

## SOME DOMINATION INEQUALITIES FOR SPECTRAL ZETA KERNELS ON CLOSED RIEMANNIAN MANIFOLDS

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*Abstract.* We first prove Kato's inequalities for the Laplacian and a Schrödinger-type operator on smooth functions on closed Riemannian manifolds. We then apply the result to establish some new domination inequalities for spectral zeta functions and their related spectral zeta kernels on  $n$ -dimensional unit spheres using Kato's inequalities and majorisation techniques. Our results are the generalisations of Kato's comparison inequalities for Riemannian surfaces to  $n$ -dimensional closed Riemannian manifolds.

### 1. Introduction and Preliminaries

Consider the Riemann zeta function  $\zeta_R$  defined by  $\zeta_R : \{s \in \mathbb{C} : \Re(s) > 1\} \rightarrow \mathbb{C}$  with

$$\zeta_R(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}; \quad \Re(s) > 1. \quad (1)$$

From (1), we deduce that

$$\sum_{k=1}^{\infty} \left| \frac{1}{k^s} \right| = \sum_{k=1}^{\infty} \frac{1}{k^{\Re(s)}}.$$

Hence, the series on the right-hand-side converges absolutely if and only if  $\Re(s) > 1$ . Consequently, the Riemann zeta function defined by (1) above is holomorphic in the region  $\Re(s) > 1$ . It, however, admits a meromorphic continuation to the whole  $s$ -complex plane with only simple pole at  $s = 1$  and has residue 1. For details, see [20]. A generalization of the Riemann zeta function is the Hurwitz zeta function, (see e.g. [9]), defined by

$$\zeta_H(s, a) = \sum_{k=0}^{\infty} \frac{1}{(k+a)^s}; \quad s \in \mathbb{C}, \quad 0 < a \leq 1, \quad \Re(s) > 1.$$

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We denote the Laplace-Beltrami operator simply called the Laplacian in many literature acting on smooth functions on  $n$ -dimensional Riemannian manifold  $(M, g)$  by  $\Delta_g$  where locally,

$$\Delta_g = -\operatorname{div}(\operatorname{grad}) = -\frac{1}{\sqrt{|g|}} \sum_{i,j} \frac{\partial}{\partial x^i} \left( \sqrt{|g|} g^{ij} \frac{\partial}{\partial x^j} \right) \tag{2}$$

with  $g^{ij}$  being the components of the dual metric on the cotangent bundle  $T_x^*M$ . The operator  $\Delta_g$  extends to a self-adjoint operator on  $L^2(M) \supset H^2(M) \rightarrow L^2(M)$  with compact resolvent. This implies that there exists an orthonormal basis  $\psi_k \in L^2(M)$  consisting of eigenfunctions such that

$$\Delta_g \psi_k = \lambda_k \psi_k \tag{3}$$

where the eigenvalues are listed with multiplicities; for details, one may see for example [14, 17, 10, 7, 18] and [21] among many literature. Consequently, we define the spectral zeta function which is another generalisation of the Riemann zeta function of the Laplacian on smooth functions on closed Riemannian manifolds by

$$\zeta_g(s) = \sum_{k=1}^{\infty} \frac{1}{\lambda_k^s}; \quad \Re(s) > \frac{n}{2}. \tag{4}$$

Kato in [16] in effort to prove essential self-adjointness for Schrödinger operators under very mild restrictions on the potential term introduced Kato’s inequality and the Kato class of potentials. These were combined to give new insight in analysis and geometry of Riemannian manifolds. It computed explicit majorisation Kato’s inequalities for the trace of the heat operator and the semigroup generated by Bochner Laplacian on forms.

A characterisation of the generators of positive semigroups were done in [1]. It showed that the semigroup consists of positive operators if and only if it satisfies the abstract version of the kato’s inequality. It also characterised the domination of semigroups by an inequality for their generators. In continuation of the characterisation, Hess, et al [12] constructed Kato’s inequalities for several Laplacians on tangent bundles of some Riemannian manifolds and studied their semigroups dominations.

In a similar study, Bär [5] used Kato’s comparison principle for heat semi-groups to derive estimates for trace of the heat operator on surfaces with variable curvature. These estimates are from above for positively curved surfaces of genus 0 and from below for genus greater or equal to 2. It was shown that the estimates are asymptotically sharp for small time and for large time in the case of positive curvature. As a consequence, it estimated the corresponding spectral zeta function by the Riemann zeta function for the surfaces. Specifically, Bär [5] derived bounds of the Laplace spectrum on closed oriented surfaces. This line of research has continued to develop, for instance, [6] had demonstrated Kato’s inequality on different Riemannian geometries. Most recently, [2] discussed some interesting consequences of Kato’s inequality on some differential operators.

In this paper, we extend these results by constructing Kato’s bounds for the spectral zeta kernel of a perturbed Laplacian of the Schrödinger-type in comparison with spectral zeta kernel of the Laplacian on  $(M, g)$ . We have the following results.

THEOREM 1. Let  $\psi \in L^1_{loc}(M)$  and suppose that the distributional Laplacian  $\Delta_g \psi \in L^1_{loc}(M)$ . Then

$$\Delta_g |\psi| \geq \Re \left( \text{sgn}(\psi) \Delta_g \psi \right) \tag{5}$$

in the sense of distribution.

THEOREM 2. The semigroup  $T$  satisfies Kato's inequality, that is,

$$\langle \text{sgn}(\psi) A \psi, \phi \rangle \leq \langle |\psi|, A^* \phi \rangle$$

for  $\psi \in \text{domain}(A)$ ,  $\phi \in \text{domain}(A^*)$  where  $T^*$  is the adjoint of  $A$ .

THEOREM 3. For all  $\psi \in C^\infty(M)$ ,  $H|\psi| \geq \Re \left( \text{sgn}(\psi) V \psi \right)$ .

THEOREM 4. Let  $Z_{S^n}(s, \rho_n)$  be the spectral zeta kernel of the Schrödinger-type operator,  $H = \Delta + c$ , on smooth functions of the  $n$ -dimensional unit sphere,  $S^n$ , where  $c$  is the potential operator that multiplies by  $\rho_n = \frac{n-1}{2}$ . Then

$$Z_{S^n}(s, x, y) \prec \zeta_{S^n}(s, x, y). \tag{6}$$

Our results show that the trace,  $\text{Tr}(\exp(-tH))$ , of the operator  $\exp(-tH)$  where  $H = \Delta_g + V$  for smooth potential  $V$  is majorised by  $\text{Tr}(\exp(-t\Delta_g))$  on  $(M, g)$ . We used Kato's inequality to prove that the spectral zeta kernel of  $H$  is majorised by that of  $\Delta_g$  on the  $n$ -dimensional unit sphere. The domination of the traces of the semi-group  $e^{-tH}$  by  $e^{-t\Delta_g}$  is shown to be

$$\text{Tre}^{-tH} \leq n \text{Tre}^{-t\Delta}, \quad t > 0. \tag{7}$$

This result is in tandem with those of [11] and references therein. This leads to the comparison of the spectra of the generators  $H$  and  $\Delta_g$  on  $M$ . The inequality (7) of course yields inequalities for the associated Riemann zeta as well as the spectral zeta functions and the associated spectral zeta kernel on manifolds of certain dimensions.

### 2. Majorisation and the Kato's inequalities

Majorisation techniques in conjunction with Kato's inequalities are used to compare heat operators, the spectral zeta functions and the zeta kernels. Majorisation is a pre-order of sequences of real numbers. We make the following formal definitions on majorisation and specify its connotation as it is used here.

DEFINITION 1. Let  $x, y \in \mathbb{R}^n$  and let  $x^\downarrow$  and  $y^\downarrow$  be vectors with the same components as  $x$  and  $y$  respectively. We say that  $x$  weakly majorises  $y$  and write this as  $x \succ_w y$  if and only if

$$\sum_{j=1}^k x^\downarrow \geq \sum_{j=1}^k y^\downarrow \quad \text{for } k = 1, 2, \dots, n-1. \tag{8}$$

That is, if  $x = (x_1, x_2, \dots, x_n)$  and  $y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$  then  $x \succ_w y$  if and only if

$$\begin{aligned} x_1 &\geq y_1, \\ x_1 + x_2 &\geq y_1 + y_2, \\ &\vdots \\ x_1 + x_2 + \dots + x_k &\geq y_1 + y_2 + \dots + y_k. \end{aligned}$$

Equivalently, if  $y$  weakly majorises  $x$  we write  $x \prec_w y$  or  $y \succ_w x$ .

If in addition to (8), we get that  $\sum_{j=1}^n x_j = \sum_{j=1}^n y_j$  then we say that  $x$  majorises  $y$  and write this as  $x \succeq y$ .

DEFINITION 2. Let  $\Omega \subset \mathbb{R}^n$  and let  $\psi : \Omega \rightarrow \mathbb{R}$ . We call the function  $\psi : \Omega \rightarrow \mathbb{R}$  Schur convex if  $x \succeq y$  implies that  $\psi(x) \geq \psi(y) \forall x \in \Omega$ ; see e.g. [26, 24, 25].

DEFINITION 3. We say  $x$  dominates  $y$  and write  $x \succ y$  whenever (8) is satisfied and call  $\psi : \Omega \rightarrow \mathbb{R}$  Schur convex if  $\psi''(x) > 0 \forall x \in \Omega \subset \mathbb{R}^n$  and  $\psi(x) \geq \psi(y)$ .

We will also use the fact that if  $x \succ y$  then  $x^{-1} \prec y^{-1}$ . We make the next definitions following [13] and [1].

DEFINITION 4. Let  $\psi \in C^\infty(M)$  be any function. Define the sign function  $sgn(\psi)$  by

$$sgn(\psi)(x) = \begin{cases} \bar{\psi}|\psi|^{-1} & ; \text{ if } \psi(x) \neq 0, \\ 0 & ; \text{ if } \psi(x) = 0. \end{cases} \tag{9}$$

For any  $\psi, \phi \in C^\infty(M)$ , the following properties are satisfied:

$$\begin{aligned} sgn(\psi)\psi &= \frac{\bar{\psi}\psi(x)}{|\psi|} = |\psi|. \\ sgn(\psi)\phi &= 0, \text{ if } \psi \perp \phi. \\ sgn(\psi)\phi &= \frac{\bar{\psi}\psi}{|\phi|} \text{ if } \psi \not\perp \phi. \\ |sgn(\psi)\phi| &\leq |\phi|. \\ \langle sgn(\psi)T\psi, \phi \rangle &\leq \langle |\psi|, T^*\phi \rangle \end{aligned}$$

where  $T$  a generator of a strongly continuous semigroup such as the heat operator on the manifold and  $T^*$  is the adjoint of  $T$ ; see e.g [1].

For any  $\varepsilon > 0$ , we also define a regularised absolute value of  $\psi$  by

$$\psi_\varepsilon(x) := \sqrt{|\psi(x)|^2 + \varepsilon^2}. \tag{10}$$

So,  $\lim_{\varepsilon \rightarrow 0} \psi_\varepsilon(x) = |\psi(x)|$  point-wise. Thus, the regularised sign function is

$$\operatorname{sgn}_\varepsilon \psi := \begin{cases} \psi |\psi|^{-1} & ; \text{ on } \operatorname{supp}(\psi), \\ \varepsilon & ; \text{ otherwise.} \end{cases} \quad (11)$$

for  $\psi \in C^\infty(T_x M)$ .

We now give the Kato's inequality for the case of the Laplacian.

LEMMA 1. *Let  $\Delta_g$  be the Laplacian defined by (2) and let  $\psi \in C^\infty(M)$ . Then,*

$$\Delta_g |\psi| \geq \Re \left( \operatorname{sgn}(\psi) \Delta_g \psi \right) \quad (12)$$

except where  $|\psi|$  is not differentiable.

*Proof of Lemma 1.* Observe from (10) that  $\psi_\varepsilon \geq |\psi|$ . Differentiating

$$\begin{aligned} \psi_\varepsilon^2 &= |\psi|^2 + \varepsilon^2 \text{ gives} \\ \psi_\varepsilon \nabla_g \psi_\varepsilon &= \Re(\bar{\psi} \nabla_g \psi). \end{aligned} \quad (13)$$

Squaring (13) and using that  $\psi_\varepsilon \geq |\psi|$  gives

$$|\nabla_g \psi_\varepsilon| \leq \psi_\varepsilon^{-1} |\psi| |\nabla_g \psi| \leq |\nabla_g \psi|. \quad (14)$$

Now take the divergence of (13) to obtain

$$|\nabla_g \psi_\varepsilon|^2 + \psi_\varepsilon \Delta_g \psi_\varepsilon = |\nabla_g \psi|^2 + \Re(\bar{\psi} \Delta_g \psi).$$

By (14), this is equivalent to

$$\psi_\varepsilon \Delta_g \psi_\varepsilon \geq \Re(\bar{\psi} \Delta_g \psi). \quad (15)$$

So, using that  $\operatorname{sgn}_\varepsilon \psi = \bar{\psi} |\psi_\varepsilon|^{-1}$  by (11), equation (15) becomes

$$\Delta_g \psi_\varepsilon \geq \Re \left( \operatorname{sgn}_\varepsilon(\psi) \Delta_g \psi \right). \quad (16)$$

Now, since  $\Delta_g \psi_\varepsilon \rightarrow \Delta_g |\psi|$  point-wise and  $\operatorname{sgn}_\varepsilon(\psi) \rightarrow \operatorname{sgn}(\psi)$  point-wise, take limit in (16) as  $\varepsilon \rightarrow 0$  to conclude that

$$\Delta_g |\psi| \geq \Re \left( \operatorname{sgn}(\psi) \Delta_g \psi \right).$$

except where  $|\psi|$  is not differentiable.  $\square$

Our goal is to extend the Kato's inequality (12) to a more general class of functions which would prove our first main result given as Theorem 1. We make one more definition.

DEFINITION 5. (Approximate identity) Let  $\omega \in C^\infty(\mathbb{R}^n)$   $\omega \geq 0$ , and  $\int \omega(x)dx = 1$ . For  $\varepsilon > 0$ , we define

$$\omega_\varepsilon(x) := \frac{\omega(\frac{x}{\varepsilon})}{\varepsilon^n}.$$

Then,

$$\int \omega_\varepsilon(x)dx = 1.$$

We define a map  $I_\varepsilon$  by

$$I_\varepsilon \psi := \omega_\varepsilon * \psi \tag{17}$$

whenever the right-hand-side of (17) exists and where

$$(f * g)(x) := \int f(x - y)g(y)dy$$

is the convolution of  $f$  and  $g$ . The map  $I_\varepsilon$  is called an approximation of the identity, or simply, an approximate identity.

The approximate identity  $I_\varepsilon$  has the following properties.

- (1.) If  $\psi \in L^1_{loc}(\mathbb{R}^n)$  then  $I_\varepsilon \psi \in C^\infty(\mathbb{R}^n)$ .
- (2.) If  $\psi$  is differentiable then  $\frac{\partial}{\partial x_i}(I_\varepsilon \psi) = I_\varepsilon \frac{\partial \psi}{\partial x_i}$ ; that is, the approximate identity  $I_\varepsilon$  commutes with the differentiation operator  $\frac{\partial}{\partial x_i}$ .
- (3.) The map  $I_\varepsilon : L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)$  is bounded; and  $\|I_\varepsilon\| \leq 1$ .
- (4.) For any  $\psi \in L^p(\mathbb{R}^n)$ ,  $\lim_{\delta \rightarrow 0} \|I_\varepsilon \psi - \psi\|_p = 0$ .
- (5.) For any  $\psi \in L^1(\mathbb{R}^n)$ ,  $I_\varepsilon \psi \rightarrow \psi$  as  $\varepsilon \rightarrow 0$  in the sense of distribution;

see e.g. [10], [13] and [15] for details.

*Proof of Theorem 1.* Let  $\psi \in L^1_{loc}(M)$ . From the properties of  $I_\varepsilon$  we see that  $I_\varepsilon \psi$  is smooth for any  $\varepsilon > 0$ . So, inserting  $I_\varepsilon \psi$  in the theorem in place of  $\psi$ , we obtain for any  $\varepsilon > 0$  that

$$\Delta_g(I_\varepsilon \psi)_\varepsilon \geq \mathfrak{R}(sgn_\varepsilon(I_\varepsilon \psi)\Delta_g(I_\varepsilon \psi)). \tag{18}$$

To remove the approximate identity in equation (18), we use the fact that since  $sgn_\varepsilon(I_\varepsilon \psi)\Delta_g(I_\varepsilon \psi)$  is a sequence in  $L^1_{loc}(M)$ , there exists a subsequence of  $sgn_\varepsilon(I_\varepsilon \psi)\Delta_g(I_\varepsilon \psi)$  which converges to  $sgn_\varepsilon(\psi)\Delta_g(\psi)$  except possibly on a set of measure zero (almost everywhere).

Besides, since  $\Delta_g \psi \in L^1_{loc}(M)$ , it follows from the properties of  $I_\varepsilon$  that the limit of  $\Delta_g(I_\varepsilon \psi)$  as  $\varepsilon \rightarrow 0^+$  is  $\Delta_g \psi \in L^1_{loc}(M)$ . Also by the boundedness of  $sgn_\varepsilon(I_\varepsilon \psi)$  we have that

$$\lim_{\varepsilon \rightarrow 0^+} sgn_\varepsilon(I_\varepsilon \psi) \left( \Delta_g(I_\varepsilon \psi) - \Delta_g \psi \right) = 0$$

in the sense of distribution.

It is now left to prove that there is a subsequence such that  $sgn_\epsilon(I_\epsilon \psi)\Delta_g(I_\epsilon \psi)$  converges to  $sgn_\epsilon(\psi\Delta_g \psi)$ . But Lebesgue dominated convergence theorem ensures this. Hence, taking this subsequential limit in (18) gives

$$\Delta_g|\psi|_\epsilon \geq \Re\left((sgn_\epsilon \psi)\Delta_g \psi\right).$$

Therefore,  $\lim_{\epsilon \rightarrow 0} \left(\Delta_g|\psi|_\epsilon \geq \Re\left((sgn_\epsilon \psi)\Delta_g \psi\right)\right)$  gives  $\Delta_g|\psi| \geq \Re\left(sgn(\psi)\Delta_g \psi\right)$  in the sense of distribution as required.  $\square$

**COROLLARY 1.** *Suppose  $\psi$  and  $\Delta\psi \in L^2(M)$ . Then,*

$$\Delta|\psi| \geq \Re\left(\bar{\psi}\psi^{-1}\Delta\psi\right).$$

*Proof of Corollary 1.* Since  $\psi$  and  $\Delta\psi \in L^2(M)$  then there exists a sequence  $\{\psi_j\} \in \mathcal{D}$  which converges in  $\mathcal{D}$  such that in  $L^2$ -norm  $\nabla\psi_j \rightarrow \nabla\psi$  and  $\Delta\psi_j \rightarrow \Delta\psi$ . Moreover,

$$\|\nabla\psi\|^2 = \langle \nabla\psi, \nabla\psi \rangle_g = -\langle \psi, \Delta\psi \rangle_g \leq \|\psi\|_2 \|\Delta\psi\|_2.$$

By the continuity of these estimates, it follows that  $\Delta|\psi| \geq \Re\left(sgn(\psi)\Delta\psi\right) \forall \psi \in L^2(M)$ .  $\square$

We can extend this concept to semigroups. Let  $T$  be a strongly continuous semigroup with differential operator generator  $A$  defined by

$$Ax = \lim_{t \downarrow 0} \frac{1}{t} \left(T(t) - id\right)x$$

for  $x \in M$ . That is  $T$  satisfies

1.  $T(0) = id$  that is an identity operator on  $M$ .
2. For all  $0 \leq t, s \in M$ ,  $T(t+s) = T(t)T(s)$ .
3. For all  $x_0 \in M$ ,  $\|T(t)x_0 - x_0\| \rightarrow 0$  as  $t \downarrow 0$ . That is the strong operator topology. See e.g. [10] and [22].

*Proof of Theorem 2.* Let  $\psi \in \text{domain}(A)$  and  $\phi \in \text{domain}(A^*)$  then

$$\begin{aligned} \langle sgn(\psi)A\psi, \phi \rangle &= \lim_{t \downarrow 0} \frac{1}{t} \langle sgn(\psi)(T(t)\psi - \psi), \phi \rangle \\ &= \lim_{t \downarrow 0} \frac{1}{t} \langle sgn(\psi)T(t)\psi - |\psi|, \phi \rangle \\ &\leq \lim_{t \downarrow 0} \frac{1}{t} \langle |T(t)\psi| - |\psi|, \phi \rangle \\ &\leq \lim_{t \downarrow 0} \frac{1}{t} \langle |\psi|, (T^*(t)\phi - \phi) \rangle = \langle |\psi|, T^*\phi \rangle. \quad \square \end{aligned}$$

Of course, in the case that  $A$  is self-adjoint, the the Kato's inequality for the semi-group is

$$\langle \text{sgn}(\psi)A\psi, \phi \rangle \leq \langle |\psi|, A\phi \rangle.$$

Now consider a Schrödinger type operator  $H = \Delta + V$  on  $C^\infty(M)$  where  $V$  is a nonnegative multiplication potential. We now prove the result tagged Theorem (3):

*Proof of Theorem 3.* By definition of  $I_\epsilon$  we have

$$HI_\epsilon|\psi| = \Delta I_\epsilon|\psi| + VI_\epsilon|\psi| = I_\epsilon(\Delta|\psi| + V|\psi|) \geq 0 \tag{19}$$

since  $\Delta|\psi| + V|\psi| \geq 0$ . But,

$$\langle \Delta I_\epsilon|\psi|, I_\epsilon|\psi| \rangle_g = -\|\nabla(I_\epsilon|\psi|)\|_2^2 \leq 0. \tag{20}$$

Since by (19) the left side of (20) is nonnegative then  $\nabla(I_\epsilon|\psi|) = 0$  in  $L^2$ -sense. Therefore  $I_\epsilon|\psi| = V|\psi| = c \geq 0$  with  $c$  a constant. Since  $|\psi| \in L^2(M)$  and  $|I_\epsilon|\psi| \rightarrow |\psi|$  in the  $L^2$ -sense, we conclude that  $c = 0$  and so  $I_\epsilon|\psi| = 0$ ,  $|\psi| = 0$  and so  $\psi = 0$ . Hence,  $H|\psi| \geq \Delta|\psi| \geq \Re(\text{sgn}(\psi)\Delta\psi) \geq \Re(\text{sgn}(\psi)V\psi)$ .  $\square$

We obtain corollaries of this result by considering more general operators on a Hilbert space. Let  $X$  and  $Y$  be operators on Hilbert space of functions  $\mathcal{H}$ . We assume that  $X$  and  $Y$  are well defined generators of the heat semigroups  $e^{-tX}$  and  $e^{-tY}$  satisfying

$$\left. \begin{aligned} (\frac{\partial}{\partial t} + X)e^{-tX}f &= 0 \\ \lim_{t \rightarrow 0} e^{-tX}f &= f \end{aligned} \right\} \tag{21}$$

and similarly

$$\left. \begin{aligned} (\frac{\partial}{\partial t} + Y)e^{-tY}f &= 0 \\ \lim_{t \rightarrow 0} e^{-tY}f &= f \end{aligned} \right\} \tag{22}$$

for some  $f \in \mathcal{H}$ . In the lemma that follows, we denote  $e^{-tX}f$  just by  $e^{-tX}$ .

From here on, we suppress the subscript  $g$  in  $\zeta_g(s)$  and  $\Delta_g$ . We simply write  $\zeta(s)$  and  $\Delta$  for  $\zeta_g(s)$  and  $\Delta_g$  respectively, unless for purpose of emphasis.

LEMMA 2. *The Laplacian  $\Delta$  in (2) satisfies  $\Delta e^{-t\Delta} = e^{-t\Delta}\Delta$ .*

*Proof of Lemma 2.* A direct computation shows this. That is,

$$\begin{aligned} \Delta e^{-t\Delta}f(x) &= \Delta_x \left( \int_M K(t,x,y)f(y)dV(y) \right) \\ &= \int_M \Delta_x K(t,x,y)f(y)dV(y) \\ &= -\partial_t \int_M K(t,x,y)f(y)dV(y) \end{aligned}$$

and by symmetry of  $K(t, x, y)$  in  $x$  and  $y$  we have

$$\begin{aligned} e^{-t\Delta}\Delta f(x) &= \int_M K(t, x, y)\Delta_y f(y)dV(y) \\ &= \int_M \Delta_y K(t, x, y)f(y)dV(y) \\ &= -\partial_t \int_M K(t, x, y)f(y)dV(y) \end{aligned}$$

which proves the lemma.  $\square$

We note that the heat semigroup  $e^{-t(X+Y)}$  satisfies the Duhamel's formula

$$e^{-t(X+Y)} = e^{-tX} - \int_0^t e^{-(t-s)(X+Y)} Y e^{-sX} ds \quad (23)$$

where  $X$  and  $Y$  are Laplacian-like operators. This is proved as a Theorem in [23] and [8].

We now highlight the generalisation of the Riemann zeta function, namely, the spectral zeta function, which is the function of interest in this paper. The spectral zeta function is explicitly defined through the operator  $\Delta_g^{-s}$  and its integral kernel  $\zeta_g(s, x, y)$ , also called the zeta kernel. The operator  $\Delta_g^{-s}$  is uniquely defined by the following properties (see e.g [20]):

- (1.) it is linear on  $L^2(M)$  with 1-dimensional null space consisting of constant functions. This ensures that the smallest eigenvalue of  $\Delta_g^{-s}$  is 0 of multiplicity 1 with corresponding eigenfunction  $\frac{1}{\sqrt{V}}$  where  $V$  is the volume of  $M$ ;
- (2.) the image of  $\Delta_g^{-s}$  is contained in the orthogonal complement of constant functions in  $L^2(M)$  i.e.

$$\int_M \Delta_g^{-s} \psi dV_g = 0 \quad \forall \psi \in L^2(M) \text{ constant; and}$$

- (3.)  $\Delta_g^{-s} \psi_k(x) = \lambda_k^{-s} \psi_k(x)$  for all  $\psi_k$ ;  $k > 0$  an orthonormal basis of eigenfunction of  $\Delta_g$ .

So, for  $\Re(s) > \frac{n}{2}$ , we see by property (3.) that  $\Delta_g^{-s}$  is trace class, with trace given by the spectral zeta function, namely

$$\zeta_g(s) = \sum_{k=1}^{\infty} \frac{1}{\lambda_k^s} = \text{Tr}(\Delta_g^{-s}) = \int_M \zeta_g(s, x, x) dV; \quad \Re(s) > \frac{n}{2}. \quad (24)$$

Let  $\{\psi_k\}_{k=1}^{\infty}$  be an orthonormal eigenbasis for  $\Delta_g$  corresponding to the eigenvalues  $\{\lambda_k\}_{k=1}^{\infty}$  listed with multiplicities. It is proved in [20] that the zeta kernel,  $\zeta_g(s, x, y)$ , equals

$$\zeta_g(s, x, y) = \sum_{k=1}^{\infty} \frac{\psi_k(x)\bar{\psi}_k(y)}{\lambda_k^s}; \quad \Re(s) > \frac{n}{2}. \quad (25)$$

A relationship between the zeta kernel and the heat kernel enables one to define the spectral zeta kernel explicitly. The heat kernel,  $K(t, x, y) : (0, \infty) \times M \times M \rightarrow R$ , is a continuous function on  $(0, \infty) \times M \times M$ . It is the so-called fundamental solution to the heat equation, i.e, it is the unique solution to the following system of equations:

$$\left. \begin{aligned} \left(\frac{\partial}{\partial t} + \Delta_x\right)K(t, x, y) &= 0 \\ \lim_{t \rightarrow 0} \int_M K(t, x, y) \psi(y) dV_y &= \psi(x) \end{aligned} \right\} \tag{26}$$

for  $t > 0$ ;  $x, y \in M$  and  $\Delta_x$  is the Laplacian acting on any  $\psi \in L^2(M)$ , where the limit in the second equation of (26) is uniform for every  $\psi \in C^\infty(M)$ .

The heat operator  $e^{-t\Delta} : L^2(M) \rightarrow L^2(M)$  is the operator defined by the integral kernel  $K(t, x, y)$  as

$$(e^{-t\Delta}\psi)(y) := \int_M K(t, x, y)\psi(x)dV_x$$

for  $\psi \in L^2(M)$ . The heat kernel is symmetric in the space variables, that is  $K(t, x, y) = K(t, y, x) \forall x, y \in M$ . Thus the heat operator is self-adjoint, that is, for  $\psi_1, \psi_2 \in L^2(M)$  we have

$$\begin{aligned} \langle e^{-t\Delta}\psi_1, \psi_2 \rangle_{L^2(M)} &= \int_M \left\{ \int_M K(t, x, y)\psi_1(y)dV_y \right\} \bar{\psi}_2(x)dV_x \\ &= \int_M \left\{ \int_M K(t, y, x)\psi_2(x)dV_x \right\} \bar{\psi}_1(y)dV_y = \langle \psi_1, e^{-t\Delta}\psi_2 \rangle_{L^2(M)}. \end{aligned}$$

Now returning to the heat kernel, let  $\{\psi_k\}_{k=0}^\infty$  with  $\int_M \psi_k(x)\bar{\psi}_l(x)dV_g(x) = \delta_{kl}$  be orthonormal basis of eigenfunctions of  $\Delta$  with corresponding eigenvalues  $\{\lambda_k\}$  listed with multiplicities. Then  $\{\psi_k\}_{k=0}^\infty$  are also eigenfunctions of the heat operator with corresponding eigenvalues  $\{e^{-\lambda_k t}\}$ . In terms of these eigenfunctions, Mercer’s theorem, (see e.g. [7]), implies that  $e^{-t\Delta}$  is trace-class for all  $t > 0$ . Thus, one can write the heat kernel as

$$K(t, x, y) = \sum_{k=0}^\infty e^{-\lambda_k t} \psi_k(x)\bar{\psi}_k(y).$$

The convergence for all  $t > 0$  is uniform on  $M \times M$ . In particular, the trace of the heat operator

$$\text{Tr}(e^{-\Delta_g t}) = \sum_{k=0}^\infty e^{-\lambda_k t} |\psi_k(x)|^2 = \sum_{k=0}^\infty e^{-\lambda_k t} = \int_M K(t, x, x)dV_g(x) < \infty. \tag{27}$$

The zeta kernel and the heat kernel are related by

$$\zeta_g(s, x, y) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \left( K(t, x, y) - \frac{1}{V} \right) dt; \quad \Re(s) > \frac{n}{2}.$$

To see this, we observe that for any  $x > 0$  and  $\Re(s) > 0$ ,

$$x^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty e^{-xt} t^{s-1} dt$$

since a change of variable from, say,  $xt$  to  $\tau$  gives  $x^{-s}$  and since  $\Gamma(s)$  is holomorphic for  $\Re(s) > 0$ . Consequently,

$$\lambda_k^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty e^{-\lambda_k t} t^{s-1} dt.$$

Thus,

$$\zeta_g(s, x, y) = \sum_{k=1}^\infty \left[ \psi_k(x) \overline{\psi}_k(y) \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-\lambda_k t} dt \right]; \quad \Re(s) > \frac{n}{2}.$$

Therefore,

$$\zeta_g(s, x, y) = \frac{1}{\Gamma(s)} \int_0^\infty \left( \sum_{k=1}^\infty e^{-\lambda_k t} \psi_k(x) \overline{\psi}_k(y) \right) t^{s-1} dt.$$

Thus,

$$\zeta_g(s, x, y) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \left( K(t, x, y) - \frac{1}{V} \right) dt, \quad \Re(s) > \frac{n}{2}.$$

We have another result as the corollary that follows.

**COROLLARY 2.** *The Schrödinger-like operator  $H = \Delta + V$ , where  $V \in L^2_{loc}$  and  $V \geq 0$ , is essentially self-adjoint on  $C^\infty_0(M)$ .*

*Proof of Corollary 2.* Since the domain  $D(H^*) \subset L^2(M)$ , it suffices to show that  $\ker(H^* + 1) = \{0\}$ . This implies that if

$$(\Delta + V + 1)\psi = 0, \quad \text{for } \psi \in L^2(M) \tag{28}$$

then  $\psi = 0$ . We prove (28) by Kato’s inequality. Since  $\psi \in L^2(M)$  and  $V \in L^2_{loc}(M)$  it follows by Cauchy-Schwarz inequality that  $V\psi \in L^1_{loc}(M)$  following the inclusion  $L^2 \subset L^2_{loc} \subset L^1_{loc}$  from the estimate

$$\int_M 1 \cdot |\psi(x)| dV_g \leq V_g \sqrt{\int_M |\psi(x)|^2 dV_g}$$

where  $V_g$  is the volume of  $M$ . This implies that  $\psi \in L^1_{loc}(M)$ .

Using the Kato’s inequality, we have

$$\Delta|\psi| \geq \Re((sgn\psi)\Delta\psi) \geq \Re((sgn\psi)(V+1)\psi) = |\psi|(V+1) \geq 0.$$

Hence, the function  $\Delta|\psi| \geq 0$  and so,

$$\Delta I_\epsilon|\psi| = I_\epsilon \Delta|\psi| \geq 0. \tag{29}$$

On the other hand,  $I_\epsilon|\psi| \in D(\Delta)$  and therefore

$$\langle \Delta(I_\epsilon|\psi|), (I_\epsilon|\psi|) \rangle = -\|\nabla(I_\epsilon|\psi|)\|^2 \leq 0. \tag{30}$$

But by equation (29) the left side of (30) is nonnegative and so  $\nabla(I_\epsilon|\psi|) = 0$  in the  $L^2$ -sense and therefore  $I_\epsilon|\psi| = c \geq 0$ . But  $|\psi| \in L^2$  and  $I_\epsilon|\psi| \rightarrow |\psi|$  in  $L^2$ -sense; and so  $c = 0$ . Hence  $I_\epsilon|\psi| = 0 \Rightarrow |\psi| = 0$  and  $\psi = 0$ .  $\square$

### 3. Bounds for spectral kernels on the $n$ -sphere

Finally, we are set to prove another main result, namely Theorem (4), of this work. Consider the Laplacian on the unit  $n$ -dimensional sphere  $S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$  defined in polar coordinates as

$$\Delta_n = \frac{1}{\sin^{n-1} \theta} \frac{\partial}{\partial \theta} \left\{ \sin^{n-1} \theta \frac{\partial}{\partial \theta} \right\} + \frac{1}{\sin^2 \theta} \Delta_{n-1} \tag{31}$$

where  $\Delta_{n-1}$  is the Laplacian on  $S^{n-1}$ .

The harmonic homogeneous polynomials restricted to the  $n$ -sphere are the eigenfunctions of the Laplacian on  $S^n$ . A detailed treatment of these functions can be found in [19, 4] and [3]. The restriction of the harmonic polynomials to  $S^n$  are called spherical harmonic polynomials of degree  $k$ . Let  $\mathcal{H}_k$  denote the space of the spherical harmonic polynomials on  $S^n$ . They are the eigenfunctions of  $\Delta_n$  with eigenvalues  $k(k+n-1)$ .

The dimension  $d_k(n)$  of the space of harmonic polynomial  $\mathcal{H}_k$  is given by the formula

$$d_k(n) = \binom{k+n}{n} - \binom{k+n-2}{n} = \frac{(2k+n-1)(k+n-2)!}{k!(n-1)!} \tag{32}$$

where  $k \in N_0$  and  $n \geq 1$  is the dimension of the manifold  $S^n$ . For proof, one may see [19, 20] and [18].

We now consider the Schrödinger operator  $\Delta_g + c$  where  $c = \frac{n-1}{2}$  and  $n$  is the dimension of the unit sphere  $S^n$ . One expresses the associated spectral zeta function in terms of the Hurwitz zeta function. Denote by  $\{\mu_k\}$  the spectrum of  $\Delta_g + c$  on  $S^n$  :

$$\mu_k = k(k+n-1) + c \tag{33}$$

with the same eigenfunctions and multiplicities,  $d_k(n)$  as for  $\Delta_g$ .

We define the regularized zeta function as

$$Z_{S^n}(s) = \sum_{k=1}^{\infty} \frac{d_k(n)}{\mu_k^s} = \sum_{k=1}^{\infty} \frac{d_k(n)}{(k + \frac{n-1}{2})^{2s}}; \quad \Re(s) > \frac{n}{2}. \tag{34}$$

The regularized zeta function (34) of the operator  $\Delta_g + c$  on  $S^n$  can then be expressed in terms of the Riemann zeta function see e.g. Elizalde, et al [9]. We prove that  $Z_{S^n}(s, c) \prec \zeta_{S^n}(s)$  which is Theorem 4 of this work.

*Proof of Theorem 4.* Let  $\{\psi_{k,j} : 1 \leq j \leq d_k(n)\}$  be an orthonormal basis of the space of  $n$ -dimensional spherical harmonics  $\mathcal{H}_k(S^n)$ . By Kato's inequality of Theorem (2) it suffices to use

$$\langle \text{sgn}(\psi_{k,j}) H \psi_{k,j}, \psi_{k,l} \rangle \leq \langle |\psi_{k,j}|, \Delta \psi_{k,l} \rangle.$$

So,

$$\begin{aligned} \langle \text{sgn}(\psi_{k,j}) H \psi_{k,j}, \psi_{k,l} \rangle &= \langle \text{sgn}(\psi_{k,j}) (\Delta_x + c) \psi_{k,j}, \psi_{k,l} \rangle \\ &= \langle \text{sgn}(\psi_{k,j}) \Delta_x \psi_{k,j}, \psi_{k,l} \rangle + \langle \text{sgn}(\psi_{k,j}) c \psi_{k,j}, \psi_{k,l} \rangle \\ &= \lambda_k \langle \text{sgn}(\psi_{k,j}) \psi_{k,j}, \psi_{k,l} \rangle + \rho_n \langle \text{sgn}(\psi_{k,j}) \psi_{k,j}, \psi_{k,l} \rangle \\ &= (\lambda_k + \rho_n) \langle \text{sgn}(\psi_{k,j}) \psi_{k,j}, \psi_{k,l} \rangle. \end{aligned}$$

Hence,

$$\int_{S^n} \Delta^{-s} \langle \text{sgn}(\psi_{k,j}) H \psi_{k,j}, \psi_{k,l} \rangle dV_y = \int_{S^n} \Delta^{-s} (\lambda_k + \rho_n) \langle \text{sgn}(\psi_{k,j}) \psi_{k,j}, \psi_{k,l} \rangle dV_y.$$

Since  $\Delta^{-s}$  is trace class with trace given by (24) we have

$$\begin{aligned} \int_{S^n} \Delta^{-s} \langle \text{sgn}(\psi_{k,j}) H \psi_{k,j}, \psi_{k,l} \rangle dV_y &= \sum_{k=1}^{\infty} \frac{1}{(\lambda_k + \rho_n)^s} \int_{S^n} \langle \text{sgn}(\psi_{k,j}) \psi_{k,j}, \psi_{k,l} \rangle dV_y \\ &\leq \sum_{k=1}^{\infty} \frac{1}{(\lambda_k + \rho_n)^s} \int_{S^n} \psi_{k,j} \bar{\psi}_{k,l} dV_y \\ &= \sum_{k=1}^{\infty} \frac{1}{(\lambda_k + \rho_n)^s}. \end{aligned}$$

However, for  $|z| < 1$  the following binomial expansion holds

$$(1-z)^{-2s} = \sum_{m=0}^{\infty} \frac{\Gamma(2s+m)}{m! \Gamma(2s)} z^m.$$

So for  $\Re(2s) > 1$ , we have

$$\begin{aligned} \zeta_H(2s, \rho_n) &= \frac{1}{\rho_n^{2s}} + \sum_{k=1}^{\infty} \frac{1}{k^{2s}} \frac{1}{(1 + \frac{\rho_n}{k})^{2s}} \\ &= \frac{1}{\rho_n^{2s}} + \sum_{k=1}^{\infty} \frac{1}{k^{2s}} \sum_{m=0}^{\infty} (-1)^m \frac{\Gamma(2s+m)}{m! \Gamma(2s)} \left(\frac{\rho_n}{k}\right)^m \\ &= \frac{1}{\rho_n^{2s}} + \sum_{m=0}^{\infty} (-1)^m \frac{\Gamma(2s+m)}{m! \Gamma(2s)} (\rho_n)^m \sum_{k=1}^{\infty} \frac{1}{k^{2s+m}} \end{aligned}$$

which gives the expansion

$$Z_{S^n}(s, \rho_n) = \frac{1}{\rho_n^{2s}} + \sum_{m=0}^{\infty} (-1)^m \frac{\Gamma(2s+m)}{m! \Gamma(2s)} \rho_n^m \zeta_R(2s+m)$$

provided  $0 < \rho_n \leq 1$  and where  $\zeta_R$  is the Riemann zeta function.

Thus, since

$$\frac{d_k(n) \downarrow}{\mu_k^s} \preceq \frac{d_k(n) \downarrow}{\lambda_k^s}$$

and summation operator is Schur convex (see e.g. [26]), it follows that

$$Z_{S^n}(s, x, y) \leq \zeta_g(s, x, y).$$

Therefore,

$$\sum_{k=1}^{\infty} \frac{d_k(n)}{(k + \frac{n-1}{2})^{2s}} \preceq \sum_{k=1}^{\infty} \frac{d_k(n)}{(k(k+n-1))^s} \text{ for } \Re(s) > \frac{n}{2}. \quad \square$$

#### 4. Conclusion

We have constructed Kato's bounds for the spectral zeta kernel of a Schrödinger-type operator in terms of the spectral zeta kernel of the Laplacian and Riemann zeta function on closed Riemannian manifolds. We proved that  $\text{Tr exp}(-tH) \preceq \text{Tr exp}(-t\Delta)$  for  $H = \Delta_g + V$  on smooth functions on  $(M, g)$ . Several illustrations were done on the  $n$ -dimensional unit sphere. A similar study can be done on other Riemannian manifolds of higher genus and with boundary.

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