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HERMITE–HADAMARD TYPE INEQUALITIES FOR RIEMANN–LIOUVILLE FRACTIONAL INTEGRALS VIA STRONGLY h–CONVEX FUNCTIONS

YI XING, CHAOQUN JIANG AND JIANMIAO RUAN*

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Abstract. In this paper, the Hermite-Hadamard type inequalitis for Riemann-Liouville fractional integrals via strongly h-convex functions are established. Furthermore, we obtain some identities related to the fractional integrals with n-times differentiable functions, and then gain midpoint type and trapezoid type error estimates connected with the Hermite-Hadamard type inequalities, which generalize some known results.

1. Introduction

Let I be an interval in $\mathbb{R}$ and $h:[0,1]\to[0,\infty)$ be a given function. A function $f:I\to\mathbb{R}$ is called h*-convex* provided that

$$f(tx+(1-t)y)\leqslant h(t)f(x)+h(1-t)f(y) \tag{1.1}$$

holds for all $x,y\in I$ and $t\in(0,1)$. This notation was introduced by Varošanec [33] and generalizes the classes of *convex functions, s-convex functions (in the second sence), Godunova-Levin functions* and *P-functions*, which are obtained by taking in (1.1) $h(t)=t$, $h(t)=t^s$ $(s\in(0,1))$, $h(t)=1/t$ and $h(t)=1$, respectively. Many properties of them can be found, for instance, in [10, 12, 16, 25, 26, 38].

A significant application of the convex function is the well-known Hermite-Hadamard inequality, if $f:[a,b]\to\mathbb{R}$ is a convex function, then

$$f\left(\frac{a+b}{2}\right)\leqslant\frac{1}{b-a}\int_a^b f(x)dx\leqslant\frac{f(a)+f(b)}{2}. \tag{1.2}$$

This inequality was studied extensively and had been extended under various convex type functions. In 1995, Dragomir, Pečarić and Persson [11] established similar results for Godunova-Levin functions and P-functions. In 1999, Dragomir and Fitzpatrick [9]

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* Corresponding author.

obtained an analogous inequality for s-convex functions (in the second sence). In 2008, Sarikaya, Saglam and Yildirim [29] extended it to h-convex functions.

Following Polyak [23], a function $f: I \to \mathbb{R}$ is said to be *strongly convex with modulus* $\beta > 0$ if

$$f(tx+(1-t)y) \leqslant tf(x)+(1-t)f(y)-\beta t(1-t)(x-y)^2, \tag{1.3}$$

for all $x,y \in I$ and $t \in (0,1)$. The function played an important role in optimization theory, mathematical economics and so on (see e.g. [17, 18, 19, 23, 24, 34, 35]). In 2011, Angulo, Gimenez, Moros and Nikodem [3] introduced the strongly h-convex function, which unified the classes of strongly convex functions and h-convex functions. And then they extended (1.2) to these new functions.

DEFINITION 2. [3] Let $h: [0,1] \to [0,\infty)$ be a given function and β be a positive constant. We say that $f: I \to \mathbb{R}$ is strongly h-convex with modulus β, or f belongs to the class $SX(h,\beta,I)$, if

$$f(tx+(1-t)y) \leqslant h(t)f(x)+h(1-t)f(y)-\beta t(1-t)(x-y)^2, \tag{1.4}$$

for all $x,y \in I$ and $t \in (0,1)$.

Particularly, if f satisfies (1.4) with $h(t)=t$, $h(t)=t^s$ $(s \in (0,1))$, $h(t)=1/t$ and $h(t)=1$, then f is said to be *strongly convex functions, strongly s-convex functions, strongly Godunova-Levin functions* and *strongly P-function*, respectively. Moreover, it is not difficult to check that $h(1/2)>0$ if $f \in SX(h,\beta,I)$ and $f \geqslant 0$. As an application, the authors [3] established the following Hermite-Hadamard inequality.

THEOREM A. *Let $f \in SX(h,\beta,[a,b])$ and h be Lebesgue integrable on $(0,1)$ with $h(1/2)>0$. If f is Lebesgue integrable on $[a,\ b]$, then*

$$\frac{1}{2h(1/2)}\left[f\left(\frac{a+b}{2}\right)+\frac{\beta}{12}(b-a)^2\right] \leqslant \frac{1}{b-a}\int_a^b f(x)dx$$
$$\leqslant (f(a)+f(b))\int_0^1 h(t)dt-\frac{\beta}{6}(b-a)^2. \tag{1.5}$$

It is notable that Theorem A reduces to Theorem 6 in [29] with $\beta \to 0$.

Let $\alpha > 0$ and $f \in L([a,b])$. *The left-sided and the right-sided Riemann-Liouville fractional integrals $J_{a^+}^{\alpha}f$ and $J_{b^-}^{\alpha}f$ of order α* are defined by

$$J_{a^+}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt,\ \ x \in (a,b], \tag{1.6}$$

and

$$J_{b^-}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)dt,\ \ x \in [a,b), \tag{1.7}$$

respectively, where $\Gamma(\alpha)=\int_0^{\infty} t^{\alpha-1}e^{-t}dx$ is the Gamma function.

In recent years, Hermite-Hadamard type inequalities via the fractional integrals are studied extensively, for instance, see [5, 8, 27, 30, 32, 36, 37] and the references therein.

In 2017, Sarikaya and Yildirim [28] first obtained a remarkable inequality of Hermite-Hadamard type involving the fractional integrals.

THEOREM B. [28] *Let* $f:[a,b]\to\mathbb{R}$ *be a convex function and* $f\in L([a,b])$. *Then*

$$f\left(\frac{a+b}{2}\right)\leqslant\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^-}f(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)^+}f(b)\right]\leqslant\frac{f(a)+f(b)}{2}.$$

In 2020, Budak, Ertuğral and Sarikaya [6] extended it to more generalized fractional integrals. In 2021, Zhang, Farid and Akbar [39] obtained an analogue inequality as Theorem B for strongly (s,m)-convex functions.

In 2000, Pearce and Pečarić [25] proved an important equality connect with the left part of (1.2).

LEMMA A. [25] *Let* $f:[a,b]\to\mathbb{R}$ *be a differentiable function and* $f'\in L([a,b])$. *Then*

$$\begin{aligned}&\frac{1}{b-a}\int_a^b f(x)dx-f\left(\frac{a+b}{2}\right)\\&=\frac{1}{b-a}\left[\int_{(a+b)/2}^b(b-x)f'(x)dx-\int_a^{(a+b)/2}(x-a)f'(x)dx\right]\\&=(b-a)\left[\int_0^{1/2}tf'(ta+(1-t)b)dt-\int_{1/2}^1(1-t)f'(ta+(1-t)b)dt\right].\end{aligned}$$

By the lemma, the authors showed the following result.

THEOREM C. [25] *Let* $f:[a,b]\to\mathbb{R}$ *be a differentiable function. If* $|f'|^q$ *is convex on* $[a,b]$ *with* $1\leqslant q<\infty$, *then*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx-f\left(\frac{a+b}{2}\right)\right|\leqslant\frac{b-a}{4}\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{1/q}. \tag{1.8}$$

Furthermore, some estimates for concave functions are also achieved in [15] and [25]. In 2004, Kirmaci [14] rediscovered Lemma A and established some other estimates similar to Theorem C. In 2011, Alomori, Darus and Kirmaci [2] obtained analogue results for s-convex functions. And more results about the difference estimates connected with (1.2) can be referred, for instance, to [4, 13, 25].

In 2017, Sarikaya and Yildirim [28] found an important identity related to Riemann-Liouville integrals as follows.

LEMMA B. *Let* $f:[a,b]\to\mathbb{R}$ *be a differentiable function and* $f'\in L([a,b])$. *Then*

$$\begin{aligned}&\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^-}f(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)^+}f(b)\right]-f\left(\frac{a+b}{2}\right)\\&=\frac{b-a}{4}\int_0^1(1-t)^{\alpha}\left[f'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)-f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right]dt.\end{aligned}$$

It is not difficult to check that Lemma B becomes Lemma A with $\alpha=1$. As a consequence, they obtained the following midpoint type inequalities for differentiable functions.

THEOREM D. [28] *Let* $f:[a,b]\to\mathbb{R}$ *be a differentiable function. Suppose that* $|f'|^q$ *is convex on* $[a,b]$ *for* $1\leqslant q<\infty$.

(i) *If* $1\leqslant q<\infty$, *then*

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^-}f(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)^+}f(b)\right]-f\left(\frac{a+b}{2}\right)\right|$$
$$\leqslant\frac{b-a}{4(\alpha+1)}\left(\frac{1}{2(\alpha+2)}\right)^{1/q}\left\{\left[(\alpha+1)|f'(a)|^q+(\alpha+3)|f'(b)|^q\right]^{1/q}\right.$$
$$\left.+\left[(\alpha+3)|f'(a)|^q+(\alpha+1)|f'(b)|^q\right]^{1/q}\right\}.$$

(ii) *If* $1<q<\infty$, *then*

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^-}f(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)^+}f(b)\right]-f\left(\frac{a+b}{2}\right)\right|$$
$$\leqslant\frac{b-a}{4}\left(\frac{1}{\alpha p+1}\right)^{1/p}\left[\left(\frac{|f'(a)|+3|f'(b)|}{4}\right)^{1/q}+\left(\frac{3|f'(a)|+4|f'(b)|}{4}\right)^{1/q}\right]$$
$$\leqslant\frac{b-a}{4}\left(\frac{4}{\alpha p+1}\right)^{1/p}\left[|f'(a)|+|f'(b)|\right],$$

where $1/p+1/q=1$.

It is notable that Theorem D reduces to the theorems in [14]. In 2017, Set, Sarikaya and Gözpinart [31] generalized the proceeding theorem to conformal fractional integrals. In 2020, the authors [6] extended them to more generalized fractional integrals. In 2021, the authors [39] gained similar inequalities for strongly (s,m)-convex functions. Moreover, Noor and Awan [20] proved an equality for twice differentiable functions.

LEMMA C. [20] *Let* $f:[a,b]\to\mathbb{R}$ *be a twice differentiable function and* $f''\in L([a,b])$. *Then*

$$\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^-}f(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)^+}f(b)\right]-f\left(\frac{a+b}{2}\right)$$
$$=\frac{(b-a)^2}{8(\alpha+1)}\int_0^1(1-t)^{\alpha+1}\left[f''\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)+f''\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right]dt.$$

Consequently, they established the following inequalities for s-convex functions.

THEOREM E. [20] *Let* $f\in C^2([a,b])$ *and* $f''\in L([a,b])$. *Suppose that* $|f''|^q$ $(1\leqslant q<\infty)$ *is an* s*-convex function (in the second sense).*

(i) *If* $1 \leqslant q < \infty$, *then*

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^-}f(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)^+}f(b)\right]-f\left(\frac{a+b}{2}\right)\right|$$
$$\leqslant \frac{(b-a)^2}{2^{3+s/q}(\alpha+1)}\left(\frac{1}{\alpha+2}\right)^{1-1/q}$$
$$\times\left\{\left(\int_0^1(1-t)^{\alpha+1}(1+t)^s dt\left|f''(a)\right|^q+\frac{1}{s+\alpha+2}\left|f''(b)\right|^q\right)^{1/q}\right.$$
$$\left.+\left(\frac{1}{s+\alpha+2}\left|f''(a)\right|^q+\int_0^1(1-t)^{\alpha+1}(1+t)^s dt\left|f''(b)\right|^q\right)^{1/q}\right\}.$$

(ii) *If* $1 < q < \infty$, *then*

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^-}f(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)^+}f(b)\right]-f\left(\frac{a+b}{2}\right)\right|$$
$$\leqslant \frac{(b-a)^2}{2^{3+s/q}(\alpha+1)}\left(\frac{1}{p(\alpha+1)+1}\right)^{1/p}\left(\frac{1}{s+1}\right)^{1/q}$$
$$\times\left\{\left[\left(2^{s+1}-1\right)\left|f''(a)\right|^q+\left|f''(b)\right|^q\right]^{\frac{1}{q}}+\left[\left|f''(a)\right|^q+\left(2^{s+1}-1\right)\left|f''(b)\right|^q\right]^{1/q}\right\},$$

where $1/p+1/q=1$.

On the other hand, in 1998, Dragomir and Agarwal [7] established the following identity for the right hand side of (1.2), and then they gained error estimates related to it. Some more studies please refer to, for examples, [1, 13, 14, 22].

LEMMA D. [7] *Let* $f \in C^1([a,b])$ *and* $f' \in L([a,b])$. *Then*

$$\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx=\frac{b-a}{2}\int_0^1(1-2t)f'(ta+(1-t)b)dt.$$

In 2016, Özdemir, M. Avci-Ardinç and H. Kavurmaci-Önalan [21] (Lemma 2 for $x=(a+b)/2$) proved a trapezoid type equality for differentiable function via fractional integral.

LEMMA E. [21] *Let* $f \in C^1([a,b])$ *and* $f' \in L([a,b])$. *Then*

$$\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^-}f(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)^+}f(b)\right]-\frac{f(a)+(b)}{2}$$
$$=\frac{(b-a)}{4}\int_0^1\left[1-(1-t)^{\alpha}\right]\left[f'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)-f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right]dt.$$

Thereafter, Budak [5] obtained it for generalized fractional integral in 2019 and Budak, Ertuğral and Sarikaya [6] extended it to other fractional integrals in 2020. As a consequence, the authors obtained the following results.

THEOREM F. [5, 6] *Let $f \in C^1([a,b])$ and $f' \in L([a,b])$. Suppose that $|f'|^q$ is convex on $[a,b]$ for $1 \leqslant q < \infty$.*

(i) *If $1 \leqslant q < \infty$, then*

$$\begin{aligned}&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^-}f(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)^+}f(b)\right]-\frac{f(a)+f(b)}{2}\right|\\&\leqslant \frac{b-a}{2^{2+1/q}}\frac{\alpha}{\alpha+1}\left[\left(\frac{\alpha+1}{2(\alpha+2)}|f'(a)|^q+\frac{3\alpha+7}{2(\alpha+2)}|f'(b)|^q\right)^{1/q}\right.\\&\quad\left.+\left(\frac{3\alpha+7}{2(\alpha+2)}|f'(a)|^q+\frac{\alpha+1}{2(\alpha+2)}|f'(b)|^q\right)^{1/q}\right].\end{aligned}$$

(ii) *If $1 < q < \infty$, then*

$$\begin{aligned}&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^-}f(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)^+}f(b)\right]-\frac{f(a)+f(b)}{2}\right|\\&\leqslant \frac{b-a}{4}\left(\frac{1}{\alpha p+1}\right)^{1/p}\left[\left(\frac{|f'(a)|+3|f'(b)|}{4}\right)^{1/q}+\left(\frac{3|f'(a)|+4|f'(b)|}{4}\right)^{1/q}\right]\\&\leqslant \frac{b-a}{4}\left(\frac{4}{\alpha p+1}\right)^{1/p}\left[|f'(a)|+|f'(b)|\right],\end{aligned}$$

where $1/p+1/q=1$.

In the sequel, we assume that the function h in the above definitions is always Lebesgue integrable on $[0,1]$. Denote $L(I)$ be the set of Lebesgue integrable functions on the interval I and let $C^n(I)$ be the space of functions f with its derivatives $f^{(k)}$ continuous on I for all $0 \leqslant k \leqslant n$. The aim of this paper is to extend the above results to strongly h-convex functions and obtain some error estimates related to these inequalities.

2. New Hermite-Hadamard inequality via Riemann-Liouville fractional integrals

In this section, we establish a similar results as Theorem B for strongly h-convex functions.

THEOREM 1. *Let $f \in L([a,b])$ and $f \in SX(h,\beta,[a,b])$ with $h(1/2) > 0$. Then*

$$\begin{aligned}&\frac{1}{2h(1/2)}\left[f\left(\frac{a+b}{2}\right)+\frac{\beta(b-a)^2}{2(\alpha+1)(\alpha+2)}\right]\\&\leqslant \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^-}f(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)^+}f(b)\right]\\&\leqslant \frac{f(a)+f(b)}{2}\alpha\int_0^1 t^{\alpha-1}\left[h\left(\frac{t}{2}\right)+h\left(1-\frac{t}{2}\right)\right]dt-\frac{\beta\alpha(\alpha+3)(b-a)^2}{4(\alpha+1)(\alpha+2)}.\end{aligned}$$

Proof. Since f is a strongly h-covex function with modulus β, for any $t \in [0,1]$ we have

$$f\left(\frac{a+b}{2}\right) = f\left[\frac{1}{2}\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)+\frac{1}{2}\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right]$$
$$\leqslant h(1/2)f\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)+h(1/2)f\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)-\frac{\beta}{4}t^2(b-a)^2,$$

with means that

$$\begin{aligned}
&\frac{1}{\alpha h(1/2)}f\left(\frac{a+b}{2}\right)\\
&\leqslant \int_0^1 (1-t)^{\alpha-1} f\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)dt+\int_0^1 (1-t)^{\alpha-1} f\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)dt\\
&\quad -\frac{\beta(b-a)^2}{4h(1/2)}\int_0^1 (1-t)^{\alpha-1}t^2 dt\\
&= \frac{2^\alpha}{(b-a)^\alpha}\int_{(a+b)/2}^b (b-u)^{\alpha-1}f(u)du+\frac{2^\alpha}{(b-a)^\alpha}\int_a^{(a+b)/2}(u-a)^{\alpha-1}f(u)du \qquad (2.1)\\
&\quad -\frac{\beta(b-a)^2}{4h(1/2)\alpha(\alpha+1)(\alpha+2)}\\
&= \frac{2^\alpha\Gamma(\alpha)}{(b-a)^\alpha}\left[J^\alpha_{\left(\frac{a+b}{2}\right)^+}f(b)+J^\alpha_{\left(\frac{a+b}{2}\right)^-}f(a)\right]-\frac{\beta(b-a)^2}{4h(1/2)\alpha(\alpha+1)(\alpha+2)}. \qquad (2.2)
\end{aligned}$$

Therefore we finish the first inequality of the theorem.

On the other hand, it is easy to see that

$$f\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\leqslant h\left(\frac{1-t}{2}\right)f(a)+h\left(\frac{1+t}{2}\right)f(b)-\frac{\beta(1-t)(1+t)}{4}(b-a)^2,$$
$$f\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\leqslant h\left(\frac{1+t}{2}\right)f(a)+h\left(\frac{1-t}{2}\right)f(b)-\frac{\beta(1-t)(1+t)}{4}(b-a)^2,$$

which, combing with (2.1) and (2.2), imply that

$$\begin{aligned}
&\frac{2^\alpha\Gamma(\alpha)}{(b-a)^\alpha}\left[J^\alpha_{\left(\frac{a+b}{2}\right)^-}f(a)+J^\alpha_{\left(\frac{a+b}{2}\right)^+}f(b)\right]\\
&= \int_0^1 (1-t)^{\alpha-1} f\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)dt+\int_0^1 (1-t)^{\alpha-1} f\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)dt\\
&\leqslant [f(a)+f(b)]\int_0^1 (1-t)^{\alpha-1}\left[h\left(\frac{1-t}{2}\right)+h\left(\frac{1+t}{2}\right)\right]dt-\frac{\beta(b-a)^2}{2}\frac{\alpha+3}{(\alpha+1)(\alpha+2)}.
\end{aligned}$$

Thus we finish the proof of Theorem 1. □

If taking $h(t)=t$ and $h(t)=t^s$, then Theorem 1 reduces to Corollary 3 and Corollary 4 in [39], respectively. And, it is not difficult to see that the theorem is Theorem A for $\alpha=1$. Letting $\beta\to 0$ in Theorem 1, we have the following result.

COROLLARY 1. *Let $f:[a,b]\to\mathbb{R}$ be an h-covex function with $h(1/2)>0$ and $f\in L([a,b])$. Then*

$$\frac{1}{2h(1/2)}f\left(\frac{a+b}{2}\right)\leqslant\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^-}f(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)^+}f(b)\right]$$
$$\leqslant\frac{f(a)+f(b)}{2}\alpha\int_0^1 t^{\alpha-1}\left[h\left(\frac{t}{2}\right)+h\left(1-\frac{t}{2}\right)\right]dt.$$

Especially, if $h(t)=t$, Corollary 1 becomes Theorem B.

3. Midpoint type inequalities for n times differentiable functions

In this section, we will extend the midpoint type inequalities in Theorem D and Theorem E to the case of strongly h-convex functions with n order derivatives. For the sake of convenience, if $f:[a,b]\to\mathbb{R}$ is an n-times differentiable function, we denote

$$\mathfrak{L}\left(f,\frac{a+b}{2}\right)=\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^-}f(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)^+}f(b)\right]-f\left(\frac{a+b}{2}\right)$$
$$-\sum_{j=1}^{n-1}\frac{\left[1+(-1)^j\right](b-a)^j}{2^{j+1}\prod_{k=1}^{j}(\alpha+k)}f^{(j)}\left(\frac{a+b}{2}\right). \tag{3.1}$$

It is easy to see that if $n=1$ or 2, $\mathfrak{L}\left(f,\frac{a+b}{2}\right)$ have the same concise form:

$$\mathfrak{L}\left(f,\frac{a+b}{2}\right)=\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^-}f(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)^+}f(b)\right]-f\left(\frac{a+b}{2}\right). \tag{3.2}$$

Now we introduce the following key lemma.

LEMMA 1. *Let $f\in C^n([a,b])$ and $f^{(n)}\in L([a,b])$, $n\in\mathbb{Z}^+$. Then*

$$\mathfrak{L}\left(f,\frac{a+b}{2}\right)=\frac{(b-a)^n}{2^{n+1}\prod_{k=1}^{n-1}(\alpha+k)}\left[(-1)^n\int_0^1(1-t)^{\alpha+n-1}f^{(n)}\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)dt\right.$$
$$\left.+\int_0^1(1-t)^{\alpha+n-1}f^{(n)}\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)dt\right],$$

here we denote $\prod_{k=1}^{0}(\alpha+k)\equiv 1$.

It is not difficult to check that Lemma 1 reduces to Lemma B and Lemma C for $n=1$ and $n=2$, respectively.

Proof. Without loss of generality, we may assume $n\geqslant 2$. Integration by parts n times show that

$$\int_0^1(1-t)^{\alpha+n-1}f^{(n)}\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)dt$$
$$=\frac{2}{b-a}f^{(n-1)}\left(\frac{a+b}{2}\right)-\frac{2(\alpha+n-1)}{b-a}\int_0^1(1-t)^{\alpha+n-2}f^{(n-1)}\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)dt$$

$$
\begin{aligned}
&= \frac{2}{b-a} f^{(n-1)}\left(\frac{a+b}{2}\right) - \frac{2^2(\alpha+n-1)}{(b-a)^2} f^{(n-2)}\left(\frac{a+b}{2}\right) \\
&\quad + \frac{2^2(\alpha+n-1)(\alpha+n-2)}{(b-a)^2} \int_0^1 (1-t)^{\alpha+n-3} f^{(n-2)}\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right) dt \\
&= \cdots \\
&= \frac{2}{b-a} f^{(n-1)}\left(\frac{a+b}{2}\right) - \frac{2^2(\alpha+n-1)}{(b-a)^2} f^{(n-2)}\left(\frac{a+b}{2}\right) \\
&\quad + \frac{2^3(\alpha+n-1)(\alpha+n-2)}{(b-a)^3} f^{(n-3)}\left(\frac{a+b}{2}\right) \\
&\quad + \cdots + \frac{(-1)^{n-1}2^n(\alpha+n-1)(\alpha+n-2)\cdots(\alpha+1)}{(b-a)^n} f\left(\frac{a+b}{2}\right) \\
&\quad + \frac{(-1)^n 2^n(\alpha+n-1)(\alpha+n-2)\cdots(\alpha+1)\alpha}{(b-a)^{\alpha+n}} \int_0^1 (1-t)^{\alpha-1} f\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right) dt \\
&= \sum_{j=0}^{n-1} \frac{(-1)^{n-j-1}2^{n-j}\prod_{k=j+1}^{n-1}(\alpha+k)}{(b-a)^{n-j}} f^{(j)}\left(\frac{a+b}{2}\right) \\
&\quad + \frac{(-1)^n 2^{\alpha+n}\prod_{k=0}^{n-1}(\alpha+k)\Gamma(\alpha)}{(b-a)^{\alpha+n}} J^{\alpha}_{\left(\frac{a+b}{2}\right)^-} f(a).
\end{aligned}
$$

Multiplying both sides of the proceeding equality by $\frac{(-1)^n(b-a)^n}{2^{n+1}\prod_{k=1}^{n-1}(\alpha+k)}$, we obtain

$$
\begin{aligned}
&\frac{(-1)^n(b-a)^n}{2^{n+1}\prod_{k=1}^{n-1}(\alpha+k)} \int_0^1 (1-t)^{\alpha+n-1} f^{(n)}\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right) dt \\
&= \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}} J^{\alpha}_{\left(\frac{a+b}{2}\right)^-} f(a) - \frac{1}{2} f\left(\frac{a+b}{2}\right) - \sum_{j=1}^{n-1} \frac{(-1)^j(b-a)^j}{2^{j+1}\prod_{k=1}^{j}(\alpha+k)} f^{(j)}\left(\frac{a+b}{2}\right).
\end{aligned}
\tag{3.3}
$$

Similarly, using integration by parts n times again,

$$
\begin{aligned}
&\int_0^1 (1-t)^{\alpha+n-1} f^{(n)}\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right) dt \\
&= -\sum_{j=0}^{n-1} \frac{2^{n-j}\prod_{k=j+1}^{n-1}(\alpha+k)}{(b-a)^{n-j}} f^{(j)}\left(\frac{a+b}{2}\right) + \frac{2^{\alpha+n}\prod_{k=0}^{n-1}(\alpha+k)\Gamma(\alpha)}{(b-a)^{\alpha+n}} J^{\alpha}_{\left(\frac{a+b}{2}\right)^+} f(b),
\end{aligned}
$$

which means that

$$
\begin{aligned}
&\frac{(b-a)^n}{2^{n+1}\prod_{k=1}^{n-1}(\alpha+k)} \int_0^1 (1-t)^{\alpha+n-1} f^{(n)}\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right) dt \\
&= \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}} J^{\alpha}_{\left(\frac{a+b}{2}\right)^+} f(b) - \frac{1}{2} f\left(\frac{a+b}{2}\right) - \sum_{j=1}^{n-1} \frac{(b-a)^j}{2^{j+1}\prod_{k=1}^{j}(\alpha+k)} f^{(j)}\left(\frac{a+b}{2}\right).
\end{aligned}
\tag{3.4}
$$

Therefore we complete the proof of the lemma by (3.3) and (3.4). □

Using Lemma 1, we obtain the following fractional integral inequalities. For simplicity, we first denote

$$\mathscr{A}=\int_0^{1/2} t^{\alpha+n-1}h(t)dt,\ \ \mathscr{B}=\int_{1/2}^{1}(1-t)^{\alpha+n-1}h(t)dt.$$

THEOREM 2. *Let $f\in C^n([a,b])$ and $f^{(n)}\in L([a,b]),n\in\mathbb{Z}^+$. Suppose that $\left|f^{(n)}\right|^q \in SX(h,\beta,[a,b]),1\leqslant q<\infty$.*

(i) *If $q=1$, then*

$$\begin{aligned}&\left|\mathfrak{L}\left(f,\frac{a+b}{2}\right)\right|\\ &\leqslant \frac{2^{\alpha-1}(b-a)^n}{\prod_{k=1}^{n-1}(\alpha+k)}\left\{(\mathscr{A}+\mathscr{B})\left(\left|f^{(n)}(a)\right|+\left|f^{(n)}(b)\right|\right)-\frac{\beta(\alpha+n+3)(b-a)^2}{2^{n+\alpha+1}(\alpha+n+1)(\alpha+n+2)}\right\}.\end{aligned}$$

(ii) *If $1<q<\infty$, then*

$$\begin{aligned}&\left|\mathfrak{L}\left(f,\frac{a+b}{2}\right)\right|\\ &\leqslant \frac{2^{(\alpha+n)/q-n-1}(\alpha+n)^{1/q}(b-a)^n}{\prod_{k=1}^{n}(\alpha+k)}\\ &\quad\times\left\{\left[\mathscr{B}\left|f^{(n)}(a)\right|^q+\mathscr{A}\left|f^{(n)}(b)\right|^q-\frac{\beta(\alpha+n+3)(b-a)^2}{2^{\alpha+n+2}(\alpha+n+1)(\alpha+n+2)}\right]^{1/q}\right.\\ &\quad\left.+\left[\mathscr{B}\left|f^{(n)}(b)\right|^q+\mathscr{A}\left|f^{(n)}(a)\right|^q-\frac{\beta(\alpha+n+3)(b-a)^2}{2^{\alpha+n+2}(\alpha+n+1)(\alpha+n+2)}\right]^{1/q}\right\}\\ &\leqslant \frac{2^{(\alpha+n)/q-n-1}(\alpha+n)^{1/q}(b-a)^n}{\prod_{k=1}^{n}(\alpha+k)}\\ &\quad\times\left\{\left(\mathscr{A}^{1/q}+\mathscr{B}^{1/q}\right)\left(\left|f^{(n)}(a)\right|+\left|f^{(n)}(b)\right|\right)+2\left[\frac{\beta(\alpha+n+3)(b-a)^2}{2^{\alpha+n+2}(\alpha+n+1)(\alpha+n+2)}\right]^{1/q}\right\}.\end{aligned}$$

(iii) *If $1<q<\infty$, then*

$$\begin{aligned}&\left|\mathfrak{L}\left(f,\frac{a+b}{2}\right)\right|\\ &\leqslant \frac{(b-a)^n}{2^{n+1/p}\prod_{k=1}^{n-1}(\alpha+k)}\left(\frac{1}{p(\alpha+n-1)+1}\right)^{1/p}\\ &\quad\times\left\{\left(\int_{1/2}^{1}h(t)dt\left|f^{(n)}(a)\right|^q+\int_0^{1/2}h(t)dt\left|f^{(n)}(b)\right|^q-\frac{\beta(b-a)^2}{12}\right)^{1/q}\right.\\ &\quad\left.+\left(\int_{1/2}^{1}h(t)dt\left|f^{(n)}(b)\right|^q+\int_0^{1/2}h(t)dt\left|f^{(n)}(a)\right|^q-\frac{\beta(b-a)^2}{12}\right)^{1/q}\right\}\end{aligned}$$

$$\leqslant \frac{(b-a)^n}{2^{n+1/p}\prod_{k=1}^{n-1}(\alpha+k)}\left(\frac{1}{p(\alpha+n-1)+1}\right)^{1/p}$$
$$\times\left\{\left[\left(\int_0^{1/2}h(t)dt\right)^{1/q}+\left(\int_{1/2}^1 h(t)dt\right)^{1/q}\right]\left(\left|f^{(n)}(a)\right|+\left|f^{(n)}(b)\right|\right)\right.$$
$$\left.+2\left[\frac{\beta(b-a)^2}{12}\right]^{1/q}\right\},$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. (1) If $q=1$, then it follows from the fact of $|f|\in SX(h,\beta,[a,b])$ that

$$\left|\int_0^1(1-t)^{\alpha+n-1}f^{(n)}\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)dt\right|$$
$$\leqslant \int_0^1(1-t)^{\alpha+n-1}\left[h\left(\frac{1+t}{2}\right)\left|f^{(n)}(a)\right|+h\left(\frac{1-t}{2}\right)\left|f^{(n)}(b)\right|\right.$$
$$\left.-\beta\frac{(1-t)(1+t)}{4}(b-a)^2\right]dt$$
$$=2^{\alpha+n}\int_{1/2}^1(1-t)^{\alpha+n-1}h(t)dt\left|f^{(n)}(a)\right|+2^{\alpha+n}\int_0^{1/2}t^{\alpha+n-1}h(t)dt\left|f^{(n)}(b)\right|$$
$$-\frac{\beta(\alpha+n+3)(b-a)^2}{4(\alpha+n+1)(\alpha+n+2)}$$
$$=2^{\alpha+n}\mathscr{B}\left|f^{(n)}(a)\right|+2^{\alpha+n}\mathscr{A}\left|f^{(n)}(b)\right|-\frac{\beta(\alpha+n+3)(b-a)^2}{4(\alpha+n+1)(\alpha+n+2)}.$$

Similarly,

$$\left|\int_0^1(1-t)^{\alpha+n-1}f^{(n)}\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)dt\right|$$
$$\leqslant 2^{\alpha+n}\mathscr{A}\left|f^{(n)}(a)\right|+2^{\alpha+n}\mathscr{B}\left|f^{(n)}(b)\right|-\frac{\beta(\alpha+n+3)(b-a)^2}{4(\alpha+n+1)(\alpha+n+2)}.$$

Then we complete the proof of (i) by the proceeding two inequalities and Lemma 1.

(2) If $1<q<\infty$, then power-mean inequality and the fact of $|f|^q\in SX(h,\beta,[a,b])$ show that

$$\left|\int_0^1(1-t)^{\alpha+n-1}f^{(n)}\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)dt\right|$$
$$\leqslant\left(\int_0^1(1-t)^{\alpha+n-1}dt\right)^{1-1/q}\left(\int_0^1(1-t)^{\alpha+n-1}\left|f^{(n)}\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q dt\right)^{1/q}$$

$$
\begin{aligned}
&\leqslant \left(\frac{1}{\alpha+n}\right)^{1-1/q}\left[2^{\alpha+n}\int_{1/2}^{1}(1-t)^{\alpha+n-1}h(t)dt\left|f^{(n)}(a)\right|^{q}\right.\\
&\quad\left.+2^{\alpha+n}\int_{0}^{1/2}t^{\alpha+n-1}h(t)dt\left|f^{(n)}(b)\right|^{q}-\frac{\beta(\alpha+n+3)(b-a)^2}{4(\alpha+n+1)(\alpha+n+2)}\right]^{1/q}\\
&=\frac{2^{(\alpha+n)/q}}{(\alpha+n)^{1-1/q}}\left[\mathscr{B}\left|f^{(n)}(a)\right|^{q}+\mathscr{A}\left|f^{(n)}(b)\right|^{q}-\frac{\beta(\alpha+n+3)(b-a)^2}{2^{\alpha+n+2}(\alpha+n+1)(\alpha+n+2)}\right]^{1/q},
\end{aligned}
$$

and

$$
\begin{aligned}
&\left|\int_{0}^{1}(1-t)^{\alpha+n-1}f^{(n)}\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)dt\right|\\
&\leqslant\frac{2^{(\alpha+n)/q}}{(\alpha+n)^{1-1/q}}\left[\mathscr{B}\left|f^{(n)}(b)\right|^{q}+\mathscr{A}\left|f^{(n)}(a)\right|^{q}-\frac{\beta(\alpha+n+3)(b-a)^2}{2^{\alpha+n+2}(\alpha+n+1)(\alpha+n+2)}\right]^{1/q},
\end{aligned}
$$

which finish the proof of the first inequality in (ii) by Lemma 1 again.

For the proof of the second inequality, let

$$
\begin{gathered}
a_1=\textstyle\int_{1/2}^{1}(1-t)^{\alpha+n-1}h(t)dt\left|f^{(n)}(a)\right|^{q},\ b_1=2\int_{0}^{1/2}t^{\alpha+n-1}h(t)dt\left|f^{(n)}(b)\right|^{q},\\
a_2=\textstyle\int_{1/2}^{1}(1-t)^{\alpha+n-1}h(t)dt\left|f^{(n)}(b)\right|^{q},\ b_2=\int_{0}^{1/2}t^{\alpha+n-1}h(t)dt\left|f^{(n)}(a)\right|^{q},\\
c_1=c_2=-\tfrac{\beta(\alpha+n+3)}{2^{\alpha+n+2}(\alpha+n+1)(\alpha+n+2)}(b-a)^2.
\end{gathered}
$$

According to the fact that

$$
\sum_{k=1}^{m}(|a_k|+|b_k|+|c_k|)^s\leqslant\sum_{k=1}^{m}|a_k|^s+\sum_{k=1}^{m}|b_k|^s+\sum_{k=1}^{m}|c_k|^s,\ \ 0\leqslant s<1,
$$

then the desired result can be obtained easily.

(3) If $1<q<\infty$, then the Hölder inequality and the fact of $|f|^q\in SX(h,\beta,[a,b])$ tell us that

$$
\begin{aligned}
&\left|\int_{0}^{1}(1-t)^{\alpha+n-1}f^{(n)}\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)dt\right|\\
&\leqslant\left(\int_{0}^{1}(1-t)^{p(\alpha+n-1)}dt\right)^{1/p}\left(\int_{0}^{1}\left|f^{(n)}\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^{q}dt\right)^{q}\\
&\leqslant\left(\frac{1}{p(\alpha+n-1)+1}\right)^{1/p}\left\{\int_{0}^{1}\left[h\left(\frac{1+t}{2}\right)\left|f^{(n)}(a)\right|^{q}+h\left(\frac{1-t}{2}\right)\left|f^{(n)}(b)\right|^{q}\right.\right.\\
&\quad\left.\left.-\beta\frac{(1-t)(1+t)}{4}(b-a)^2\right]dt\right\}^{1/q}
\end{aligned}
$$

$$
\begin{aligned}
&= \left(\frac{1}{p(\alpha+n-1)+1}\right)^{1/p} \left(2\int_{1/2}^{1} h(t)dt \left|f^{(n)}(a)\right|^q \right.\\
&\left. +2\int_{0}^{1/2} h(t)dt \left|f^{(n)}(b)\right|^q - \frac{\beta(b-a)^2}{6}\right)^{1/q}.
\end{aligned}
$$

By the same way, we have

$$
\begin{aligned}
&\left|\int_0^1 (1-t)^{\alpha+n-1} f^{(n)}\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)dt\right|\\
&\leqslant \left(\frac{1}{p(\alpha+n-1)+1}\right)^{1/p} \left(2\int_{1/2}^{1} h(t)dt \left|f^{(n)}(b)\right|^q \right.\\
&\left. +2\int_{0}^{1/2} h(t)dt \left|f^{(n)}(a)\right|^q - \frac{\beta(b-a)^2}{6}\right)^{1/q}.
\end{aligned}
$$

Then we complete the proof of the first inequality in (ii) by the above two inequalities and Lemma 1.

The second inequality is proved by a similar way as (2), we leave the details to readers. □

Letting $\beta \to 0$. We have the following results.

COROLLARY 2. *Let $f \in C^n([a,b])$ and $f^{(n)} \in L([a,b])$, $n \in \mathbb{Z}^+$. Suppose that $\left|f^{(n)}\right|^q$ is an h-convex function with $1 \leqslant q < \infty$.*

(i) *If $1 \leqslant q < \infty$, then*

$$
\begin{aligned}
&\left|\mathfrak{L}\left(f,\frac{a+b}{2}\right)\right|\\
&\leqslant \frac{2^{(\alpha+n)/q-n-1}(\alpha+n)^{1/q}(b-a)^n}{\prod_{k=1}^{n}(\alpha+k)}\\
&\quad\times\left\{\left[\mathscr{B}\left|f^{(n)}(a)\right|^q+\mathscr{A}\left|f^{(n)}(b)\right|^q\right]^{1/q}+\left[\mathscr{B}\left|f^{(n)}(b)\right|^q+\mathscr{A}\left|f^{(n)}(a)\right|^q\right]^{1/q}\right\}\\
&\leqslant \frac{2^{(\alpha+n)/q-n-1}(\alpha+n)^{1/q}(b-a)^n}{\prod_{k=1}^{n}(\alpha+k)}\left(\mathscr{A}^{1/q}+\mathscr{B}^{1/q}\right)\left(\left|f^{(n)}(a)\right|+\left|f^{(n)}(b)\right|\right).
\end{aligned}
$$

(ii) *If $1 < q < \infty$, then*

$$
\begin{aligned}
\left|\mathfrak{L}\left(f,\frac{a+b}{2}\right)\right| &\leqslant \frac{(b-a)^n}{2^{n+1/p}\prod_{k=1}^{n-1}(\alpha+k)}\left(\frac{1}{p(\alpha+n-1)+1}\right)^{1/p}\\
&\quad\times\left\{\left(\int_{1/2}^{1} h(t)dt\left|f^{(n)}(a)\right|^q+\int_0^{1/2} h(t)dt\left|f^{(n)}(b)\right|^q\right)^{1/q}\right.\\
&\quad\left.+\left(\int_{1/2}^{1} h(t)dt\left|f^{(n)}(b)\right|^q+\int_0^{1/2} h(t)dt\left|f^{(n)}(a)\right|^q\right)^{1/q}\right\}
\end{aligned}
$$

$$\leqslant \frac{(b-a)^n}{2^{n+1/p}\prod_{k=1}^{n-1}(\alpha+k)}\left(\frac{1}{p(\alpha+n-1)+1}\right)^{1/p}$$
$$\times\left[\left(\int_0^{1/2}h(t)dt\right)^{1/q}+\left(\int_{1/2}^{1}h(t)dt\right)^{1/q}\right]\left(\left|f^{(n)}(a)\right|+\left|f^{(n)}(b)\right|\right),$$

where $\frac{1}{p}+\frac{1}{q}=1$.

REMARK. By changing of variable, it is not difficult to check that Corollary 2 extends Theorem D and Theorem E for $n=1$, $h(t)=t$ and $n=2$, $h(t)=t^s$, respectively.

4. Trapezoid type inequalities for n times differentiable functions

In this section, we will prove some similar results as Theorem F for strongly h-convex functions with n order derivatives. For simplicity, if $f\in C^n([a,b])$, we denote

$$\mathfrak{R}\left(f,\frac{a+b}{2}\right)=\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{\left(\frac{a+b}{2}\right)^-}f(a)+J^\alpha_{\left(\frac{a+b}{2}\right)^+}f(b)\right]-\frac{f(a)+f(b)}{2}$$
$$+\sum_{j=1}^{n-1}\frac{\left[1+(-1)^j\right](b-a)^j}{2^{j+1}}\frac{\prod_{k=1}^{j}(\alpha+k)-j!}{j!\prod_{k=1}^{j}(\alpha+k)}f^{(j)}\left(\frac{a+b}{2}\right). \quad (4.1)$$

It is easy to check that if $n=1$ or 2, $\mathfrak{R}\left(f,\frac{a+b}{2}\right)$ has the simplified form:

$$\mathfrak{R}\left(f,\frac{a+b}{2}\right)=\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{\left(\frac{a+b}{2}\right)^-}f(a)+J^\alpha_{\left(\frac{a+b}{2}\right)^+}f(b)\right]-\frac{f(a)+f(b)}{2}. \quad (4.2)$$

Now we introduce the following key lemma.

LEMMA 2. *Let* $f\in C^n([a,b])$ *and* $f^{(n)}\in L([a,b])$, $n\in\mathbb{Z}^+$. *Then*

$$\mathfrak{R}\left(f,\frac{a+b}{2}\right)=-\frac{(b-a)^n}{2^{n+1}\prod_{k=1}^{n-1}(\alpha+k)}\int_0^1\left[\frac{\prod_{k=1}^{n-1}(\alpha+k)}{(n-1)!}(1-t)^{n-1}-(1-t)^{\alpha+n-1}\right]$$
$$\times\left[(-1)^n f^{(n)}\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)+f^{(n)}\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right]dt.$$

It is not difficult to check that Lemma 2 reduces to Lemma E by (4.2) for $n=1$.

Proof. Without loss of generality, we assume that $n\geqslant 2$. Integrating by parts n times show that

$$\int_0^1\left[\frac{\prod_{k=1}^{n-1}(\alpha+k)}{(n-1)!}(1-t)^{n-1}-(1-t)^{\alpha+n-1}\right]f^{(n)}\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)dt$$
$$=\frac{2}{b-a}\left(\frac{\prod_{k=1}^{n-1}(\alpha+k)}{(n-1)!}-1\right)f^{(n-1)}\left(\frac{a+b}{2}\right)$$
$$-\frac{2(\alpha+n-1)}{b-a}\int_0^1\left[\frac{\prod_{k=1}^{n-2}(\alpha+k)}{(n-2)!}(1-t)^{n-2}-(1-t)^{\alpha+n-2}\right]f^{(n-1)}\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)dt$$

$$
\begin{aligned}
&= \frac{2}{b-a}\left(\frac{\prod_{k=1}^{n-1}(\alpha+k)}{(n-1)!}-1\right) f^{(n-1)}\left(\frac{a+b}{2}\right) \\
&\quad -\frac{2^2(\alpha+n-1)}{(b-a)^2}\left(\frac{\prod_{k=1}^{n-2}(\alpha+k)}{(n-2)!}-1\right) f^{(n-2)}\left(\frac{a+b}{2}\right) \\
&\quad +\frac{2^2(\alpha+n-1)(\alpha+n-2)}{(b-a)^2} \\
&\quad \times \int_0^1\left[\frac{\prod_{k=1}^{n-3}(\alpha+k)}{(n-3)!}(1-t)^{n-3}-(1-t)^{\alpha+n-3}\right] f^{(n-2)}\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right) dt \\
&= \cdots \\
&= \sum_{j=1}^{n-1} \frac{(-1)^{n-j+1}2^{n-j}\prod_{i=j+1}^{n-1}(\alpha+i)}{(b-a)^{n-j}}\left(\frac{\prod_{k=1}^{j}(\alpha+k)}{j!}-1\right) f^{(j)}\left(\frac{a+b}{2}\right) \\
&\quad +\frac{(-1)^n 2^{n-1}\prod_{k=1}^{n-1}(\alpha+k)}{(b-a)^{n-1}}\int_0^1[1-(1-t)^{\alpha}]f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right) dt \\
&= \sum_{j=1}^{n-1} \frac{(-1)^{n-j+1}2^{n-j}\prod_{i=j+1}^{n-1}(\alpha+i)}{(b-a)^{n-j}}\left(\frac{\prod_{k=1}^{j}(\alpha+k)}{j!}-1\right) f^{(j)}\left(\frac{a+b}{2}\right) \\
&\quad +\frac{(-1)^n 2^{n}\prod_{k=1}^{n-1}(\alpha+k)}{(b-a)^{n}} f(a)+\frac{(-1)^{n+1} 2^{n}\prod_{k=0}^{n-1}(\alpha+k)}{(b-a)^{n}} \\
&\quad \times \int_0^1 (1-t)^{\alpha-1} f\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right) dt \\
&= \sum_{j=1}^{n-1} \frac{(-1)^{n-j+1}2^{n-j}\prod_{i=j+1}^{n-1}(\alpha+i)}{(b-a)^{n-j}}\left(\frac{\prod_{k=1}^{j}(\alpha+k)}{j!}-1\right) f^{(j)}\left(\frac{a+b}{2}\right) \\
&\quad +\frac{(-1)^n 2^{n}\prod_{k=1}^{n-1}(\alpha+k)}{(b-a)^{n}} f(a)+\frac{(-1)^{n+1} 2^{n+\alpha}\Gamma(\alpha)\prod_{k=0}^{n-1}(\alpha+k)}{(b-a)^{n+\alpha}} J^{\alpha}_{\left(\frac{a+b}{2}\right)^-} f(a),
\end{aligned}
$$

which means that

$$
\begin{aligned}
&\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}} J^{\alpha}_{\left(\frac{a+b}{2}\right)^-} f(a)-\frac{f(a)}{2} \\
&\quad +\sum_{j=1}^{n-1}\frac{(-1)^j(b-a)^j}{2^{j+1}}\frac{\prod_{k=1}^{j}(\alpha+k)-j!}{j!\prod_{k=1}^{j}(\alpha+k)} f^{(j)}\left(\frac{a+b}{2}\right) \\
&= \frac{(-1)^{n+1}(b-a)^n}{2^{n+1}\prod_{k=1}^{n-1}(\alpha+k)} \\
&\quad \times \int_0^1\left[\frac{\prod_{k=1}^{n-1}(\alpha+k)}{(n-1)!}(1-t)^{n-1}-(1-t)^{\alpha+n-1}\right] f^{(n)}\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right) dt.
\end{aligned} \tag{4.3}
$$

Similarly, integrating by parts n times again tell us that

$$
\begin{aligned}
&\int_0^1 \left[\frac{\prod_{k=1}^{n-1}(\alpha+k)}{(n-1)!}(1-t)^{n-1}-(1-t)^{\alpha+n-1}\right] f^{(n)}\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)dt\\
&=-\frac{2}{b-a}\left(\frac{\prod_{k=1}^{n-1}(\alpha+k)}{(n-1)!}-1\right)f^{(n-1)}\left(\frac{a+b}{2}\right)+\frac{2(\alpha+n-1)}{b-a}\\
&\quad\times\int_0^1 \left[\frac{\prod_{k=1}^{n-2}(\alpha+k)}{(n-2)!}(1-t)^{n-2}-(1-t)^{\alpha+n-2}\right] f^{(n-1)}\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)dt\\
&=\cdots\\
&=-\sum_{j=1}^{n-1}\frac{2^{n-j}\prod_{i=j+1}^{n-1}(\alpha+i)}{(b-a)^{n-j}}\left(\frac{\prod_{k=1}^{j}(\alpha+k)}{j!}-1\right)f^{(j)}\left(\frac{a+b}{2}\right)\\
&\quad+\frac{2^{n-1}\prod_{k=1}^{n-1}(\alpha+k)}{(b-a)^{n-1}}\int_0^1[1-(1-t)^{\alpha}]f'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)dt\\
&=-\sum_{j=1}^{n-1}\frac{2^{n-j}\prod_{i=j+1}^{n-1}(\alpha+i)}{(b-a)^{n-j}}\left(\frac{\prod_{k=1}^{j}(\alpha+k)}{j!}-1\right)f^{(j)}\left(\frac{a+b}{2}\right)\\
&\quad+\frac{2^{n}\prod_{k=1}^{n-1}(\alpha+k)}{(b-a)^{n}}f(b)-\frac{2^{n+\alpha}\Gamma(\alpha)\prod_{k=0}^{n-1}(\alpha+k)}{(b-a)^{n+\alpha}}J^{\alpha}_{\left(\frac{a+b}{2}\right)^+}f(b),
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}J^{\alpha}_{\left(\frac{a+b}{2}\right)^-}f(a)-\frac{f(b)}{2}\\
&\quad+\sum_{j=1}^{n-1}\frac{(b-a)^j}{2^{j+1}}\frac{\prod_{k=1}^{j}(\alpha+k)-j!}{j!\prod_{k=1}^{j}(\alpha+k)}f^{(j)}\left(\frac{a+b}{2}\right)\\
&=-\frac{(b-a)^n}{2^{n+1}\prod_{k=1}^{n-1}(\alpha+k)}\\
&\quad\times\int_0^1 \left[\frac{\prod_{k=1}^{n-1}(\alpha+k)}{(n-1)!}(1-t)^{n-1}-(1-t)^{\alpha+n-1}\right] f^{(n)}\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)dt.
\end{aligned}
\tag{4.4}
$$

Then we complete the proof by (4.3) and (4.4). □

Using Lemma 2, we obtain the following error estimates. For convenience, we first set

$$
\begin{aligned}
\mathscr{C}&=\int_0^{1/2}\left(\frac{\prod_{k=1}^{n-1}(\alpha+k)}{(n-1)!}t^{n-1}-2^{\alpha}t^{\alpha+n-1}\right)h(t)\,dt,\\
\mathscr{D}&=\int_{1/2}^{1}\left(\frac{\prod_{k=1}^{n-1}(\alpha+k)}{(n-1)!}(1-t)^{n-1}-2^{\alpha}(1-t)^{\alpha+n-1}\right)h(t)\,dt.
\end{aligned}
$$

THEOREM 3. *Let* $f \in C^n([a,b])$ *and* $f^{(n)} \in L([a,b]), n \in \mathbb{Z}^+$. *Suppose that* $\left|f^{(n)}\right|^q \in SX(h,\beta,[a,b]), 1 \leqslant q < \infty$.

(i) *If* $q = 1$, *then*

$$\left|\mathfrak{R}\left(f,\frac{a+b}{2}\right)\right| \leqslant \frac{(b-a)^n}{2\prod_{k=1}^{n-1}(\alpha+k)}\left[(\mathscr{C}+\mathscr{D})\left(\left|f^{(n)}(a)\right|+\left|f^{(n)}(b)\right|\right)\right.$$
$$\left.-\beta\frac{n(n+3)\prod_{k=1}^{n+2}(\alpha+k)-(n+2)!(\alpha+n+3)}{2^{n+1}(n+2)!(\alpha+n+1)(\alpha+n+2)}(b-a)^2\right].$$

(ii) *If* $1 < q < \infty$, *then*

$$\begin{aligned}
&\left|\mathfrak{R}\left(f,\frac{a+b}{2}\right)\right| \\
&\leqslant \frac{(b-a)^n}{2^{n(1-1/q)+1}\prod_{k=1}^{n-1}(\alpha+k)}\left(\frac{\prod_{k=1}^{n-1}(\alpha+k)}{n!}-\frac{1}{\alpha+n}\right)^{1-1/q} \\
&\quad\times\left\{\left[\mathscr{C}\left|f^{(n)}(a)\right|^q+\mathscr{D}\left|f^{(n)}(b)\right|^q\right.\right. \\
&\quad\left.-\beta\frac{n(n+3)\prod_{k=1}^{n+2}(\alpha+k)-(n+2)!(\alpha+n+3)}{2^{n+2}(n+2)!(\alpha+n+1)(\alpha+n+2)}(b-a)^2\right]^{1/q} \\
&\quad+\left[\mathscr{C}\left|f^{(n)}(b)\right|^q+\mathscr{D}\left|f^{(n)}(a)\right|^q\right. \\
&\quad\left.\left.-\beta\frac{n(n+3)\prod_{k=1}^{n+2}(\alpha+k)-(n+2)!(\alpha+n+3)}{2^{n+2}(n+2)!(\alpha+n+1)(\alpha+n+2)}(b-a)^2\right]^{1/q}\right\} \\
&\leqslant \frac{(b-a)^n}{2^{n(1-1/q)+1}\prod_{k=1}^{n-1}(\alpha+k)}\left(\frac{\prod_{k=1}^{n-1}(\alpha+k)}{n!}-\frac{1}{\alpha+n}\right)^{1-1/q} \\
&\quad\times\left\{\left(\mathscr{C}^{1/q}+\mathscr{D}^{1/q}\right)\left(\left|f^{(n)}(a)\right|+\left|f^{(n)}(b)\right|\right)\right. \\
&\quad\left.+2\left[\beta\frac{n(n+3)\prod_{k=1}^{n+2}(\alpha+k)-(n+2)!(\alpha+n+3)}{2^{n+2}(n+2)!(\alpha+n+1)(\alpha+n+2)}(b-a)^2\right]^{1/q}\right\}.
\end{aligned}$$

(iii) *If* $1 < q < \infty$, *then*

$$\begin{aligned}
&\left|\mathfrak{R}\left(f,\frac{a+b}{2}\right)\right| \\
&\leqslant \frac{(b-a)^n}{2^{n+1-1/q}\prod_{k=1}^{n-1}(\alpha+k)}\left(\int_0^1\left[\frac{\prod_{k=1}^{n-1}(\alpha+k)}{(n-1)!}t^{n-1}-t^{\alpha+n-1}\right]^p dt\right)^{1/p}
\end{aligned}$$

$$\times\left\{\left[\int_{1/2}^{1} h(t)dt\left|f^{(n)}(b)\right|^q+\int_0^{1/2} h(t)dt\left|f^{(n)}(a)\right|^q-\frac{\beta(b-a)^2}{12}\right]^{1/q}\right.$$

$$\left.+\left[\int_{1/2}^{1} h(t)dt\left|f^{(n)}(a)\right|^q+\int_0^{1/2} h(t)dt\left|f^{(n)}(b)\right|^q-\frac{\beta(b-a)^2}{12}\right]^{1/q}\right\}$$

$$\leqslant \frac{(b-a)^n}{2^{n+1-1/q}\prod_{k=1}^{n-1}(\alpha+k)}\left(\int_0^1\left[\frac{\prod_{k=1}^{n-1}(\alpha+k)}{(n-1)!}t^{n-1}-t^{\alpha+n-1}\right]^p dt\right)^{1/p}$$

$$\times\left\{\left[\left(\int_0^{1/2} h(t)dt\right)^{1/q}+\left(\int_{1/2}^{1} h(t)dt\right)^{1/q}\right]\left(\left|f^{(n)}(a)\right|+\left|f^{(n)}(b)\right|\right)\right.$$

$$\left.+2\left[\frac{\beta(b-a)^2}{12}\right]^{1/q}\right\},$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. (1) If $q=1$, then it follows from the fact of $|f|\in SX(h,\beta,[a,b])$ that

$$\left|\int_0^1\left[\frac{\prod_{k=1}^{n-1}(\alpha+k)}{(n-1)!}(1-t)^{n-1}-(1-t)^{\alpha+n-1}\right]f^{(n)}\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)dt\right|$$

$$\leqslant \int_0^1\left[\frac{\prod_{k=1}^{n-1}(\alpha+k)}{(n-1)!}(1-t)^{n-1}-(1-t)^{\alpha+n-1}\right]h\left(\frac{1+t}{2}\right)dt\left|f^{(n)}(a)\right|$$

$$+\int_0^1\left[\frac{\prod_{k=1}^{n-1}(\alpha+k)}{(n-1)!}(1-t)^{n-1}-(1-t)^{\alpha+n-1}\right]h\left(\frac{1-t}{2}\right)dt\left|f^{(n)}(b)\right|$$

$$-\int_0^1\left[\frac{\prod_{k=1}^{n-1}(\alpha+k)}{(n-1)!}(1-t)^{n-1}-(1-t)^{\alpha+n-1}\right]\beta\frac{(1-t)(1+t)}{4}(b-a)^2dt$$

$$=2^n\int_{1/2}^1\left[\frac{\prod_{k=1}^{n-1}(\alpha+k)}{(n-1)!}(1-t)^{n-1}-2^{\alpha}(1-t)^{\alpha+n-1}\right]h(t)\,dt\left|f^{(n)}(a)\right|$$

$$+2^n\int_0^{1/2}\left[\frac{\prod_{k=1}^{n-1}(\alpha+k)}{(n-1)!}t^{n-1}-2^{\alpha}t^{\alpha+n-1}\right]h(t)\,dt\left|f^{(n)}(b)\right|$$

$$-\frac{n(n+3)\prod_{k=1}^{n+2}(\alpha+k)-(n+2)!(\alpha+n+3)}{4(n+2)!(\alpha+n+1)(\alpha+n+2)}\beta(b-a)^2$$

$$=2^n\left(\mathscr{C}\left|f^{(n)}(a)\right|+\mathscr{D}\left|f^{(n)}(b)\right|\right)$$

$$-\frac{n(n+3)\prod_{k=1}^{n+2}(\alpha+k)-(n+2)!(\alpha+n+3)}{4(n+2)!(\alpha+n+1)(\alpha+n+2)}\beta(b-a)^2.$$

Similarly,

$$\begin{aligned}
&\left|\int_0^1 \left[\frac{\prod_{k=1}^{n-1}(\alpha+k)}{(n-1)!}(1-t)^{n-1}-(1-t)^{\alpha+n-1}\right] f^{(n)}\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)dt\right| \\
&\leqslant 2^n\left(\mathscr{D}\left|f^{(n)}(a)\right|+\mathscr{C}\left|f^{(n)}(b)\right|\right) \\
&\quad -\frac{n(n+3)\prod_{k=1}^{n+2}(\alpha+k)-(n+2)!(\alpha+n+3)}{4(n+2)!(\alpha+n+1)(\alpha+n+2)}\beta(b-a)^2.
\end{aligned}$$

Then we finish the proof of (i) by the above inequalities and Lemma 2.

(2) If $1<q<\infty$, then power-mean inequality and the fact of $|f|^q \in SX(h,\beta,[a,b])$ show that

$$\begin{aligned}
&\left|\int_0^1 \left[\frac{\prod_{k=1}^{n-1}(\alpha+k)}{(n-1)!}(1-t)^{n-1}-(1-t)^{\alpha+n-1}\right] f^{(n)}\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)dt\right| \\
&\leqslant \left(\int_0^1 \left[\frac{\prod_{k=1}^{n-1}(\alpha+k)}{(n-1)!}(1-t)^{n-1}-(1-t)^{\alpha+n-1}\right]dt\right)^{1-1/q} \\
&\quad \times\left(\int_0^1 \left[\frac{\prod_{k=1}^{n-1}(\alpha+k)}{(n-1)!}(1-t)^{n-1}-(1-t)^{\alpha+n-1}\right]\left|f^{(n)}\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q dt\right)^{1/q} \\
&\leqslant 2^{n/q}\left(\frac{\prod_{k=1}^{n-1}(\alpha+k)}{n!}-\frac{1}{\alpha+n}\right)^{1-1/q}\left\{\mathscr{D}\left|f^{(n)}(a)\right|^q+\mathscr{C}\left|f^{(n)}(b)\right|^q\right. \\
&\quad \left.-\frac{n(n+3)\prod_{k=1}^{n+2}(\alpha+k)-(n+2)!(\alpha+n+3)}{2^{n+2}(n+2)!(\alpha+n+1)(\alpha+n+2)}\beta(b-a)^2\right\}^{1/q},
\end{aligned}$$

and

$$\begin{aligned}
&\left|\int_0^1 \left[\frac{\prod_{k=1}^{n-1}(\alpha+k)}{(n-1)!}(1-t)^{n-1}-(1-t)^{\alpha+n-1}\right] f^{(n)}\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)dt\right| \\
&\leqslant 2^{n/q}\left(\frac{\prod_{k=1}^{n-1}(\alpha+k)}{n!}-\frac{1}{\alpha+n}\right)^{1-1/q}\left\{\mathscr{D}\left|f^{(n)}(b)\right|^q+\mathscr{C}\left|f^{(n)}(a)\right|^q\right. \\
&\quad \left.-\frac{n(n+3)\prod_{k=1}^{n+2}(\alpha+k)-(n+2)!(\alpha+n+3)}{2^{n+2}(n+2)!(\alpha+n+1)(\alpha+n+2)}\beta(b-a)^2\right\}^{1/q},
\end{aligned}$$

which finish the proof of the first inequality in (ii) by Lemma 2 again.

The proof of the second inequality can be obtained by a similar method as in Theorem 2 (ii), we omit the details.

(3) If $1<q<\infty$, then Hölder's inequality and $|f|^q \in SX(h,\beta,[a,b])$ imply that

$$\begin{aligned}
&\left|\int_0^1 \left[\frac{\prod_{k=1}^{n-1}(\alpha+k)}{(n-1)!}(1-t)^{n-1}-(1-t)^{\alpha+n-1}\right] f^{(n)}\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)dt\right| \\
&\leqslant \left(\int_0^1 \left[\frac{\prod_{k=1}^{n-1}(\alpha+k)}{(n-1)!}(1-t)^{n-1}-(1-t)^{\alpha+n-1}\right]^p dt\right)^{1/p} \\
&\quad\times\left(\int_0^1 \left|f^{(n)}\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q dt\right)^{1/q} \\
&\leqslant 2^{1/q}\left(\int_0^1 \left[\frac{\prod_{k=1}^{n-1}(\alpha+k)}{(n-1)!}t^{n-1}-t^{\alpha+n-1}\right]^p dt\right)^{1/p} \\
&\quad\times\left[\int_{1/2}^1 h(t)dt\left|f^{(n)}(a)\right|^q+\int_0^{1/2} h(t)dt\left|f^{(n)}(b)\right|^q-\frac{\beta(b-a)^2}{12}\right]^{1/q},
\end{aligned}$$

and

$$\begin{aligned}
&\left|\int_0^1 \left[\frac{\prod_{k=1}^{n-1}(\alpha+k)}{(n-1)!}(1-t)^{n-1}-(1-t)^{\alpha+n-1}\right] f^{(n)}\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)dt\right| \\
&\leqslant 2^{1/q}\left(\int_0^1 \left[\frac{\prod_{k=1}^{n-1}(\alpha+k)}{(n-1)!}t^{n-1}-t^{\alpha+n-1}\right]^p dt\right)^{1/p} \\
&\quad\times\left[\int_{1/2}^1 h(t)dt\left|f^{(n)}(b)\right|^q+\int_0^{1/2} h(t)dt\left|f^{(n)}(a)\right|^q-\frac{\beta(b-a)^2}{12}\right]^{1/q}.
\end{aligned}$$

Therefore, we obtained the first inequality of (iii) by the above two inequalities and Lemma 2.

The second inequality is achieved by the same way in Theorem 2 (iii), we leave it to readers. □

Letting $\beta\to 0$, we take the following conclusion.

COROLLARY 3. *Let $f\in C^n([a,b])$ and $f^{(n)}\in L([a,b]), n\in\mathbb{Z}^+$. Suppose that $\left|f^{(n)}\right|^q)$ is an h-convex function with $1\leqslant q<\infty$.*

(i) *If $1\leqslant q<\infty$, then*

$$\begin{aligned}
&\left|\Re\left(f,\frac{a+b}{2}\right)\right| \\
&\leqslant \frac{(b-a)^n}{2^{n(1-1/q)+1}\prod_{k=1}^{n-1}(\alpha+k)}\left(\frac{\prod_{k=1}^{n-1}(\alpha+k)}{n!}-\frac{1}{\alpha+n}\right)^{1-1/q} \\
&\quad\times\left[\left(\mathscr{C}\left|f^{(n)}(a)\right|^q+\mathscr{D}\left|f^{(n)}(b)\right|^q\right)^{1/q}+\left(\mathscr{D}\left|f^{(n)}(a)\right|^q+\mathscr{C}\left|f^{(n)}(b)\right|^q\right)^{1/q}\right]
\end{aligned}$$

$$\leqslant \frac{(b-a)^n}{2^{n(1-1/q)+1}\prod_{k=1}^{n-1}(\alpha+k)}\left(\frac{\prod_{k=1}^{n-1}(\alpha+k)}{n!}-\frac{1}{\alpha+n}\right)^{1-1/q}$$
$$\times\left(\mathscr{C}^{1/q}+\mathscr{D}^{1/q}\right)\left(\left|f^{(n)}(a)\right|+\left|f^{(n)}(b)\right|\right).$$

(ii) *If* $1<q<\infty$, *then*

$$\left|\mathfrak{R}\left(f,\frac{a+b}{2}\right)\right|$$
$$\leqslant \frac{(b-a)^n}{2^{n+1-1/q}\prod_{k=1}^{n-1}(\alpha+k)}\left(\int_0^1\left[\frac{\prod_{k=1}^{n-1}(\alpha+k)}{(n-1)!}t^{n-1}-t^{\alpha+n-1}\right]^p dt\right)^{1/p}$$
$$\times\left\{\left[\int_{1/2}^1 h(t)dt\left|f^{(n)}(b)\right|^q+\int_0^{1/2}h(t)dt\left|f^{(n)}(a)\right|^q\right]^{1/q}\right.$$
$$\left.+\left[\int_{1/2}^1 h(t)dt\left|f^{(n)}(a)\right|^q+\int_0^{1/2}h(t)dt\left|f^{(n)}(b)\right|^q\right]^{1/q}\right\}$$
$$\leqslant \frac{(b-a)^n}{2^{n+1-1/q}\prod_{k=1}^{n-1}(\alpha+k)}\left(\int_0^1\left[\frac{\prod_{k=1}^{n-1}(\alpha+k)}{(n-1)!}t^{n-1}-t^{\alpha+n-1}\right]^p dt\right)^{1/p}$$
$$\times\left[\left(\int_0^{1/2}h(t)dt\right)^{1/q}+\left(\int_{1/2}^1 h(t)dt\right)^{1/q}\right]\left(\left|f^{(n)}(a)\right|+\left|f^{(n)}(b)\right|\right),$$

where $\frac{1}{p}+\frac{1}{q}=1$.

REMARK. If $n=1$ and $h(t)=t$, then Corollary 3 reduces to Theorem F.

As a special case of Corollary 3, we have the following results.

COROLLARY 4. *Let* $f\in C^2([a,b])$ *and* $f''\in L([a,b])$. *Suppose that* $|f''|^q$ *is a convex function with* $1\leqslant q<\infty$.

(i) *If* $1\leqslant q<\infty$, *then*

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{\left(\frac{a+b}{2}\right)^-}f(a)+J^\alpha_{\left(\frac{a+b}{2}\right)^+}f(b)\right]-\frac{f(a)+f(b)}{2}\right|$$
$$\leqslant \frac{(b-a)^2}{2^{3+1/q}(\alpha+1)}\left(\frac{\alpha+1}{2}-\frac{1}{\alpha+2}\right)^{1-1/q}$$
$$\times\left\{\left[\left(\frac{\alpha+1}{3}-\frac{1}{\alpha+3}\right)|f''(a)|^q+\left(\frac{2(\alpha+1)}{3}-\frac{2}{\alpha+2}+\frac{1}{\alpha+3}\right)|f''(b)|^q\right]^{1/q}\right.$$
$$\left.+\left[\left(\frac{\alpha+1}{3}-\frac{1}{\alpha+3}\right)|f''(b)|^q+\left(\frac{2(\alpha+1)}{3}-\frac{2}{\alpha+2}+\frac{1}{\alpha+3}\right)|f''(a)|^q\right]^{1/q}\right\}$$

$$\leqslant \frac{(b-a)^2}{2^{3+1/q}(\alpha+1)}\left(\frac{\alpha+1}{2}-\frac{1}{\alpha+2}\right)^{1-1/q}$$
$$\times\left[\left(\frac{\alpha+1}{3}-\frac{1}{\alpha+3}\right)^{1/q}+\left(\frac{2(\alpha+1)}{3}-\frac{2}{\alpha+2}+\frac{1}{\alpha+3}\right)^{1/q}\right]\left(\left|f''(a)\right|+\left|f''(b)\right|\right).$$

(ii) *If* $1<q<\infty$, *then*

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{\left(\frac{a+b}{2}\right)^-}f(a)+J^\alpha_{\left(\frac{a+b}{2}\right)^+}f(b)\right]-\frac{f(a)+f(b)}{2}\right|$$
$$\leqslant \frac{(b-a)^2}{2^{3+2/q}(\alpha+1)}\left(\int_0^1\left[(\alpha+1)t-t^{\alpha+1}\right]^p dt\right)^{1/p}$$
$$\times\left[\left(3\left|f''(b)\right|^q+\left|f''(a)\right|^q\right)^{1/q}+\left(3\left|f''(a)\right|^q+\left|f''(b)\right|^q\right)^{1/q}\right]$$
$$\leqslant \frac{\left(1+3^{1/q}\right)(b-a)^2}{2^{3+2/q}(\alpha+1)}\left(\int_0^1\left[(\alpha+1)t-t^{\alpha+1}\right]^p dt\right)^{1/p}\left(\left|f''(a)\right|+\left|f''(b)\right|\right)$$
$$\leqslant \frac{(b-a)^2}{2^{1+2/q}(\alpha+1)}\left(\int_0^1\left[(\alpha+1)t-t^{\alpha+1}\right]^p dt\right)^{1/p}\left(\left|f''(a)\right|+\left|f''(b)\right|\right),$$

where $\frac{1}{p}+\frac{1}{q}=1$.

REMARK. Especially, if taking $q=1$ in Corollary 4 (i), we have

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{\left(\frac{a+b}{2}\right)^-}f(a)+J^\alpha_{\left(\frac{a+b}{2}\right)^+}f(b)\right]-\frac{f(a)+f(b)}{2}\right|$$
$$\leqslant \frac{(b-a)^2}{16}\left[1-\frac{2}{(\alpha+1)(\alpha+2)}\right]\left(\left|f''(a)\right|+\left|f''(b)\right|\right).$$

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Yi Xing
Department of Mathematics
Zhejiang International Studies University
Hangzhou 310014, China
e-mail: 947228976@qq.com

Chaoqun Jiang
Department of Mathematics
Zhejiang International Studies University
Hangzhou 310014, China
e-mail: 932963926@qq.com

Jianmiao Ruan
Department of Mathematics
Zhejiang International Studies University
Hangzhou 310014, China
e-mail: rjmath@163.com