

COMPLETE CONVERGENCE AND COMPLETE MOMENT CONVERGENCE FOR ARRAYS OF ROWWISE NEGATIVELY DEPENDENT RANDOM VARIABLES UNDER SUB-LINEAR EXPECTATIONS

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Abstract. In this paper, under some suitable conditions, we study the complete convergence and complete moment convergence for arrays of rowwise negatively dependent random variables in sub-linear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$. Some general results on complete convergence and complete moment convergence for arrays of rowwise negatively dependent random variables under sub-linear expectations are established, which extend the corresponding ones in classical probability space to the case of sub-linear expectation space.

1. Introduction

As is known to all, the limit theorems play very important roles in probability and mathematical statistics, and classical limit theorems only hold in the case of model certainty. However, there exist uncertainties, such as measures of risk, nonlinear stochastic calculus, and statistics in the process of finance. In the classical limit theorems, probabilities and expectations are additive while many uncertainty phenomena do not satisfy the linear additivity condition. At this time, nonadditive probabilities and nonadditive expectations are useful tools for studying uncertainties and nonlinear stochastic calculus in the process of finance. Therefore, in order to solve the limitation of the application of classical limit theorems in practice, Peng (2005, 2006, 2008a) introduced and established the basic framework and concept of sub-linear expectation space as a natural extension of the classical linear expectation space.

Under the framework of Peng (2005, 2006, 2008a), many scholars have paid attention to and studied the relevant properties of sub-linear expectations, and many excellent results have been achieved. For example, Gao and Xu (2011, 2012) studied the large deviations and moderate deviations for quasi-continuous random variables in a complete separable metric space under the Choquet capacity generalized by a regular

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sub-linear expectation; Zhang (2016a, 2016b, 2022) obtained the exponential inequalities, Rosenthal’s inequalities and some limit theorems in sub-linear expectation space; Zhong and Wu (2017) studied the complete convergence and complete moment convergence for weighted sums of extended negatively dependent (END, for short) random variables in $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$; Zhang and Lin (2018) gained Marcinkiewicz’s strong law of large numbers (SLLN, for short) for nonlinear expectations; Xi et al. (2019) studied the complete convergence for arrays of rowwise END random variables; Kuczmaszewska (2020) obtained the exponential inequalities, Hoffmann-Jørgensen type inequalities and the complete convergence for widely acceptable (WA, for short) random variables; Song (2020) gives an estimate of the convergence rate of this central limit theorem by Stein’s method under sub-linear expectations; Zhang (2021) build Heyde’s theorem under the sub-linear expectations; Guo and Li (2021) introduced the concept of pseudo-independence under sub-linear expectations and derived the weak laws of large numbers and SLLN with nonadditive probabilities generated by sub-linear expectations; Chen and Liu (2021) obtained SLLN for negatively dependent (ND, for short) random variables under sub-linear expectations, and so on.

The complete convergence is one of the important problems in limit theorems. The concept of the complete convergence was first introduced by Hsu and Robbins (1947) as follows: A sequence $\{X_n, n \geq 1\}$ of random variables is said to converge completely to a constant θ if

$$\sum_{n=1}^{\infty} P(|X_n - \theta| > \varepsilon) < \infty \text{ for all } \varepsilon > 0.$$

By the Borel-Cantelli lemma, this implies that $X_n \rightarrow \theta$ almost surely (*a.s.*, for short). Hence the complete convergence implies *a.s.* convergence. The concept of the complete q -th moment convergence was introduced by Chow (1988), which is a stronger concept than complete convergence as follows: Let $\{X_n, n \geq 1\}$ be a sequence of random variables and $a_n > 0, b_n > 0, q > 0$. If

$$\sum_{n=1}^{\infty} a_n E\{b_n^{-1} |X_n| - \varepsilon\}_+^q < \infty \text{ for all } \varepsilon > 0,$$

then the above result was said to be complete moment convergence.

Now let us recall the concept of ND random variables in the classical probability space (Ω, \mathcal{F}, P) , which was first introduced by Lehmann (1966) in the following way.

DEFINITION 1.1. Random variables X_1, X_2, \dots, X_n are said to be ND if for each $n \geq 2$, the following two inequalities hold:

$$P(X_1 \leq x_1, X_2 \leq x_2, \dots, X_n \leq x_n) \leq \prod_{i=1}^n P(X_i \leq x_i),$$

and

$$P(X_1 > x_1, X_2 > x_2, \dots, X_n > x_n) \leq \prod_{i=1}^n P(X_i > x_i),$$

for all $x_1, \dots, x_n \in \mathbb{R}$. An infinite family $\{X_n, n \geq 1\}$ of random variables is ND if every finite subfamily is ND.

An array $\{X_{nk}, k \geq 1, n \geq 1\}$ of random variables is said to be rowwise ND if for every $n \geq 1$, $\{X_{nk}, k \geq 1\}$ is ND.

In many statistical frameworks, such as multinomial, convolution of unlike multinomial, multivariate hypergeometric, Dirichlet, permutation distribution, negatively correlated normal distribution, random sampling without replacement and joint distribution of ranks, the ND assumption is more reasonable than the independent structures. A number of limit theorems for ND random variables have been established by many scholars. We refer to Joag-Dev and Proschan (1983), Amini and Bozorgnia (2000, 2003), Taylor et al. (2002), Volodin (2002), Amini et al. (2004, 2007), Klesov et al. (2005), Asadian et al. (2006), Li et al. (2006), Kuczmaszewska (2006), Zarei and Jabbari (2011), Wu (2010, 2011), Sung (2012, 2015), Chen and Sung (2016), Xu (2019), and so on.

Let $\{k_n, n \geq 1\}$ be a non-decreasing sequence of positive integers such that $\lim_{n \rightarrow \infty} k_n = \infty$. Qiu et al. (2011) studied the complete convergence for arrays of rowwise ND random variables, and obtained the following interesting result.

THEOREM 1.1. *Let $\{X_{nk}, 1 \leq k \leq k_n, n \geq 1\}$ be an array of rowwise ND random variables, and $\{c_n, n \geq 1\}$ be a sequence of positive constants. Suppose that for every $\tau > 0$ and some $\delta > 0$:*

- (i) $\sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} P(|X_{nk}| > \tau) < \infty$;
- (ii) *there exists $\eta \geq 1$ such that*

$$\sum_{n=1}^{\infty} c_n \left(\sum_{k=1}^{k_n} \text{Var}(X_{nk} I(|X_{nk}| \leq \delta)) \right)^\eta < \infty.$$

Then for all $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} c_n P \left(\left| \sum_{k=1}^{k_n} (X_{nk} - EX_{nk} I(|X_{nk}| \leq \delta)) \right| > \varepsilon \right) < \infty.$$

In this paper, under appropriate conditions, we will study the complete convergence and complete moment convergence for arrays of rowwise ND random variables in sub-linear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$, and obtain some simple corollaries. The results obtained by Qiu et al. (2011) from the classical probability space will be generalized to the case of sub-linear expectation space.

The structure of this article is as follows. Some preliminary lemmas are stated in Section 2. Main results and their proofs are provided in Section 3.

Throughout this paper, the symbol C represents positive some constant which may be different in various places. Let $I(A)$ be the indicator function of the set A .

2. Preliminaries

In this section, we introduce some basic notations and concepts. We use the notations established by Peng (2008b).

Let (Ω, \mathcal{F}) be a given measurable space and let \mathcal{H} be a linear space of real functions defined on (Ω, \mathcal{F}) such that if $X_1, X_2, \dots, X_n \in \mathcal{H}$, then $\varphi(X_1, \dots, X_n) \in \mathcal{H}$ for each $\varphi \in C_{l,Lip}(\mathbb{R}^n)$, where $C_{l,Lip}(\mathbb{R}^n)$ denotes the linear space of (local Lipschitz) functions φ satisfying

$$|\varphi(x) - \varphi(y)| \leq C(1 + |x|^m + |y|^m)|x - y|, \text{ for any } x, y \in \mathbb{R}^n,$$

for some $C > 0$, $m \in \mathbb{N}$ depending on φ . \mathcal{H} is considered as a space of “random variables”. In this case, we denote $X \in \mathcal{H}$.

DEFINITION 2.1. A sub-linear expectation $\hat{\mathbb{E}}$ on \mathcal{H} is a functional $\hat{\mathbb{E}} : \mathcal{H} \rightarrow \bar{\mathbb{R}} := [-\infty, \infty]$ satisfying the following properties: for all $X, Y \in \mathcal{H}$, we have

- (i) monotonicity: if $X \geq Y$, then $\hat{\mathbb{E}}[X] \geq \hat{\mathbb{E}}[Y]$;
- (ii) constant preserving: $\hat{\mathbb{E}}[c] = c$ for $c \in \mathbb{R}$;
- (iii) sub-additivity: $\hat{\mathbb{E}}[X + Y] \leq \hat{\mathbb{E}}[X] + \hat{\mathbb{E}}[Y]$, whenever $\hat{\mathbb{E}}[X] + \hat{\mathbb{E}}[Y]$ is not of the form $+\infty - \infty$ or $-\infty + \infty$;
- (iv) positive homogeneity: $\hat{\mathbb{E}}[\lambda X] = \lambda \hat{\mathbb{E}}[X]$, $\lambda \geq 0$.

The triple $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called a sub-linear expectation space. Given a sub-linear expectation $\hat{\mathbb{E}}$, let us denote the conjugate expectation $\hat{\mathbb{e}}$ of $\hat{\mathbb{E}}$ by

$$\hat{\mathbb{e}}[X] := -\hat{\mathbb{E}}[-X], \text{ for any } X \in \mathcal{H}.$$

Obviously, for all $X \in \mathcal{H}$, $\hat{\mathbb{e}}[X] \leq \hat{\mathbb{E}}[X]$. We also call $\hat{\mathbb{E}}[X]$ and $\hat{\mathbb{e}}[X]$ the upper-expectation and lower-expectation of X , respectively.

Next, we consider the capacities corresponding to the sub-linear expectations. Let $\mathcal{G} \subset \mathcal{F}$. A function $V : \mathcal{G} \rightarrow [0, 1]$ is called a capacity if

$$V(\emptyset) = 0, V(\Omega) = 1, V(A) \leq V(B), \text{ for any } A \subseteq B, A, B \in \mathcal{G}.$$

If the capacity V satisfies $V(A \cup B) \leq V(A) + V(B)$ for all $A, B \in \mathcal{G}$, then it is called to be sub-additive. In the sub-linear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$, we denote a pair $(\mathbb{V}, \mathcal{V})$ of capacities by

$$\mathbb{V}(A) := \inf\{\hat{\mathbb{E}}[\xi] : I_A \leq \xi, \xi \in \mathcal{H}\}, \mathcal{V}(A) := 1 - \mathbb{V}(A^c), \text{ for any } A \in \mathcal{F},$$

where A^c is the complement set of A . By definition of \mathbb{V} and \mathcal{V} , it is obvious that \mathbb{V} is sub-additive, and

$$\hat{\mathbb{E}}[f] \leq \mathbb{V}(A) \leq \hat{\mathbb{E}}[g], \hat{\mathbb{e}}[f] \leq \mathcal{V}(A) \leq \hat{\mathbb{e}}[g], \text{ if } f \leq I_A \leq g, f, g \in \mathcal{H}.$$

The corresponding Choquet integrals/expectations $(C_{\mathbb{V}}, C_{\mathcal{V}})$ are defined by

$$C_V(X) = \int_0^\infty V(X > t)dt + \int_{-\infty}^0 [V(X > t) - 1]dt,$$

with V being replaced by ∇ and \mathcal{V} , respectively.

Zhang (2016b) introduced the following concept of ND random variables in the sub-linear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$.

DEFINITION 2.2. (i) In a sub-linear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$, a random vector $Y = (Y_1, \dots, Y_n)$, $Y_i \in \mathcal{H}$ is said to be ND to another random vector $X = (X_1, \dots, X_m)$, $X_i \in \mathcal{H}$ under $\hat{\mathbb{E}}$, if for each pair of functions $\varphi_1 \in C_{l,Lip}(\mathbb{R}^m)$ and $\varphi_2 \in C_{l,Lip}(\mathbb{R}^n)$, we have

$$\hat{\mathbb{E}}[\varphi_1(X)\varphi_2(Y)] \leq \hat{\mathbb{E}}[\varphi_1(X)]\hat{\mathbb{E}}[\varphi_2(Y)],$$

whenever $\varphi_1(X) \geq 0$, $\hat{\mathbb{E}}[\varphi_2(Y)] \geq 0$, $\hat{\mathbb{E}}[|\varphi_1(X)\varphi_2(Y)|] < \infty$, $\hat{\mathbb{E}}[|\varphi_1(X)|] < \infty$, $\hat{\mathbb{E}}[|\varphi_2(Y)|] < \infty$, and either φ_1 and φ_2 are both coordinatewise non-decreasing or non-increasing.

(ii) Let $\{X_n, n \geq 1\}$ be a sequence of random variables in the sub-linear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$. X_1, X_2, \dots are said to be ND, if X_{i+1} is ND to (X_1, \dots, X_i) for each $i \geq 1$.

It is obvious that if $\{X_n, n \geq 1\}$ is a sequence of ND random variables in $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ and functions $f_1(x), f_2(x), \dots \in C_{l,Lip}(\mathbb{R}^n)$ are all non-decreasing (resp. all non-increasing), then $\{f_n(X_n), n \geq 1\}$ is also a sequence of ND random variables.

In the next part of this section, we present the main tools and preliminary lemmas that are needed for the proofs of main results. The first one is about basic inequalities under sub-linear expectations, which can be proved easily.

LEMMA 2.1. For any $X, Y \in \mathcal{H}$, the following inequalities hold:

- (i) $|\hat{\mathbb{E}}[X]| \leq \hat{\mathbb{E}}[|X|]$;
- (ii) $|\hat{\mathbb{E}}[X] - \hat{\mathbb{E}}[Y]| \leq \hat{\mathbb{E}}[|X - Y|]$, $\hat{\mathbb{E}}[X] - \hat{\mathbb{E}}[Y] \leq \hat{\mathbb{E}}[X - Y]$;
- (iii) Markov's inequality: $\nabla(|X| > x) \leq \frac{\hat{\mathbb{E}}[|X|^p]}{x^p}$, for any $p > 0, x > 0$;
- (iv) Jensen's inequality: Let $f(\cdot)$ be a convex function on \mathbb{R} . Suppose that $\hat{\mathbb{E}}[X]$ and $\hat{\mathbb{E}}[f(X)]$ exist, then $f(\hat{\mathbb{E}}[X]) \leq \hat{\mathbb{E}}[f(X)]$.

The next one can be referred to Theorem 3.1 in Zhang (2022).

LEMMA 2.2. Let (X_1, \dots, X_n) be a sequence of ND random variables in $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ with $\hat{\mathbb{E}}[X_k] \leq 0$. Let $S_n = \sum_{k=1}^n X_k$, $B_n = \sum_{k=1}^n \hat{\mathbb{E}}[X_k^2]$. Then, for all $x, y > 0$,

$$\nabla(S_n \geq x) \leq \nabla\left(\max_{1 \leq k \leq n} X_k \geq y\right) + \exp\left\{-\frac{x^2}{2(xy + B_n)}\left(1 + \frac{2}{3} \ln\left(1 + \frac{xy}{B_n}\right)\right)\right\};$$

in particular,

$$\nabla(S_n \geq x) \leq C \frac{B_n}{x^2} \text{ for any } x > 0.$$

In the sub-linear expectation space, because of the uncertainty of expectation and capacity, the study of complete convergence and complete moment convergence are both much more complex and difficult. As is known to all, in the classical probability space (Ω, \mathcal{F}, P) , there is an equality: $E I_A = P(A)$, $A \in \mathcal{F}$. However, in the sub-linear expectation space, this equality is not defined. We need to modify the indicator function by functions in $C_{l,Lip}(\mathbb{R})$. To this end, for $0 < \mu < 1$, we define the even

function $h(x) \in C_{l,Lip}(\mathbb{R})$, $0 \leq h(x) \leq 1$ for all x , $h(x) = 1$ if $|x| \leq \mu$, $h(x) = 0$ if $|x| > 1$ and $h(x) \downarrow$ as $x > 0$. For this function we have

$$I(|x| \leq \mu) \leq h(x) \leq I(|x| \leq 1), \quad I(|x| > 1) \leq 1 - h(x) \leq I(|x| > \mu).$$

3. Main results and their proofs

THEOREM 3.1. *Let $\{X_{nk}, 1 \leq k \leq k_n, n \geq 1\}$ be an array of rowwise ND random variables in $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$, and $\{c_n, n \geq 1\}$ be a sequence of positive constants. Suppose that:*

- (i) $\sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} \mathbb{V}(|X_{nk}| > \tau) < \infty$ for every $\tau > 0$;
- (ii) there exist constants $\eta \geq 1$ and $\delta > 0$ such that

$$\sum_{n=1}^{\infty} c_n \left(\sum_{k=1}^{k_n} \hat{\mathbb{E}} \left[X_{nk}^2 h \left(\frac{\mu X_{nk}}{\delta} \right) \right] \right)^{\eta} < \infty.$$

Then for all $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} c_n \mathbb{V} \left(\sum_{k=1}^{k_n} \left(X_{nk} - \hat{\mathbb{E}} \left[X_{nk} h \left(\frac{X_{nk}}{\delta} \right) \right] \right) > \varepsilon \right) < \infty, \tag{3.1}$$

and

$$\sum_{n=1}^{\infty} c_n \mathbb{V} \left(\sum_{k=1}^{k_n} \left(X_{nk} - \hat{\varepsilon} \left[X_{nk} h \left(\frac{X_{nk}}{\delta} \right) \right] \right) < -\varepsilon \right) < \infty. \tag{3.2}$$

In particular, if $\hat{\mathbb{E}} \left[X_{nk} h \left(\frac{X_{nk}}{\delta} \right) \right] = \hat{\varepsilon} \left[X_{nk} h \left(\frac{X_{nk}}{\delta} \right) \right]$, then for all $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} c_n \mathbb{V} \left(\left| \sum_{k=1}^{k_n} \left(X_{nk} - \hat{\mathbb{E}} \left[X_{nk} h \left(\frac{X_{nk}}{\delta} \right) \right] \right) \right| > \varepsilon \right) < \infty. \tag{3.3}$$

Proof. If $\hat{\mathbb{E}} \left[X_{nk} h \left(\frac{X_{nk}}{\delta} \right) \right] = \hat{\varepsilon} \left[X_{nk} h \left(\frac{X_{nk}}{\delta} \right) \right]$, then it is easy to check that

$$\begin{aligned} & \sum_{n=1}^{\infty} c_n \mathbb{V} \left(\left| \sum_{k=1}^{k_n} \left(X_{nk} - \hat{\mathbb{E}} \left[X_{nk} h \left(\frac{X_{nk}}{\delta} \right) \right] \right) \right| > \varepsilon \right) \\ & \leq \sum_{n=1}^{\infty} c_n \mathbb{V} \left(\sum_{k=1}^{k_n} \left(X_{nk} - \hat{\mathbb{E}} \left[X_{nk} h \left(\frac{X_{nk}}{\delta} \right) \right] \right) > \varepsilon \right) \\ & \quad + \sum_{n=1}^{\infty} c_n \mathbb{V} \left(\sum_{k=1}^{k_n} \left(X_{nk} - \hat{\varepsilon} \left[X_{nk} h \left(\frac{X_{nk}}{\delta} \right) \right] \right) < -\varepsilon \right). \end{aligned}$$

Considering $\{-X_{nk}, 1 \leq k \leq k_n, n \geq 1\}$ instead of $\{X_{nk}, 1 \leq k \leq k_n, n \geq 1\}$ in (3.1), we can obtain (3.2) and (3.3) immediately. Therefore, we just need to prove (3.1). It is easy to see that

$$\begin{aligned} & \mathbb{V} \left(\sum_{k=1}^{k_n} \left(X_{nk} - \hat{\mathbb{E}} \left[X_{nk} h \left(\frac{X_{nk}}{\delta} \right) \right] \right) \right) > \varepsilon \\ & \leq \mathbb{V} \left(\sum_{k=1}^{k_n} \left(X_{nk} - \hat{\mathbb{E}} \left[X_{nk} h \left(\frac{X_{nk}}{\delta} \right) \right] \right) > \varepsilon, \bigcup_{k=1}^{k_n} \{|X_{nk}| > \delta\} \right) \\ & \quad + \mathbb{V} \left(\sum_{k=1}^{k_n} \left(X_{nk} - \hat{\mathbb{E}} \left[X_{nk} h \left(\frac{X_{nk}}{\delta} \right) \right] \right) > \varepsilon, \bigcap_{k=1}^{k_n} \{|X_{nk}| \leq \delta\} \right) \\ & \leq \sum_{k=1}^{k_n} \mathbb{V}(|X_{nk}| > \delta) + \mathbb{V} \left(\sum_{k=1}^{k_n} \left(X_{nk} I(|X_{nk}| \leq \delta) - \hat{\mathbb{E}} \left[X_{nk} h \left(\frac{X_{nk}}{\delta} \right) \right] \right) > \varepsilon \right). \end{aligned}$$

By condition (i), it is enough to prove that

$$\sum_{n=1}^{\infty} c_n \mathbb{V} \left(\sum_{k=1}^{k_n} \left(X_{nk} I(|X_{nk}| \leq \delta) - \hat{\mathbb{E}} \left[X_{nk} h \left(\frac{X_{nk}}{\delta} \right) \right] \right) > \varepsilon \right) < \infty.$$

For fixed $n \geq 1$, denote for $1 \leq k \leq k_n$ that

$$Y_{nk} = -\delta I(X_{nk} < -\delta) + X_{nk} I(|X_{nk}| \leq \delta) + \delta I(X_{nk} > \delta).$$

Then, $\{Y_{nk}, 1 \leq k \leq k_n\}$ and $\{Y_{nk} - \hat{\mathbb{E}}[Y_{nk}], 1 \leq k \leq k_n\}$ are both ND random variables by the definition of ND random variables for each $n \geq 1$. Hence, we have

$$\begin{aligned} & \sum_{n=1}^{\infty} c_n \mathbb{V} \left(\sum_{k=1}^{k_n} \left(X_{nk} I(|X_{nk}| \leq \delta) - \hat{\mathbb{E}} \left[X_{nk} h \left(\frac{X_{nk}}{\delta} \right) \right] \right) > \varepsilon \right) \\ & = \sum_{n=1}^{\infty} c_n \mathbb{V} \left(\sum_{k=1}^{k_n} \left(Y_{nk} - \hat{\mathbb{E}}[Y_{nk}] + \hat{\mathbb{E}}[Y_{nk}] - \hat{\mathbb{E}} \left[X_{nk} h \left(\frac{X_{nk}}{\delta} \right) \right] \right) \right. \\ & \quad \left. + (X_{nk} - \delta) I(X_{nk} > \delta) + (X_{nk} + \delta) I(X_{nk} < -\delta) - X_{nk} I(|X_{nk}| > \delta) \right) > \varepsilon \\ & \leq \sum_{n=1}^{\infty} c_n \mathbb{V} \left(\sum_{k=1}^{k_n} \left(Y_{nk} - \hat{\mathbb{E}}[Y_{nk}] + \hat{\mathbb{E}}[Y_{nk}] - \hat{\mathbb{E}} \left[X_{nk} h \left(\frac{X_{nk}}{\delta} \right) \right] \right) \right. \\ & \quad \left. + (X_{nk} - \delta) I(X_{nk} > \delta) + |X_{nk}| I(|X_{nk}| > \delta) \right) > \varepsilon \\ & \leq \sum_{n=1}^{\infty} c_n \mathbb{V} \left(\sum_{k=1}^{k_n} (Y_{nk} - \hat{\mathbb{E}}[Y_{nk}]) > \frac{\varepsilon}{4} \right) \\ & \quad + \sum_{n=1}^{\infty} c_n \mathbb{V} \left(\sum_{k=1}^{k_n} \left(\hat{\mathbb{E}}[Y_{nk}] - \hat{\mathbb{E}} \left[X_{nk} h \left(\frac{X_{nk}}{\delta} \right) \right] \right) > \frac{\varepsilon}{4} \right) \end{aligned}$$

$$\begin{aligned}
 & + \sum_{n=1}^{\infty} c_n \mathbb{V} \left(\sum_{k=1}^{k_n} (X_{nk} - \delta) I(X_{nk} > \delta) > \frac{\varepsilon}{4} \right) \\
 & + \sum_{n=1}^{\infty} c_n \mathbb{V} \left(\sum_{k=1}^{k_n} |X_{nk}| I(|X_{nk}| > \delta) > \frac{\varepsilon}{4} \right) \\
 =: & I_1 + I_2 + I_3 + I_4.
 \end{aligned}$$

For the estimate of I_2 , note that

$$\begin{aligned}
 & \left| Y_{nk} - X_{nk} h \left(\frac{X_{nk}}{\delta} \right) \right| \\
 = & \left| \delta I(X_{nk} > \delta) - \delta I(X_{nk} < -\delta) + X_{nk} I(|X_{nk}| \leq \delta) - X_{nk} h \left(\frac{X_{nk}}{\delta} \right) \right| \\
 \leq & \delta I(|X_{nk}| > \delta) + |X_{nk}| \left(I(|X_{nk}| \leq \delta) - h \left(\frac{X_{nk}}{\delta} \right) \right) \\
 \leq & \delta I(|X_{nk}| > \delta) + |X_{nk}| I(\mu \delta < |X_{nk}| \leq \delta) \\
 \leq & \delta I(|X_{nk}| > \mu \delta) \leq \delta \left(1 - h \left(\frac{X_{nk}}{\mu \delta} \right) \right).
 \end{aligned}$$

We have by Lemma 2.1 and condition (i) that,

$$\begin{aligned}
 I_2 & = \sum_{n=1}^{\infty} c_n \mathbb{V} \left(\sum_{k=1}^{k_n} \left(\hat{\mathbb{E}}[Y_{nk}] - \hat{\mathbb{E}} \left[X_{nk} h \left(\frac{X_{nk}}{\delta} \right) \right] \right) > \frac{\varepsilon}{4} \right) \\
 & \leq \sum_{n=1}^{\infty} c_n \mathbb{V} \left(\sum_{k=1}^{k_n} \hat{\mathbb{E}} \left[Y_{nk} - X_{nk} h \left(\frac{X_{nk}}{\delta} \right) \right] > \frac{\varepsilon}{4} \right) \\
 & \leq C \sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} \left| \hat{\mathbb{E}} \left[Y_{nk} - X_{nk} h \left(\frac{X_{nk}}{\delta} \right) \right] \right| \\
 & \leq C \sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} \hat{\mathbb{E}} \left[\left| Y_{nk} - X_{nk} h \left(\frac{X_{nk}}{\delta} \right) \right| \right] \\
 & \leq C \sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} \hat{\mathbb{E}} \left[1 - h \left(\frac{X_{nk}}{\mu \delta} \right) \right] \\
 & \leq C \sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} \mathbb{V}(|X_{nk}| > \mu^2 \delta) \\
 & < \infty.
 \end{aligned}$$

In the following, we will show $I_3 < \infty$. It is easy to obtain by condition (i) that

$$I_3 \leq \sum_{n=1}^{\infty} c_n \mathbb{V} \left(\max_{1 \leq k \leq k_n} |X_{nk}| > \delta \right) \leq \sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} \mathbb{V}(|X_{nk}| > \delta) < \infty.$$

Similar to the proof of $I_3 < \infty$, we can obtain $I_4 < \infty$ immediately.

Finally, we will show $I_1 < \infty$. Let $B_n = \sum_{k=1}^{k_n} \hat{\mathbb{E}} \left[(Y_{nk} - \hat{\mathbb{E}}[Y_{nk}])^2 \right]$. For any $\varepsilon > 0$ and $y > 0$, set

$$\begin{aligned} N_1 &= \left\{ n : B_n > \frac{y\varepsilon}{4} \right\}, \\ N_2 &= \left\{ n : \sum_{k=1}^{k_n} \mathbb{V}(|X_{nk}| > \mu\delta) > \min \left\{ 1, \frac{y\varepsilon}{32\delta^2}, \frac{y}{4\delta} \right\} \right\}, \\ N_3 &= \left\{ n : \sum_{k=1}^{k_n} \hat{\mathbb{E}} \left[X_{nk}^2 h \left(\frac{\mu X_{nk}}{\delta} \right) \right] > \min \left\{ \frac{y\varepsilon}{32}, \frac{y^2}{16} \right\} \right\}, \\ N_4 &= (N_2 \cup N_3)^c. \end{aligned}$$

Note that

$$Y_{nk}^2 = \delta^2 I(|X_{nk}| > \delta) + X_{nk}^2 I(|X_{nk}| \leq \delta) \leq \delta^2 \left(1 - h \left(\frac{X_{nk}}{\delta} \right) \right) + X_{nk}^2 h \left(\frac{\mu X_{nk}}{\delta} \right).$$

By C_r -inequality and Jensen's inequality, it is easy to check that

$$\begin{aligned} B_n &\leq 4 \sum_{k=1}^{k_n} \hat{\mathbb{E}} [Y_{nk}^2] \\ &\leq 4 \sum_{k=1}^{k_n} \hat{\mathbb{E}} \left[X_{nk}^2 h \left(\frac{\mu X_{nk}}{\delta} \right) \right] + 4\delta^2 \sum_{k=1}^{k_n} \hat{\mathbb{E}} \left[1 - h \left(\frac{X_{nk}}{\delta} \right) \right] \\ &\leq 4 \sum_{k=1}^{k_n} \hat{\mathbb{E}} \left[X_{nk}^2 h \left(\frac{\mu X_{nk}}{\delta} \right) \right] + 4\delta^2 \sum_{k=1}^{k_n} \mathbb{V}(|X_{nk}| > \mu\delta), \end{aligned}$$

which implies that $N_1 \subseteq (N_2 \cup N_3)$. By conditions (i) and (ii), we have

$$\begin{aligned} &\sum_{n \in (N_2 \cup N_3)} c_n \mathbb{V} \left(\sum_{k=1}^{k_n} (Y_{nk} - \hat{\mathbb{E}}[Y_{nk}]) > \frac{\varepsilon}{4} \right) \leq \sum_{n \in (N_2 \cup N_3)} c_n \\ &\leq \left(\min \left\{ \frac{1}{2}, \frac{y\varepsilon}{64\delta^2}, \frac{y}{8\delta} \right\} \right)^{-1} \sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} \mathbb{V}(|X_{nk}| > \mu\delta) \\ &\quad + \left(\min \left\{ \frac{y\varepsilon}{64}, \frac{y^2}{32} \right\} \right)^{-\eta} \sum_{n=1}^{\infty} c_n \left(\sum_{k=1}^{k_n} \hat{\mathbb{E}} \left[X_{nk}^2 h \left(\frac{\mu X_{nk}}{\delta} \right) \right] \right)^\eta \\ &< \infty. \end{aligned}$$

Hence, it is sufficient to prove that

$$\sum_{n \in N_4} c_n \mathbb{V} \left(\sum_{k=1}^{k_n} (Y_{nk} - \hat{\mathbb{E}}[Y_{nk}]) > \frac{\varepsilon}{4} \right) < \infty.$$

By Lemma 2.2 we have that

$$\begin{aligned} & \sum_{n \in N_4} c_n \mathbb{V} \left(\sum_{k=1}^{k_n} (Y_{nk} - \hat{\mathbb{E}}[Y_{nk}]) > \frac{\varepsilon}{4} \right) \\ & \leq \sum_{n \in N_4} c_n \mathbb{V} \left(\max_{1 \leq k \leq k_n} (Y_{nk} - \hat{\mathbb{E}}[Y_{nk}]) \geq y \right) \\ & \quad + \sum_{n \in N_4} c_n \exp \left\{ -\frac{\varepsilon^2}{8(y\varepsilon + 4B_n)} \left(1 + \frac{2}{3} \ln \left(1 + \frac{y\varepsilon}{4B_n} \right) \right) \right\} \\ & =: I_{11} + I_{12}. \end{aligned}$$

By Jensen’s inequality, we have for $n \in N_4$ that

$$\begin{aligned} & \max_{1 \leq k \leq k_n} |\hat{\mathbb{E}}[Y_{nk}]| \leq \max_{1 \leq k \leq k_n} \hat{\mathbb{E}}[|Y_{nk}|] \\ & \leq \max_{1 \leq k \leq k_n} \delta \hat{\mathbb{E}} \left[1 - h \left(\frac{X_{nk}}{\delta} \right) \right] + \max_{1 \leq k \leq k_n} \hat{\mathbb{E}} \left[|X_{nk}| h \left(\frac{\mu X_{nk}}{\delta} \right) \right] \\ & \leq \delta \sum_{k=1}^{k_n} \mathbb{V}(|X_{nk}| > \mu \delta) + \left(\sum_{k=1}^{k_n} \hat{\mathbb{E}} \left[X_{nk}^2 h \left(\frac{\mu X_{nk}}{\delta} \right) \right] \right)^{\frac{1}{2}} \\ & \leq \delta \min \left\{ 1, \frac{y\varepsilon}{32\delta^2}, \frac{y}{4\delta} \right\} + \left(\min \left\{ \frac{y\varepsilon}{32}, \frac{y^2}{16} \right\} \right)^{\frac{1}{2}} \\ & \leq \frac{y}{4} + \frac{y}{4} = \frac{y}{2}, \end{aligned}$$

which implies that for any $n \in N_4$, $\max_{1 \leq k \leq k_n} |\hat{\mathbb{E}}[Y_{nk}]| \leq \frac{y}{2}$. Combining with condition (i), we obtain

$$\begin{aligned} I_{11} & \leq \sum_{n \in N_4} c_n \mathbb{V} \left(\max_{1 \leq k \leq k_n} |Y_{nk}| \geq \frac{y}{2} \right) \\ & \leq \sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} \mathbb{V} \left(|X_{nk}| \geq \frac{y}{2} \right) \\ & < \infty. \end{aligned}$$

Next, we prove $I_{12} < \infty$. When $n \in N_4$, we have $B_n \leq \frac{y\varepsilon}{4}$ and $\sum_{k=1}^{k_n} \mathbb{V}(|X_{nk}| > \mu \delta) \leq$

1. Let $y = \frac{\varepsilon}{24\eta}$. By C_r -inequality, conditions (i) and (ii), we have

$$\begin{aligned} I_{12} & \leq \sum_{n \in N_4} c_n \exp \left\{ -\frac{\varepsilon}{16y} \left(1 + \frac{2}{3} \ln \left(1 + \frac{y\varepsilon}{4B_n} \right) \right) \right\} \\ & = \exp \left\{ -\frac{3}{2} \eta \right\} \sum_{n \in N_4} c_n \exp \left\{ -\eta \ln \left(1 + \frac{y\varepsilon}{4B_n} \right) \right\} \end{aligned}$$

$$\begin{aligned}
 &= C \sum_{n \in N_4} c_n \left(\frac{4B_n}{4B_n + y\varepsilon} \right)^\eta \leq C \sum_{n \in N_4} c_n \left(\frac{4B_n}{y\varepsilon} \right)^\eta \\
 &\leq C \sum_{n \in N_4} c_n \left(\sum_{k=1}^{k_n} \left(\delta^2 \mathbb{V}(|X_{nk}| > \mu \delta) + \hat{\mathbb{E}} \left[X_{nk}^2 h \left(\frac{\mu X_{nk}}{\delta} \right) \right] \right) \right)^\eta \\
 &\leq C \sum_{n \in N_4} c_n \left(\sum_{k=1}^{k_n} \mathbb{V}(|X_{nk}| > \mu \delta) \right)^\eta + C \sum_{n \in N_4} c_n \left(\sum_{k=1}^{k_n} \hat{\mathbb{E}} \left[X_{nk}^2 h \left(\frac{\mu X_{nk}}{\delta} \right) \right] \right)^\eta \\
 &\leq C \sum_{n=1}^\infty c_n \sum_{k=1}^{k_n} \mathbb{V}(|X_{nk}| > \mu \delta) + C \sum_{n=1}^\infty c_n \left(\sum_{k=1}^{k_n} \hat{\mathbb{E}} \left[X_{nk}^2 h \left(\frac{\mu X_{nk}}{\delta} \right) \right] \right)^\eta \\
 &< \infty.
 \end{aligned}$$

This completes the proof of the theorem. \square

By Theorem 3.1, we can obtain the following corollaries.

COROLLARY 3.1. *Let $\{X_{nk}, 1 \leq k \leq k_n, n \geq 1\}$ be an array of rowwise ND random variables in $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$, and $\{c_n, n \geq 1\}$ be a sequence of positive constants. Assume that the conditions (i) and (ii) of Theorem 3.1 are satisfied, and*

(iii₁) *if $\sum_{k=1}^{k_n} \hat{\mathbb{E}} \left[X_{nk} h \left(\frac{X_{nk}}{\delta} \right) \right] \rightarrow 0$ as $n \rightarrow \infty$, then for all $\varepsilon > 0$,*

$$\sum_{n=1}^\infty c_n \mathbb{V} \left(\sum_{k=1}^{k_n} X_{nk} > \varepsilon \right) < \infty.$$

(iii₂) *if $\sum_{k=1}^{k_n} \hat{\mathbb{E}} \left[X_{nk} h \left(\frac{X_{nk}}{\delta} \right) \right] \rightarrow 0$ as $n \rightarrow \infty$, then for all $\varepsilon > 0$,*

$$\sum_{n=1}^\infty c_n \mathbb{V} \left(\sum_{k=1}^{k_n} X_{nk} < -\varepsilon \right) < \infty.$$

(iii₃) *if $\sum_{k=1}^{k_n} \hat{\mathbb{E}} \left[X_{nk} h \left(\frac{X_{nk}}{\delta} \right) \right] \rightarrow 0$ as $n \rightarrow \infty$ and $\sum_{k=1}^{k_n} \hat{\mathbb{E}} \left[X_{nk} h \left(\frac{X_{nk}}{\delta} \right) \right] \rightarrow 0$ as $n \rightarrow \infty$, then for all $\varepsilon > 0$*

$$\sum_{n=1}^\infty c_n \mathbb{V} \left(\left| \sum_{k=1}^{k_n} X_{nk} \right| > \varepsilon \right) < \infty.$$

COROLLARY 3.2. *Let $\{X_{nk}, 1 \leq k \leq k_n, n \geq 1\}$ be an array of rowwise ND random variables in $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$, and $\{c_n, n \geq 1\}$ be a sequence of positive constants. Assume that $\hat{\mathbb{E}}[X_{nk}] = \hat{\mathbb{E}}[X_{nk}] = 0$ for all $1 \leq k \leq k_n$ and $n \geq 1$. Let $0 < \phi(x) \in C_{l,Lip}(\mathbb{R})$ such that for some $\delta > 0$,*

$$\sup_{x > \mu \delta} \frac{x}{\phi(x)} < \infty, \quad \sup_{0 \leq x \leq \frac{\delta}{\mu}} \frac{x^2}{\phi(x)} < \infty.$$

Suppose that condition (i) of Theorem 3.1 is satisfied, and the following conditions are satisfied:

(iv) there exists some $\eta \geq 1$ such that

$$\sum_{n=1}^{\infty} c_n \left(\sum_{k=1}^{k_n} \hat{\mathbb{E}}[\phi(|X_{nk}|)] \right)^{\eta} < \infty;$$

(v) $\sum_{k=1}^{k_n} \hat{\mathbb{E}}[\phi(|X_{nk}|)] \rightarrow 0$ as $n \rightarrow \infty$.

Then for all $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} c_n \mathbb{V} \left(\left| \sum_{k=1}^{k_n} X_{nk} \right| > \varepsilon \right) < \infty.$$

Proof. Obviously, to prove the above result, we only need to show that the conditions of Theorem 3.1 and Corollary 3.1 are satisfied. For condition (ii), it is easy to check that

$$\sum_{k=1}^{k_n} \hat{\mathbb{E}} \left[X_{nk}^2 h \left(\frac{\mu X_{nk}}{\delta} \right) \right] \leq \sup_{0 \leq x \leq \frac{\delta}{\mu}} \frac{x^2}{\phi(x)} \sum_{k=1}^{k_n} \hat{\mathbb{E}}[\phi(|X_{nk}|)],$$

which together with $\sup_{0 \leq x \leq \frac{\delta}{\mu}} \frac{x^2}{\phi(x)} < \infty$ and (iv) yields (ii).

Next, we will prove (iii). Since $\hat{\mathbb{E}}[X_{nk}] = 0$, it follows by $\sup_{x > \mu\delta} \frac{x}{\phi(x)} < \infty$ and (v) that

$$\begin{aligned} \left| \sum_{k=1}^{k_n} \hat{\mathbb{E}}[X_{nk}] - \sum_{k=1}^{k_n} \hat{\mathbb{E}} \left[X_{nk} h \left(\frac{X_{nk}}{\delta} \right) \right] \right| &\leq \sum_{k=1}^{k_n} \hat{\mathbb{E}} \left[|X_{nk}| \left(1 - h \left(\frac{X_{nk}}{\delta} \right) \right) \right] \\ &\leq \sup_{x > \mu\delta} \frac{x}{\phi(x)} \sum_{k=1}^{k_n} \hat{\mathbb{E}}[\phi(|X_{nk}|)] \rightarrow 0 \text{ as } n \rightarrow \infty, \end{aligned}$$

which implies that

$$\sum_{k=1}^{k_n} \hat{\mathbb{E}} \left[X_{nk} h \left(\frac{X_{nk}}{\delta} \right) \right] \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Similarly, we get

$$\sum_{k=1}^{k_n} \hat{\mathbb{E}} \left[X_{nk} h \left(\frac{X_{nk}}{\delta} \right) \right] \rightarrow 0 \text{ as } n \rightarrow \infty.$$

The proof is completed. \square

In Theorem 3.1, the condition (ii) and results (3.1), (3.2) and (3.3) are all related to the function h . In the following theorem, we will give some conditions which are all unrelated to the function h , and get some simple results. It is obvious from the proof of Theorem 3.2 that Theorem 3.1 still holds under the conditions of the Theorem 3.2.

THEOREM 3.2. *Let $\{X_{nk}, 1 \leq k \leq k_n, n \geq 1\}$ be an array of rowwise ND random variables in $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$, and $\{c_n, n \geq 1\}$ be a sequence of positive constants. Assume that the condition (i) of Theorem 3.1 is satisfied, and there exists some constant $0 < p \leq 2$ such that $\sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} \hat{\mathbb{E}}[|X_{nk}|^p] < \infty$.*

(vi₁) *If $\hat{\mathbb{E}}[X_{nk}] = 0$, then for all $\varepsilon > 0$,*

$$\sum_{n=1}^{\infty} c_n \mathbb{V} \left(\sum_{k=1}^{k_n} X_{nk} > \varepsilon \right) < \infty. \tag{3.4}$$

(vi₂) *If $\hat{\mathbb{E}}[X_{nk}] = 0$, then for all $\varepsilon > 0$,*

$$\sum_{n=1}^{\infty} c_n \mathbb{V} \left(\sum_{k=1}^{k_n} X_{nk} < -\varepsilon \right) < \infty. \tag{3.5}$$

(vi₃) *If $\hat{\mathbb{E}}[X_{nk}] = \hat{\mathbb{E}}[X_{nk}] = 0$, then for all $\varepsilon > 0$,*

$$\sum_{n=1}^{\infty} c_n \mathbb{V} \left(\left| \sum_{k=1}^{k_n} X_{nk} \right| > \varepsilon \right) < \infty. \tag{3.6}$$

Proof. It is easy to check that

$$\begin{aligned} & \sum_{n=1}^{\infty} c_n \mathbb{V} \left(\left| \sum_{k=1}^{k_n} X_{nk} \right| > \varepsilon \right) \\ & \leq \sum_{n=1}^{\infty} c_n \mathbb{V} \left(\sum_{k=1}^{k_n} X_{nk} > \varepsilon \right) + \sum_{n=1}^{\infty} c_n \mathbb{V} \left(\sum_{k=1}^{k_n} X_{nk} < -\varepsilon \right). \end{aligned}$$

Considering $\{-X_{nk}, 1 \leq k \leq k_n, n \geq 1\}$ instead of $\{X_{nk}, 1 \leq k \leq k_n, n \geq 1\}$ in (3.4), we can obtain (3.5) and (3.6) immediately. Therefore, we just need to prove (3.4). It is easy to see that

$$\begin{aligned} & \sum_{n=1}^{\infty} c_n \mathbb{V} \left(\sum_{k=1}^{k_n} X_{nk} > \varepsilon \right) \\ & = \sum_{n=1}^{\infty} c_n \mathbb{V} \left(\sum_{k=1}^{k_n} \left(X_{nk} + \hat{\mathbb{E}} \left[X_{nk} h \left(\frac{X_{nk}}{\delta} \right) \right] - \hat{\mathbb{E}} \left[X_{nk} h \left(\frac{X_{nk}}{\delta} \right) \right] \right) > \varepsilon \right) \end{aligned}$$

$$\begin{aligned} &\leq \sum_{n=1}^{\infty} c_n \mathbb{V} \left(\sum_{k=1}^{k_n} \hat{\mathbb{E}} \left[X_{nk} h \left(\frac{X_{nk}}{\delta} \right) \right] > \frac{\varepsilon}{2} \right) \\ &\quad + \sum_{n=1}^{\infty} c_n \mathbb{V} \left(\sum_{k=1}^{k_n} \left(X_{nk} - \hat{\mathbb{E}} \left[X_{nk} h \left(\frac{X_{nk}}{\delta} \right) \right] \right) > \frac{\varepsilon}{2} \right) \\ &=: H_1 + H_2. \end{aligned}$$

For the estimate of H_1 , when $0 < p \leq 1$, we note that $|X_{nk}|h\left(\frac{X_{nk}}{\delta}\right) \leq |X_{nk}|I(|X_{nk}| \leq \delta) \leq \delta^{1-p}|X_{nk}|^p$. By Lemma 2.1, we have

$$\begin{aligned} H_1 &= \sum_{n=1}^{\infty} c_n \mathbb{V} \left(\sum_{k=1}^{k_n} \hat{\mathbb{E}} \left[X_{nk} h \left(\frac{X_{nk}}{\delta} \right) \right] > \frac{\varepsilon}{2} \right) \\ &\leq C \sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} \hat{\mathbb{E}} \left[|X_{nk}| h \left(\frac{X_{nk}}{\delta} \right) \right] \\ &\leq C \sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} \hat{\mathbb{E}} [|X_{nk}|^p] \\ &< \infty. \end{aligned}$$

When $1 < p \leq 2$, we note that $|X_{nk}| \left(1 - h\left(\frac{X_{nk}}{\delta}\right)\right) \leq |X_{nk}|I(|X_{nk}| > \mu\delta) \leq (\mu\delta)^{1-p}|X_{nk}|^p$. By Lemma 2.1 and $\hat{\mathbb{E}}[X_{nk}] = 0$, we have

$$\begin{aligned} H_1 &= \sum_{n=1}^{\infty} c_n \mathbb{V} \left(\sum_{k=1}^{k_n} \left(-\hat{\mathbb{E}}[X_{nk}] + \hat{\mathbb{E}} \left[X_{nk} h \left(\frac{X_{nk}}{\delta} \right) \right] \right) > \frac{\varepsilon}{2} \right) \\ &\leq \sum_{n=1}^{\infty} c_n \mathbb{V} \left(\sum_{k=1}^{k_n} \hat{\mathbb{E}} \left[|X_{nk}| \left(1 - h \left(\frac{X_{nk}}{\delta} \right) \right) \right] > \frac{\varepsilon}{2} \right) \\ &\leq C \sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} \hat{\mathbb{E}} \left[|X_{nk}| \left(1 - h \left(\frac{X_{nk}}{\delta} \right) \right) \right] \\ &\leq C \sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} \hat{\mathbb{E}} [|X_{nk}|^p] \\ &< \infty. \end{aligned}$$

For H_2 , according to the proof of (3.1), it suffices to prove that condition (ii) of Theorem 3.1 is satisfied. Since $0 < p \leq 2$, we have

$$\begin{aligned} &\sum_{k=1}^{k_n} \hat{\mathbb{E}} \left[X_{nk}^2 h \left(\frac{\mu X_{nk}}{\delta} \right) \right] \\ &= \frac{\delta^2}{\mu^2} \sum_{k=1}^{k_n} \hat{\mathbb{E}} \left[\left(\frac{\mu X_{nk}}{\delta} \right)^2 h \left(\frac{\mu X_{nk}}{\delta} \right) \right] \end{aligned}$$

$$\begin{aligned} &\leq \frac{\delta^2}{u^2} \sum_{k=1}^{k_n} \hat{\mathbb{E}} \left[\left(\frac{\mu |X_{nk}|}{\delta} \right)^p h \left(\frac{\mu^2 X_{nk}}{\delta} \right) \right] \\ &\leq C \sum_{k=1}^{k_n} \hat{\mathbb{E}} [|X_{nk}|^p], \end{aligned}$$

which together with $\sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} \hat{\mathbb{E}} [|X_{nk}|^p] < \infty$ yields condition (ii) of Theorem 3.1 with $\eta = 1$. The proof is completed. \square

The third theorem extends the result of Theorem 3.1 from complete convergence to complete moment convergence.

THEOREM 3.3. *Let $\{X_{nk}, 1 \leq k \leq k_n, n \geq 1\}$ be an array of rowwise ND random variables in $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$, and $\{c_n, n \geq 1\}$ be a sequence of positive constants. Suppose that the following conditions hold for some $\delta > 0$:*

- (i) $\sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} \hat{\mathbb{E}} \left[X_{nk}^2 \left(1 - h \left(\frac{X_{nk}}{\tau} \right) \right) \right] < \infty$ for every $\tau > 0$;
- (ii) $\sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} \hat{\mathbb{E}} \left[X_{nk}^2 h \left(\frac{\mu X_{nk}}{\delta} \right) \right] < \infty$;
- (iii) $\sum_{k=1}^{k_n} \hat{\mathbb{E}} \left[|X_{nk}| \left(1 - h \left(\frac{X_{nk}}{\mu \delta} \right) \right) \right] \rightarrow 0$ as $n \rightarrow \infty$.

Then for all $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} c_n C_{\mathbb{V}} \left(\sum_{k=1}^{k_n} \left(X_{nk} - \hat{\mathbb{E}} \left[X_{nk} h \left(\frac{X_{nk}}{\delta} \right) \right] \right) - \varepsilon \right)_+ < \infty, \tag{3.7}$$

and

$$\sum_{n=1}^{\infty} c_n C_{\mathbb{V}} \left(\sum_{k=1}^{k_n} \left(-X_{nk} + \hat{\mathbb{E}} \left[X_{nk} h \left(\frac{X_{nk}}{\delta} \right) \right] \right) - \varepsilon \right)_+ < \infty. \tag{3.8}$$

In particular, if $\hat{\mathbb{E}} \left[X_{nk} h \left(\frac{X_{nk}}{\delta} \right) \right] = \hat{\varepsilon} \left[X_{nk} h \left(\frac{X_{nk}}{\delta} \right) \right]$, then for all $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} c_n C_{\mathbb{V}} \left(\left| \sum_{k=1}^{k_n} \left(X_{nk} - \hat{\mathbb{E}} \left[X_{nk} h \left(\frac{X_{nk}}{\delta} \right) \right] \right) \right| - \varepsilon \right)_+ < \infty. \tag{3.9}$$

Proof. If $\hat{\mathbb{E}} \left[X_{nk} h \left(\frac{X_{nk}}{\delta} \right) \right] = \hat{\varepsilon} \left[X_{nk} h \left(\frac{X_{nk}}{\delta} \right) \right]$, then it is easy to check that

$$\begin{aligned} &\sum_{n=1}^{\infty} c_n C_{\mathbb{V}} \left(\left| \sum_{k=1}^{k_n} \left(X_{nk} - \hat{\mathbb{E}} \left[X_{nk} h \left(\frac{X_{nk}}{\delta} \right) \right] \right) \right| - \varepsilon \right)_+ \\ &= \sum_{n=1}^{\infty} c_n \int_0^{\infty} \mathbb{V} \left(\left| \sum_{k=1}^{k_n} \left(X_{nk} - \hat{\mathbb{E}} \left[X_{nk} h \left(\frac{X_{nk}}{\delta} \right) \right] \right) \right| - \varepsilon > t \right) dt \end{aligned}$$

$$\begin{aligned}
 &\leq \sum_{n=1}^{\infty} c_n \int_0^{\infty} \mathbb{V} \left(\sum_{k=1}^{k_n} \left(X_{nk} - \hat{\mathbb{E}} \left[X_{nk} h \left(\frac{X_{nk}}{\delta} \right) \right] \right) > t + \varepsilon \right) dt \\
 &\quad + \sum_{n=1}^{\infty} c_n \int_0^{\infty} \mathbb{V} \left(\sum_{k=1}^{k_n} \left(-X_{nk} - \hat{\mathbb{E}} \left[-X_{nk} h \left(\frac{X_{nk}}{\delta} \right) \right] \right) > t + \varepsilon \right) dt \\
 &= \sum_{n=1}^{\infty} c_n C_{\mathbb{V}} \left(\sum_{k=1}^{k_n} \left(X_{nk} - \hat{\mathbb{E}} \left[X_{nk} h \left(\frac{X_{nk}}{\delta} \right) \right] \right) - \varepsilon \right)_+ \\
 &\quad + \sum_{n=1}^{\infty} c_n C_{\mathbb{V}} \left(\sum_{k=1}^{k_n} \left(-X_{nk} + \hat{\mathbb{E}} \left[X_{nk} h \left(\frac{X_{nk}}{\delta} \right) \right] \right) - \varepsilon \right)_+ .
 \end{aligned}$$

Considering $\{-X_{nk}, 1 \leq k \leq k_n, n \geq 1\}$ instead of $\{X_{nk}, 1 \leq k \leq k_n, n \geq 1\}$ in (3.7), we can obtain (3.8) and (3.9) immediately. So it is sufficient to prove that

$$\sum_{n=1}^{\infty} c_n \int_0^{\infty} \mathbb{V} \left(\sum_{k=1}^{k_n} \left(X_{nk} - \hat{\mathbb{E}} \left[X_{nk} h \left(\frac{X_{nk}}{\delta} \right) \right] \right) > t + \varepsilon \right) dt < \infty.$$

Note that

$$\begin{aligned}
 &\sum_{n=1}^{\infty} c_n \int_0^{\infty} \mathbb{V} \left(\sum_{k=1}^{k_n} \left(X_{nk} - \hat{\mathbb{E}} \left[X_{nk} h \left(\frac{X_{nk}}{\delta} \right) \right] \right) > t + \varepsilon \right) dt \\
 &= \sum_{n=1}^{\infty} c_n \int_0^{\delta} \mathbb{V} \left(\sum_{k=1}^{k_n} \left(X_{nk} - \hat{\mathbb{E}} \left[X_{nk} h \left(\frac{X_{nk}}{\delta} \right) \right] \right) > t + \varepsilon \right) dt \\
 &\quad + \sum_{n=1}^{\infty} c_n \int_{\delta}^{\infty} \mathbb{V} \left(\sum_{k=1}^{k_n} \left(X_{nk} - \hat{\mathbb{E}} \left[X_{nk} h \left(\frac{X_{nk}}{\delta} \right) \right] \right) > t + \varepsilon \right) dt \\
 &\leq \delta \sum_{n=1}^{\infty} c_n \mathbb{V} \left(\sum_{k=1}^{k_n} \left(X_{nk} - \hat{\mathbb{E}} \left[X_{nk} h \left(\frac{X_{nk}}{\delta} \right) \right] \right) > \varepsilon \right) \\
 &\quad + \sum_{n=1}^{\infty} c_n \int_{\delta}^{\infty} \mathbb{V} \left(\sum_{k=1}^{k_n} \left(X_{nk} - \hat{\mathbb{E}} \left[X_{nk} h \left(\frac{X_{nk}}{\delta} \right) \right] \right) > t \right) dt \\
 &=: J_1 + J_2.
 \end{aligned}$$

Obviously, to prove (3.7), it is sufficient to prove that $J_1 < \infty$ and $J_2 < \infty$. It is easily checked that

$$\infty > \sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} \hat{\mathbb{E}} \left[X_{nk}^2 \left(1 - h \left(\frac{X_{nk}}{\tau} \right) \right) \right] \geq \tau^2 \sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} \mathbb{V}(|X_{nk}| > \tau).$$

Hence, condition (i) of Theorem 3.1 holds. Note that condition (ii) of Theorem 3.1 holds for $\eta = 1$. Therefore, by Theorem 3.1 we obtain $J_1 < \infty$.

In the following, we will show $J_2 < \infty$. It is easy to see that

$$\begin{aligned} & \mathbb{V} \left(\sum_{k=1}^{k_n} \left(X_{nk} - \hat{\mathbb{E}} \left[X_{nk} h \left(\frac{X_{nk}}{\delta} \right) \right] \right) \right) > t \Big) \\ & \leq \mathbb{V} \left(\sum_{k=1}^{k_n} \left(X_{nk} - \hat{\mathbb{E}} \left[X_{nk} h \left(\frac{X_{nk}}{\delta} \right) \right] \right) > t, \bigcup_{k=1}^{k_n} \{|X_{nk}| > t\} \right) \\ & \quad + \mathbb{V} \left(\sum_{k=1}^{k_n} \left(X_{nk} - \hat{\mathbb{E}} \left[X_{nk} h \left(\frac{X_{nk}}{\delta} \right) \right] \right) > t, \bigcap_{k=1}^{k_n} \{|X_{nk}| \leq t\} \right) \\ & \leq \sum_{k=1}^{k_n} \mathbb{V}(|X_{nk}| > t) + \mathbb{V} \left(\sum_{k=1}^{k_n} \left(X_{nk} I(|X_{nk}| \leq t) - \hat{\mathbb{E}} \left[X_{nk} h \left(\frac{X_{nk}}{\delta} \right) \right] \right) > t \right), \end{aligned}$$

which derives that

$$\begin{aligned} J_2 & \leq \sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} \int_{\delta}^{\infty} \mathbb{V}(|X_{nk}| > t) dt \\ & \quad + \sum_{n=1}^{\infty} c_n \int_{\delta}^{\infty} \mathbb{V} \left(\sum_{k=1}^{k_n} \left(X_{nk} I(|X_{nk}| \leq t) - \hat{\mathbb{E}} \left[X_{nk} h \left(\frac{X_{nk}}{\delta} \right) \right] \right) > t \right) dt \\ & =: J_3 + J_4. \end{aligned}$$

It is easy to obtain by condition (i) that,

$$\begin{aligned} J_3 & = \sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} \int_{\delta}^{\infty} \mathbb{V}(|X_{nk}| > t) dt \\ & \leq \sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} \int_{\delta}^{\infty} t^{-2} \hat{\mathbb{E}} \left[X_{nk}^2 \left(1 - h \left(\frac{X_{nk}}{t} \right) \right) \right] dt \\ & \leq C \sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} \hat{\mathbb{E}} \left[X_{nk}^2 \left(1 - h \left(\frac{X_{nk}}{\delta} \right) \right) \right] \\ & < \infty. \end{aligned}$$

For fixed $n \geq 1$ and $t \geq \delta$, denote for $1 \leq k \leq k_n$ that

$$Z_{nk} = -tI(X_{nk} < -t) + X_{nk}I(|X_{nk}| \leq t) + tI(X_{nk} > t).$$

Then, $\{Z_{nk}, 1 \leq k \leq k_n\}$ and $\{Z_{nk} - \hat{\mathbb{E}}[Z_{nk}], 1 \leq k \leq k_n\}$ are both ND random variables by the definition of ND random variables for each $n \geq 1$. Hence, we have

$$\begin{aligned} J_4 & = \sum_{n=1}^{\infty} c_n \int_{\delta}^{\infty} \mathbb{V} \left(\sum_{k=1}^{k_n} \left(X_{nk} I(|X_{nk}| \leq t) - \hat{\mathbb{E}} \left[X_{nk} h \left(\frac{X_{nk}}{\delta} \right) \right] \right) > t \right) dt \\ & = \sum_{n=1}^{\infty} c_n \int_{\delta}^{\infty} \mathbb{V} \left(\sum_{k=1}^{k_n} \left(Z_{nk} - \hat{\mathbb{E}}[Z_{nk}] + \hat{\mathbb{E}}[Z_{nk}] - \hat{\mathbb{E}} \left[X_{nk} h \left(\frac{X_{nk}}{\delta} \right) \right] \right) \right. \\ & \quad \left. + (X_{nk} - t)I(X_{nk} > t) + (X_{nk} + t)I(X_{nk} < -t) - X_{nk}I(|X_{nk}| > t) \right) > t \Big) dt \end{aligned}$$

$$\begin{aligned}
 &\leq \sum_{n=1}^{\infty} c_n \int_{\delta}^{\infty} \mathbb{V} \left(\sum_{k=1}^{k_n} \left(Z_{nk} - \hat{\mathbb{E}}[Z_{nk}] + \hat{\mathbb{E}}[Z_{nk}] - \hat{\mathbb{E}} \left[X_{nk} h \left(\frac{X_{nk}}{\delta} \right) \right] \right. \right. \\
 &\quad \left. \left. + (X_{nk} - t)I(X_{nk} > t) + |X_{nk}|I(|X_{nk}| > t) \right) > t \right) dt \\
 &\leq \sum_{n=1}^{\infty} c_n \int_{\delta}^{\infty} \mathbb{V} \left(\sum_{k=1}^{k_n} (Z_{nk} - \hat{\mathbb{E}}[Z_{nk}]) > \frac{t}{4} \right) dt \\
 &\quad + \sum_{n=1}^{\infty} c_n \int_{\delta}^{\infty} \mathbb{V} \left(\sum_{k=1}^{k_n} \hat{\mathbb{E}} \left[Z_{nk} - X_{nk} h \left(\frac{X_{nk}}{\delta} \right) \right] > \frac{t}{4} \right) dt \\
 &\quad + \sum_{n=1}^{\infty} c_n \int_{\delta}^{\infty} \mathbb{V} \left(\sum_{k=1}^{k_n} (X_{nk} - t)I(X_{nk} > t) > \frac{t}{4} \right) dt \\
 &\quad + \sum_{n=1}^{\infty} c_n \int_{\delta}^{\infty} \mathbb{V} \left(\sum_{k=1}^{k_n} |X_{nk}|I(|X_{nk}| > t) > \frac{t}{4} \right) dt \\
 &=: J_5 + J_6 + J_7 + J_8.
 \end{aligned}$$

For the estimate of J_6 , note that for $t \geq \delta$,

$$\begin{aligned}
 &\left| Z_{nk} - X_{nk} h \left(\frac{X_{nk}}{\delta} \right) \right| \\
 &= \left| tI(X_{nk} > t) - tI(X_{nk} < -t) + X_{nk}I(|X_{nk}| \leq t) - X_{nk} h \left(\frac{X_{nk}}{\delta} \right) \right| \\
 &\leq tI(|X_{nk}| > t) + |X_{nk}| \left(I(|X_{nk}| \leq t) - h \left(\frac{X_{nk}}{\delta} \right) \right) \\
 &\leq tI(|X_{nk}| > t) + |X_{nk}|I(\mu\delta < |X_{nk}| \leq t) \\
 &\leq |X_{nk}|I(|X_{nk}| > \mu\delta) \leq |X_{nk}| \left(1 - h \left(\frac{X_{nk}}{\mu\delta} \right) \right).
 \end{aligned}$$

By condition (iii), for all sufficiently large n , we have that

$$\sum_{k=1}^{k_n} \hat{\mathbb{E}} \left[|X_{nk}| \left(1 - h \left(\frac{X_{nk}}{\mu\delta} \right) \right) \right] < 1.$$

Note that $|X_{nk}| \left(1 - h \left(\frac{X_{nk}}{\mu\delta} \right) \right) \leq |X_{nk}|I(|X_{nk}| > \mu^2\delta) \leq CX_{nk}^2 \left(1 - h \left(\frac{X_{nk}}{\mu^2\delta} \right) \right)$. We have by Lemma 2.1 and condition (i) that,

$$\begin{aligned}
 J_6 &\leq \sum_{n=1}^{\infty} c_n \int_{\delta}^{\infty} \mathbb{V} \left(\sum_{k=1}^{k_n} \left| \hat{\mathbb{E}} \left[Z_{nk} - X_{nk} h \left(\frac{X_{nk}}{\delta} \right) \right] \right| > \frac{t}{4} \right) dt \\
 &\leq C \sum_{n=1}^{\infty} c_n \int_{\delta}^{\infty} t^{-2} \left(\sum_{k=1}^{k_n} \hat{\mathbb{E}} \left[\left| Z_{nk} - X_{nk} h \left(\frac{X_{nk}}{\delta} \right) \right| \right] \right)^2 dt
 \end{aligned}$$

$$\begin{aligned}
 &\leq C \sum_{n=1}^{\infty} c_n \int_{\delta}^{\infty} t^{-2} \left(\sum_{k=1}^{k_n} \hat{\mathbb{E}} \left[|X_{nk}| \left(1 - h \left(\frac{X_{nk}}{\mu \delta} \right) \right) \right] \right)^2 dt \\
 &\leq C \sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} \hat{\mathbb{E}} \left[|X_{nk}| \left(1 - h \left(\frac{X_{nk}}{\mu \delta} \right) \right) \right] \\
 &\leq C \sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} \hat{\mathbb{E}} \left[X_{nk}^2 \left(1 - h \left(\frac{X_{nk}}{\mu^2 \delta} \right) \right) \right] \\
 &< \infty.
 \end{aligned}$$

Next, we will show $J_7 < \infty$. By condition (i), we have

$$\begin{aligned}
 J_7 &= \sum_{n=1}^{\infty} c_n \int_{\delta}^{\infty} \mathbb{V} \left(\sum_{k=1}^{k_n} (X_{nk} - t) I(X_{nk} > t) > \frac{t}{4} \right) dt \\
 &\leq \sum_{n=1}^{\infty} c_n \int_{\delta}^{\infty} \mathbb{V} \left(\max_{1 \leq k \leq k_n} |X_{nk}| > t \right) dt \\
 &\leq \sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} \int_{\delta}^{\infty} \mathbb{V}(|X_{nk}| > t) dt \\
 &\leq \sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} \int_{\delta}^{\infty} t^{-2} \hat{\mathbb{E}} \left[X_{nk}^2 \left(1 - h \left(\frac{X_{nk}}{t} \right) \right) \right] dt \\
 &\leq C \sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} \hat{\mathbb{E}} \left[X_{nk}^2 \left(1 - h \left(\frac{X_{nk}}{\delta} \right) \right) \right] \\
 &< \infty.
 \end{aligned}$$

Similar to the proof of $J_7 < \infty$, we can obtain $J_8 < \infty$ immediately.

Finally, we will show $J_5 < \infty$. Note that

$$\begin{aligned}
 Z_{nk}^2 &\leq X_{nk}^2 = X_{nk}^2 I(|X_{nk}| > \delta) + X_{nk}^2 I(|X_{nk}| \leq \delta) \\
 &\leq X_{nk}^2 \left(1 - h \left(\frac{X_{nk}}{\delta} \right) \right) + X_{nk}^2 h \left(\frac{\mu X_{nk}}{\delta} \right).
 \end{aligned}$$

By Lemma 2.2, C_r -inequality, Jensen's inequality and conditions (i) and (ii), it is easy to check that

$$\begin{aligned}
 J_5 &= \sum_{n=1}^{\infty} c_n \int_{\delta}^{\infty} \mathbb{V} \left(\sum_{k=1}^{k_n} (Z_{nk} - \hat{\mathbb{E}}[Z_{nk}]) > \frac{t}{4} \right) dt \\
 &\leq C \sum_{n=1}^{\infty} c_n \int_{\delta}^{\infty} t^{-2} \sum_{k=1}^{k_n} \hat{\mathbb{E}} \left[(Z_{nk} - \hat{\mathbb{E}}[Z_{nk}])^2 \right] dt \\
 &\leq C \sum_{n=1}^{\infty} c_n \int_{\delta}^{\infty} t^{-2} \sum_{k=1}^{k_n} \hat{\mathbb{E}} [Z_{nk}^2] dt \\
 &\leq C \sum_{n=1}^{\infty} c_n \int_{\delta}^{\infty} t^{-2} \sum_{k=1}^{k_n} \left(\hat{\mathbb{E}} \left[X_{nk}^2 \left(1 - h \left(\frac{X_{nk}}{\delta} \right) \right) \right] + \hat{\mathbb{E}} \left[X_{nk}^2 h \left(\frac{\mu X_{nk}}{\delta} \right) \right] \right) dt
 \end{aligned}$$

$$\leq C \sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} \hat{\mathbb{E}} \left[X_{nk}^2 \left(1 - h \left(\frac{X_{nk}}{\delta} \right) \right) \right] + C \sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} \hat{\mathbb{E}} \left[X_{nk}^2 h \left(\frac{\mu X_{nk}}{\delta} \right) \right] < \infty.$$

The proof is completed. \square

The following theorem extends the result of the Theorem 3.2 from complete convergence to complete moment convergence.

THEOREM 3.4. *Let $\{X_{nk}, 1 \leq k \leq k_n, n \geq 1\}$ be an array of rowwise ND random variables in $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$, and $\{c_n, n \geq 1\}$ be a sequence of positive constants. Suppose that:*

- (i) $\sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} \hat{\mathbb{E}} \left[X_{nk}^2 \left(1 - h \left(\frac{X_{nk}}{\tau} \right) \right) \right] < \infty$ for every $\tau > 0$;
- (ii) there exists some constant $1 \leq p \leq 2$ such that $\sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} \hat{\mathbb{E}} [|X_{nk}|^p] < \infty$;
- (iii) $\sum_{k=1}^{k_n} \hat{\mathbb{E}} \left[|X_{nk}| \left(1 - h \left(\frac{X_{nk}}{\mu \delta} \right) \right) \right] \rightarrow 0$ as $n \rightarrow \infty$, for some $\delta > 0$.

If $\hat{\mathbb{E}}[X_{nk}] = 0$, then for all $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} c_n C_V \left(\sum_{k=1}^{k_n} X_{nk} - \varepsilon \right)_+ < \infty. \tag{3.10}$$

If $\hat{\mathbb{E}}[X_{nk}] = 0$, then for all $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} c_n C_V \left(- \sum_{k=1}^{k_n} X_{nk} - \varepsilon \right)_+ < \infty. \tag{3.11}$$

If $\hat{\mathbb{E}}[X_{nk}] = \hat{\mathbb{E}}[X_{nk}] = 0$, then for all $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} c_n C_V \left(\left| \sum_{k=1}^{k_n} X_{nk} \right| - \varepsilon \right)_+ < \infty. \tag{3.12}$$

Proof. It is easy to check that

$$\begin{aligned} & \sum_{n=1}^{\infty} c_n C_V \left(\left| \sum_{k=1}^{k_n} X_{nk} \right| - \varepsilon \right)_+ \\ &= \sum_{n=1}^{\infty} c_n \int_0^{\infty} \mathbb{V} \left(\left| \sum_{k=1}^{k_n} X_{nk} \right| - \varepsilon > t \right) dt \\ &= \sum_{n=1}^{\infty} c_n \int_0^{\infty} \mathbb{V} \left(\sum_{k=1}^{k_n} X_{nk} > \varepsilon + t \right) dt + \sum_{n=1}^{\infty} c_n \int_0^{\infty} \mathbb{V} \left(- \sum_{k=1}^{k_n} X_{nk} > \varepsilon + t \right) dt \\ &= \sum_{n=1}^{\infty} c_n C_V \left(\sum_{k=1}^{k_n} X_{nk} - \varepsilon \right)_+ + \sum_{n=1}^{\infty} c_n C_V \left(- \sum_{k=1}^{k_n} X_{nk} - \varepsilon \right)_+. \end{aligned}$$

Considering $\{-X_{nk}, 1 \leq k \leq k_n, n \geq 1\}$ instead of $\{X_{nk}, 1 \leq k \leq k_n, n \geq 1\}$ in (3.10), we can obtain (3.11) and (3.12) immediately. Therefore, we just need to prove (3.10). It is sufficient to prove that

$$\sum_{n=1}^{\infty} c_n \int_0^{\infty} \mathbb{V} \left(\sum_{k=1}^{k_n} X_{nk} > \varepsilon + t \right) dt < \infty.$$

Note that

$$\begin{aligned} & \sum_{n=1}^{\infty} c_n \int_0^{\infty} \mathbb{V} \left(\sum_{k=1}^{k_n} X_{nk} > \varepsilon + t \right) dt \\ &= \sum_{n=1}^{\infty} c_n \int_0^{\delta} \mathbb{V} \left(\sum_{k=1}^{k_n} X_{nk} > \varepsilon + t \right) dt + \sum_{n=1}^{\infty} c_n \int_{\delta}^{\infty} \mathbb{V} \left(\sum_{k=1}^{k_n} X_{nk} > \varepsilon + t \right) dt \\ &\leq \delta \sum_{n=1}^{\infty} c_n \mathbb{V} \left(\sum_{k=1}^{k_n} X_{nk} > \varepsilon \right) + \sum_{n=1}^{\infty} c_n \int_{\delta}^{\infty} \mathbb{V} \left(\sum_{k=1}^{k_n} X_{nk} > t \right) dt \\ &=: L_1 + L_2. \end{aligned}$$

Obviously, to prove (3.10), it is sufficient to prove that $L_1 < \infty$ and $L_2 < \infty$. It is easily checked that

$$\infty > \sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} \hat{\mathbb{E}} \left[X_{nk}^2 \left(1 - h \left(\frac{X_{nk}}{\tau} \right) \right) \right] \geq \tau^2 \sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} \mathbb{V}(|X_{nk}| > \tau).$$

Hence, condition (i) of Theorem 3.1 holds. Therefore, by Theorem 3.2 we obtain $L_1 < \infty$.

In the following, we will show $L_2 < \infty$. It is easy to see that

$$\begin{aligned} L_2 &= \sum_{n=1}^{\infty} c_n \int_{\delta}^{\infty} \mathbb{V} \left(\sum_{k=1}^{k_n} X_{nk} > t \right) dt \\ &\leq \sum_{n=1}^{\infty} c_n \int_{\delta}^{\infty} \mathbb{V} \left(\sum_{k=1}^{k_n} \left(X_{nk} - \hat{\mathbb{E}} \left[X_{nk} h \left(\frac{X_{nk}}{\delta} \right) \right] \right) > \frac{t}{2} \right) dt \\ &\quad + \sum_{n=1}^{\infty} c_n \int_{\delta}^{\infty} \mathbb{V} \left(\sum_{k=1}^{k_n} \hat{\mathbb{E}} \left[X_{nk} h \left(\frac{X_{nk}}{\delta} \right) \right] > \frac{t}{2} \right) dt \\ &=: L_3 + L_4. \end{aligned}$$

For the estimate of L_4 , by condition (iii), for all sufficiently large n , we have that

$$\sum_{k=1}^{k_n} \hat{\mathbb{E}} \left[|X_{nk}| \left(1 - h \left(\frac{X_{nk}}{\delta} \right) \right) \right] < 1.$$

Noting that $1 \leq p \leq 2$, we have

$$|X_{nk}| \left(1 - h \left(\frac{X_{nk}}{\delta} \right) \right) \leq |X_{nk}| I(|X_{nk}| > \mu \delta) \leq (\mu \delta)^{1-p} |X_{nk}|^p.$$

By Lemma 2.1, condition (ii) and $\hat{\mathbb{E}}[X_{nk}] = 0$, we get that

$$\begin{aligned}
 L_4 &= \sum_{n=1}^{\infty} c_n \int_{\delta}^{\infty} \mathbb{V} \left(\sum_{k=1}^{k_n} \left(-\hat{\mathbb{E}}[X_{nk}] + \hat{\mathbb{E}} \left[X_{nk} h \left(\frac{X_{nk}}{\delta} \right) \right] \right) \right) > \frac{t}{2} dt \\
 &\leq \sum_{n=1}^{\infty} c_n \int_{\delta}^{\infty} \mathbb{V} \left(\sum_{k=1}^{k_n} \hat{\mathbb{E}} \left[|X_{nk}| \left(1 - h \left(\frac{X_{nk}}{\delta} \right) \right) \right] \right) > \frac{t}{2} dt \\
 &\leq C \sum_{n=1}^{\infty} c_n \int_{\delta}^{\infty} t^{-2} \left(\sum_{k=1}^{k_n} \hat{\mathbb{E}} \left[|X_{nk}| \left(1 - h \left(\frac{X_{nk}}{\delta} \right) \right) \right] \right)^2 dt \\
 &\leq C \sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} \hat{\mathbb{E}} \left[|X_{nk}| \left(1 - h \left(\frac{X_{nk}}{\delta} \right) \right) \right] \\
 &\leq C \sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} \hat{\mathbb{E}}[|X_{nk}|^p] \\
 &< \infty.
 \end{aligned}$$

For L_3 , according to the proof of J_2 in Theorem 3.3, it suffices to prove that condition (ii) of Theorem 3.3 is satisfied. Since $1 \leq p \leq 2$, we have by condition (ii) that,

$$\begin{aligned}
 \sum_{k=1}^{k_n} \hat{\mathbb{E}} \left[X_{nk}^2 h \left(\frac{\mu X_{nk}}{\delta} \right) \right] &= \frac{\delta^2}{u^2} \sum_{k=1}^{k_n} \hat{\mathbb{E}} \left[\left(\frac{\mu X_{nk}}{\delta} \right)^2 h \left(\frac{\mu X_{nk}}{\delta} \right) \right] \\
 &\leq \frac{\delta^2}{u^2} \sum_{k=1}^{k_n} \hat{\mathbb{E}} \left[\left(\frac{\mu |X_{nk}|}{\delta} \right)^p h \left(\frac{\mu^2 X_{nk}}{\delta} \right) \right] \\
 &\leq C \sum_{k=1}^{k_n} \hat{\mathbb{E}}[|X_{nk}|^p],
 \end{aligned}$$

which yields condition (ii) of Theorem 3.3. Hence, we can obtain $L_3 < \infty$. The proof is completed. \square

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