at $x = 0$, and combining with the fact that Ψ is a strictly increasing function on each of the intervals $[-r,0)$ and $(0,r]$, we deduce that

$$\psi(-r) < \psi(r),\ r \in (0,1).$$

Therefore

$$\min\{\operatorname{Re}\psi(z):|z|\leqslant r\}=\psi(-r)=\min_{|z|=r}|\psi(z)|,\ r\in(0,1),$$

which completes our proof. □

If we put in Theorem 1 $\varphi(z)=\dfrac{1+(1-2\beta)z}{1-z}$, with $0\leqslant\beta<1$, we get the next result when θ be majorized by $\Theta\in\mathcal{M}(\alpha,\beta)$:

COROLLARY 4. *Let $\theta\in\mathcal{A}$, $\Theta\in\mathcal{M}(\alpha,\beta)$, with $\alpha>0$, $0\leqslant\beta<1$, and $\theta(z)\ll\Theta(z)$. If the function*

$$\psi(z):=\frac{1}{{}_2F_1\left(\frac{2(1-\beta)}{\alpha},1,\frac{1}{\alpha}+1;\frac{z}{z-1}\right)} \tag{7}$$

is convex in $\mathbb{U}$ for some $\alpha>0$ and $\beta\in[0,1)$, then $|\theta'(z)|\leqslant|\Theta'(z)|$ for all z in the disk $|z|\leqslant r_2'$, where r_2' is the smallest positive root of the equation

$$\frac{1}{{}_2F_1\left(\frac{2(1-\beta)}{\alpha},1,\frac{1}{\alpha}+1;\frac{r}{r+1}\right)}\left(1-r^2\right)-2r=0,\ r\in(0,1),$$

and ${}_2F_1(a,b,c;z)$ is the Gaussian hypergeometric function.

Proof. Let chose $\alpha>0$ and $\beta\in[0,1)$ such that the function ψ is convex (univalent) in $\mathbb{U}$.

Setting $\varphi(z)=\dfrac{1+(1-2\beta)z}{1-z}$ in Theorem 1, which is convex (univalent) with real positive part in $\mathbb{U}$, $\varphi(\mathbb{U})$ is symmetric with respect to the real axis, and $\varphi(0)=1$, we will use Theorem 3.3d. of [18] by taking in the subordination (3.3-11) of [18, page 109] the values

$$\beta:=\frac{1}{\alpha},\quad\gamma:=0,\quad A:=1-2\beta,\quad B:=-1.$$

Thus, the assumptions of Theorem 3.3d., i.e. $\operatorname{Re}[\beta+\gamma]=1/\alpha>0$ and (3.3-10) of [18, page 108] are satisfied, and according to this theorem, combined with the relations (3.3-13) and (3.3-15) of [18, page 110], (see also [1, Theorem 2.2]) we conclude that

$$\psi(z)=\frac{1}{{}_2F_1\left(\frac{2(1-\beta)}{\alpha},1,\frac{1}{\alpha}+1;\dfrac{z}{z-1}\right)}.$$

On the other hand, to determine the best lower-bound

$$\min_{|z|=r<1}\operatorname{Re}\psi(z)$$

we will use Theorem 1 of [22] (see also [18, Theorem 3.3e]). Therefore, by choosing in this theorem

$$\beta := \frac{1}{\alpha} > 0, \quad \beta + \gamma := \frac{1}{\alpha} > 0, \quad \alpha := 0,$$

since

$$\alpha \in [\alpha_0, 1), \ \alpha_0 := \max\left\{\frac{1-\alpha}{2}; 0\right\} < 1,$$

we obtain that the best lower-bound is

$$\min_{|z|=r<1} \operatorname{Re} \psi(z) = \psi(-r) = \frac{1}{{}_2F_1\left(\frac{2(1-\beta)}{\alpha}, 1, \frac{1}{\alpha}+1; \frac{r}{r+1}\right)}.$$

Since $\psi(\mathbb{U})$ is symmetric with respect to the real axis, ψ was assumed to be convex in $\mathbb{U}$ for some α and β, and $\operatorname{Re} \psi(z) > 0$, $z \in \mathbb{U}$, it follows that

$$\min_{|z|=r} |\psi(z)| = \min_{|z|=r} \operatorname{Re} \psi(z) = \psi(-r),$$

and this completes the proof. $\square$

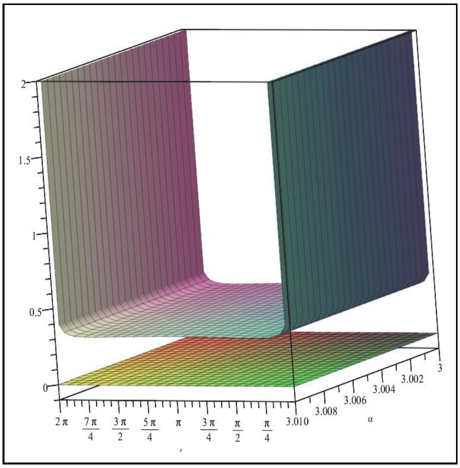

Figure 2: *The plot of* $\Phi\left(Re^{it}\right)$, $R = 1 - 10^{-6}$, $t \in [0, 2\pi]$, *for* $\beta = 1/2$ *and* $\alpha \in [3, 3.01]$

If we take $\beta = \dfrac{1}{2}$ in the above corollary, from the Fig. 2 made with MAPLE™ software we get that the function Φ defined by

$$\Phi(z) := \operatorname{Re}\left(1 + \frac{z\psi''(z)}{\psi'(z)}\right),$$

where ψ is given by (7) is positive in $\mathbb{U}$ for all $\alpha \in [3, 3.01]$. Therefore, for $\beta = \dfrac{1}{2}$ and $\alpha \in [3, 3.01]$ the function ψ is convex, and from Corollary 4 we get the next example:

EXAMPLE 1. Let $\theta \in \mathscr{A}$, $\Theta \in \mathscr{M}(\alpha, 1/2)$, with $\alpha \in [3, 3.01]$, and $\theta(z) \ll \Theta(z)$. Then, $|\theta'(z)| \leqslant |\Theta'(z)|$ for all z in the disk $|z| \leqslant \widehat{r}$, where $\widehat{r}$ is the smallest positive root of the equation

$$\frac{1}{{}_2F_1\left(\frac{1}{\alpha}, 1, \frac{1}{\alpha}+1; \frac{r}{r+1}\right)}\left(1-r^2\right) - 2r = 0,\ r \in (0,1),$$

and ${}_2F_1(a,b,c;z)$ is the *Gaussian hypergeometric function*.

Similarly, considering $\alpha = 3.001$ in the above corollary, from the Fig. 3 made with MAPLE$^{\text{TM}}$ software we get that the function Φ defined by (2), where ψ is given by (7), is positive in $\mathbb{U}$ for all $\beta \in [0.4999, 0.5001]$. If follows that for $\alpha = 3.001$ and $\beta \in [0.4999, 0.5001]$ the function ψ is convex, and from Corollary 4 we get:

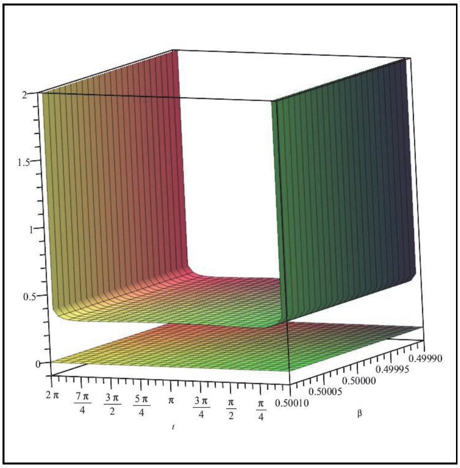

Figure 3: *The plot of* $\Phi\left(Re^{it}\right)$, $R = 1 - 10^{-6}$, $t \in [0, 2\pi]$, *for* $\alpha = 3.001$ *and* $\beta \in [0.4999, 0.5001]$

EXAMPLE 2. Let $\theta \in \mathscr{A}$, $\Theta \in \mathscr{M}(3.001, \beta)$, with $\beta \in [0.4999, 0.5001]$, and $\theta(z) \ll \Theta(z)$. Then, $|\theta'(z)| \leqslant |\Theta'(z)|$ for all z in the disk $|z| \leqslant \widetilde{r}$, where $\widetilde{r}$ is the smallest positive root of the equation

$$\frac{1}{{}_2F_1\left(\frac{2(1-\beta)}{3.001}, 1, \frac{1}{3.001}+1; \frac{r}{r+1}\right)}\left(1-r^2\right) - 2r = 0,\ r \in (0,1),$$

and ${}_2F_1(a,b,c;z)$ is the *Gaussian hypergeometric function*.

For $\beta = 0$ in Corollary 4 we get the next result when θ be majorized by Θ where $\Theta \in \mathscr{M}(\alpha)$:

COROLLARY 5. *Let $\theta \in \mathscr{A}$, $\Theta \in \mathscr{M}(\alpha)$, with $\alpha > 0$, and $\theta(z) \ll \Theta(z)$. If the function*

$$\psi(z) := \frac{1}{{}_2F_1\left(\frac{2}{\alpha}, 1, \frac{1}{\alpha}+1; \frac{z}{z-1}\right)} \tag{8}$$

is convex in $\mathbb{U}$ for some $\alpha > 0$, then $|\theta'(z)| \leqslant |\Theta'(z)|$ for all z in the disk $|z| \leqslant r_3'$, where r_3' is the smallest positive root of the equation

$$\frac{1}{{}_2F_1\left(\frac{2}{\alpha}, 1, \frac{1}{\alpha}+1; \frac{r}{r+1}\right)}\left(1-r^2\right) - 2r = 0,\ r \in (0,1),$$

and ${}_2F_1(a,b,c;z)$ is the Gaussian hypergeometric function.

If we set $\alpha \in [20, 20.01]$ in the above corollary, from the Fig. 4 made with MAPLE$^{\text{TM}}$ software we get that the function Φ defined by

$$\Phi(z) := \operatorname{Re}\left(1 + \frac{z\psi''(z)}{\psi'(z)}\right),$$

where ψ is given by (8) is positive in $\mathbb{U}$ for $\beta = 0$. Therefore, for $\beta = 0$ and $\alpha \in [20, 20.01]$ the function ψ is convex, and from Corollary 5 we get the next example:

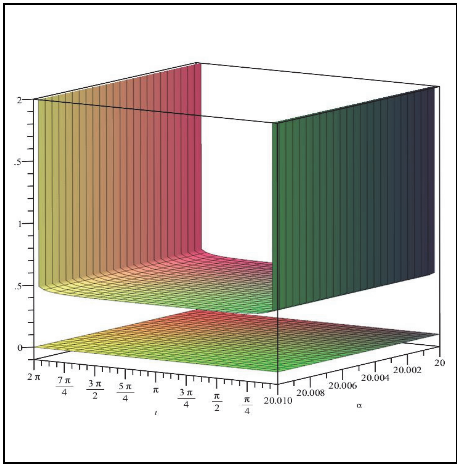

Figure 4: *The plot of* $\Phi\left(Re^{it}\right)$, $R = 1 - 10^{-6}$, $t \in [0, 2\pi]$, *for* $\beta = 0$ *and* $\alpha \in [20, 20.01]$

EXAMPLE 3. Let $\theta \in \mathscr{A}$, $\Theta \in \mathscr{M}(\alpha)$, with $\alpha \in [20, 20.01]$, and $\theta(z) \ll \Theta(z)$. Then, $|\theta'(z)| \leqslant |\Theta'(z)|$ for all z in the disk $|z| \leqslant r_*$, where r_* is the smallest positive root of the equation

$$\frac{1}{{}_2F_1\left(\frac{2}{\alpha}, 1, \frac{1}{\alpha}+1; \frac{r}{r+1}\right)}\left(1-r^2\right) - 2r = 0,\ r \in (0,1),$$

and ${}_2F_1(a,b,c;z)$ is the *Gaussian hypergeometric function*.

THEOREM 2. *Let A and B be such that $-1 \leqslant B < A \leqslant 1$, and suppose that $\theta \in \mathscr{A}$, $\Theta \in \mathscr{G}(\beta)$, where $\beta = (A-B)/(1+|A|)^2$. Then, $|\theta'(z)| \leqslant |\Theta'(z)|$ for all z in the disk $|z| \leqslant r''$, where r'' is the smallest positive root of the equation*

$$\frac{1-Ar}{1-Br}\left(1-r^2\right)-2r=0,\ r\in(0,1).$$

Proof. If $\theta \in \mathscr{A}$ and $\Theta \in \mathscr{G}(\beta)$, with $\beta = (A-B)/(1+|A|)^2$, using Lemma 3 it follows that $\Theta \in \mathscr{S}^*[A,B]$. Since $\varphi(z) = \dfrac{1+Az}{1+Bz}$ is convex (univalent) in $\mathbb{U}$, with real positive part, and $\varphi(\mathbb{U})$ is symmetric with respect to the real axis, we have

$$\min_{|z|=r}|\varphi(z)| = \varphi(-r) = \frac{1-Ar}{1-Br}.$$

Now, from Lemma 4 our result follows immediately. □

For $B=-1$ and $A=1-2\lambda$, with $0 \leqslant \lambda < 1$, the above theorem reduces to the next particular case:

COROLLARY 6. *Let $0 \leqslant \lambda < 1$, and suppose that $\theta \in \mathscr{A}$, $\Theta \in \mathscr{G}\big(2(1-\lambda)/(1+|1-2\lambda|)^2\big)$. Then,*

$$\big(1-(1-2\lambda)r\big)(1-r)-2r=0,\ r\in(0,1).$$

THEOREM 3. *Let $\theta(z) = z+\sum\limits_{n=2}^{\infty} a_n z^n \in \mathscr{A}$, and $\Theta(z) = z+\sum\limits_{n=2}^{\infty} b_n z^n \in \mathscr{M}(\alpha)$, $\alpha > 0$, with $\theta(z) \ll \Theta(z)$. Then,*

$$|a_n| \leqslant 1+\sum_{j=2}^{n}\left(\sum\frac{\Upsilon(\alpha,q-1)c_1^{x_1}c_2^{x_2}\cdots c_{j-1}^{x_{j-1}}}{x_1!x_2!\cdots x_{j-1}!}\right),\ n=2,3,\cdots,$$

where the summation is taken like in Lemma 2.

Proof. Since $\theta(z) \ll \Theta(z)$, by the majorization principle there is an analytic function $\nu(z) = \sum\limits_{n=0}^{\infty} c_n z^n$, with $|\nu(z)| \leqslant 1$ in $\mathbb{U}$, satisfying

$$\theta(z) = \nu(z)\Theta(z),\ z \in \mathbb{U},$$

that implies

$$a_n = c_0 b_n + c_1 b_{n-1} + \cdots + c_{n-2} b_2 + c_{n-1},\ n=2,3,\cdots. \tag{9}$$

If γ is an arbitrary circle $|z|=r$, with $0<r<1$, that is $z = re^{i\zeta}$, $0 \leqslant \zeta \leqslant 2\pi$, then

$$c_k = \frac{1}{2\pi \mathrm{i}}\int_{\gamma}\frac{\nu(z)}{z^{k+1}}\,\mathrm{d}z,\ k=0,1,\cdots.$$

In view of the above relation we can rewrite (9) in the form (see also [16, p. 99])

$$a_n = \frac{1}{2\pi \mathrm{i}} \int_\gamma \left[\frac{\nu(z)}{z^n} \left(1 + b_2 z + \cdots + b_n z^{n-1}\right) \right] \mathrm{d}z,\ n = 2, 3, \cdots.$$

Using the well-known fact that $|\nu(z)| \leqslant |z|$ for all $z \in \mathbb{U}$, from the above relation we get

$$\begin{aligned} |a_n| &\leqslant \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{r^{n-1}} \left| 1 + b_2 r e^{i\zeta} + \cdots + b_n r^{n-1} e^{i(n-1)\zeta} \right| \mathrm{d}\zeta \\ &\leqslant \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{r^{n-1}} \left(1 + |b_2| r + \cdots + |b_n| r^{n-1}\right) \mathrm{d}\zeta \\ &< \frac{1}{r^{n-1}} (1 + |b_2| + \cdots + |b_n|),\ n = 2, 3, \cdots. \end{aligned}$$

And since the above inequality holds for all $r \in (0,1)$, letting $r \to 1^-$ we conclude that

$$|a_n| \leqslant 1 + |b_2| + \cdots + |b_n|,\ n = 2, 3, \cdots.$$

From this inequality, using Lemma 2 we obtain the required inequality. $\square$

As a special case of this theorem we easily get the next result:

EXAMPLE 4. Let $\theta(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathscr{A}$, and $\Theta(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathscr{M}(\alpha)$, $\alpha > 0$, with $\theta(z) \ll \Theta(z)$. Then,

$$|a_2| \leqslant 1 + \frac{2}{1+\alpha}.$$

Proof. Since $\Theta(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathscr{M}(\alpha)$, according to Lemma 2 for $n = 1$ we get

$$|b_2| \leqslant \sum \frac{\Upsilon(\alpha, q-1) c_1^{x_1}}{x_1!} = \frac{2}{1+\alpha}.$$

Now, from Theorem 3 for $n = 2$ and the above inequality it follows that

$$|a_2| \leqslant 1 + \sum_{j=2}^{2} \left(\sum \frac{\Upsilon(\alpha, q-1) c_1^{x_1} c_2^{x_2} \cdots c_{j-1}^{x_{j-1}}}{x_1! x_2! \cdots x_{j-1}!} \right) = 1 + \sum \frac{\Upsilon(\alpha, q-1) c_1^{x_1}}{x_1!} = 1 + \frac{2}{1+\alpha},$$

where all the above summations are taken like in Lemma 2. $\square$

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