

## ON SOME SUFFICIENT CONDITIONS FOR $p$ -VALENTLY STARLIKENESS

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*Abstract.* In this paper we prove some properties for functions that are multivalent in the unit disc  $|z| < 1$  in the complex plane. As a corollary we obtain that if  $f(z)$  is  $p$ -valent,  $p \geq 10$ , and

$$\left| \arg \left\{ f^{(p)}(z) \right\} \right| < \pi$$

in  $|z| < 1$ , then  $f(z)$  is  $p$ -valent starlike.

### 1. Introduction

A function  $f(z)$  analytic in a domain  $D \in \mathbb{C}$  is called  $p$ -valent in  $D$ , if for every complex number  $w$ , the equation  $f(z) = w$  has at most  $p$  roots in  $D$ , so that there exists a complex number  $w_0$  such that the equation  $f(z) = w_0$  has exactly  $p$  roots in  $D$ . We denote by  $\mathcal{H}$  the class of functions  $f(z)$  which are holomorphic in the open unit unit  $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ . Denote by  $\mathcal{A}_p$ ,  $p \in \mathbb{N} = \{1, 2, \dots\}$ , the class of functions  $f(z) \in \mathcal{H}$  given by

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n, \quad (z \in \mathbb{D}).$$

Let  $\mathcal{A} = \mathcal{A}_1$ . Let  $\mathcal{S}$  denote the class of all functions in  $\mathcal{A}$  which are univalent. Also let  $\mathcal{S}_p^*(\alpha)$  and  $\mathcal{C}_p(\alpha)$  be the subclasses of  $\mathcal{A}_p$  consisting of all  $p$ -valent functions which are strongly starlike and strongly convex of order  $\alpha$ ,  $0 \leq \alpha < 1$ , defined as

$$\mathcal{S}_p^*(\alpha) = \left\{ f(z) \in \mathcal{A}_p : \left| \arg \left\{ \frac{z f'(z)}{f(z)} \right\} \right| < \frac{\alpha \pi}{2}, z \in \mathbb{D} \right\},$$

$$\mathcal{C}_p(\alpha) = \{ f(z) \in \mathcal{A}_p : z f'(z) / p \in \mathcal{S}_p^*(\alpha) \}.$$

Note that  $\mathcal{S}_1^*(1) = \mathcal{S}^*$  and  $\mathcal{C}_1(1) = \mathcal{C}$ , where  $\mathcal{S}^*$  and  $\mathcal{C}$  are usual classes of starlike and convex functions respectively.

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## 2. Preliminaries

In this paper we need the following lemmas.

LEMMA 2.1. [2, Th.5] *If  $f(z) \in \mathcal{A}_p$ , then for all  $z \in \mathbb{D}$ , we have*

$$\Re \left\{ \frac{zf^{(p)}(z)}{f^{(p-1)}(z)} \right\} > 0 \quad \Rightarrow \quad \forall k \in \{1, \dots, p\} : \quad \Re \left\{ \frac{zf^{(k)}(z)}{f^{(k-1)}(z)} \right\} > 0. \quad (2.1)$$

LEMMA 2.2. [4] *Let  $p(z)$  be analytic function in  $|z| < 1$  with  $p(0) = 1$ ,  $p(z) \neq 0$ . If there exists a point  $z_0$ ,  $|z_0| < 1$ , such that*

$$|\arg \{p(z)\}| < \pi\beta \quad \text{for } |z| < |z_0|$$

and

$$|\arg \{p(z_0)\}| = \pi\beta$$

for some  $0 < \beta$ , then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = 2ik\beta,$$

where

$$k \geq \frac{1}{2} \left( a + \frac{1}{a} \right) \quad \text{when } \arg \{p(z_0)\} = \pi\beta$$

and

$$k \leq -\frac{1}{2} \left( a + \frac{1}{a} \right) \quad \text{when } \arg \{p(z_0)\} = -\pi\beta,$$

where

$$\{p(z_0)\}^{1/(2\beta)} = \pm ia, \quad \text{and } a > 0.$$

LEMMA 2.3. [2, Th.1] *If  $f(z) \in \mathcal{A}_p$ , then for all  $z \in \mathbb{D}$ , we have*

$$\Re \left\{ p + \frac{zf^{(p+1)}(z)}{f^{(p)}(z)} \right\} > 0 \quad (z \in \mathbb{D}), \quad (2.2)$$

then  $f(z)$  is  $p$ -valent in  $\mathbb{D}$  and

$$\forall k \in \{1, \dots, p-1\} : \quad \Re \left\{ k + \frac{zf^{(k+1)}(z)}{f^{(k)}(z)} \right\} > 0 \quad (z \in \mathbb{D}).$$

### 3. Main results

THEOREM 3.1. If  $q(z) \in \mathcal{H}$  and

$$|\arg \{q(z)\}| < \alpha\pi, \quad (z \in \mathbb{D}), \quad (3.1)$$

for some  $\alpha \in (0, 1]$ , and

$$\frac{1}{z} \int_0^z q(t) dt \neq 0, \quad (z \in \mathbb{D}), \quad (3.2)$$

then

$$\left| \arg \left\{ \frac{1}{z} \int_0^z q(t) dt \right\} \right| < \beta\pi < \alpha\pi, \quad (z \in \mathbb{D}),$$

where  $\beta$ , is the unique root of of the equation

$$\beta\pi + \tan^{-1}(2\beta) = \alpha\pi. \quad (3.3)$$

*Proof.* If we put

$$s(z) = \frac{1}{z} \int_0^z q(t) dt, \quad (z \in \mathbb{D}),$$

then it follows that

$$q(z) = s(z) + zs'(z).$$

If there exists a point  $z_1 \in \mathbb{D}$ , such that

$$|\arg \{s(z)\}| < \beta\pi, \quad (|z| < |z_1|)$$

and

$$|\arg \{s(z_1)\}| = \beta\pi,$$

then from Lemma 2.2, we have

$$\frac{z_1 s'(z_1)}{s(z_1)} = 2ik\beta,$$

where

$$k \geq \frac{1}{2} \left( a + \frac{1}{a} \right) \quad \text{when } \arg \{p(z_1)\} = \pi\beta$$

and

$$k \leq -\frac{1}{2} \left( a + \frac{1}{a} \right) \quad \text{when } \arg \{p(z_1)\} = -\pi\beta,$$

where

$$\{p(z_1)\}^{1/(2\beta)} = \pm ia, \quad \text{and } a > 0.$$

Therefore, we have

$$\begin{aligned}
 |\arg\{q(z_1)\}| &= \left| \arg \left\{ s(z_1) \left( 1 + \frac{z_1 s'(z_1)}{s(z_1)} \right) \right\} \right| \\
 &= \left| \arg\{s(z_1)\} + \arg \left\{ 1 + \frac{z_1 s'(z_1)}{s(z_1)} \right\} \right| \\
 &= \left| \beta\pi + \arg \left\{ 1 + \frac{z_1 s'(z_1)}{s(z_1)} \right\} \right| \\
 &= |\beta\pi + \arg\{1 + 2i\beta k\}| \\
 &\geq \beta\pi + \tan^{-1}(2\beta) \\
 &= \alpha\pi
 \end{aligned}$$

because of (3.3) but this contradicts hypothesis (3.1). This shows that

$$|\arg\{s(z)\}| = \left| \frac{1}{z} \int_0^z q(t) dt \right| < \beta\pi, \quad (z \in \mathbb{D}). \quad \square \quad (3.4)$$

If we take  $\alpha = 1/2$  then Theorem 3.1 becomes the following corollary.

**COROLLARY 3.2.** *If  $q(z) \in \mathcal{H}$ ,  $q(z) \neq 0$  in  $\mathbb{D}$  and*

$$|\arg\{q(z)\}| < \frac{\pi}{2}, \quad (z \in \mathbb{D}) \quad (3.5)$$

and

$$\frac{1}{z} \int_0^z q(t) dt \neq 0, \quad (z \in \mathbb{D}),$$

then

$$\left| \arg \left\{ \frac{1}{z} \int_0^z q(t) dt \right\} \right| < \beta\pi < \frac{\pi}{2}, \quad (z \in \mathbb{D}),$$

where  $\beta = 0.319161\dots$ , is the unique root of of the equation

$$\beta\pi + \tan^{-1}(2\beta) = \frac{\pi}{2}. \quad (3.6)$$

Putting  $q(z) = f^{(p)}(z)$ ,  $f(z) \in \mathcal{A}_p$ , in Corollary 3.2 gives us the following result.

**COROLLARY 3.3.** *If  $f(z) \in \mathcal{A}_p$  and*

$$\left| \arg \left\{ f^{(p)}(z) \right\} \right| < \frac{\pi}{2}, \quad (z \in \mathbb{D}) \quad (3.7)$$

and

$$\frac{f^{(p-1)}(z)}{z} \neq 0, \quad (z \in \mathbb{D}),$$

then

$$\left| \arg \left\{ \frac{f^{(p-1)}(z)}{z} \right\} \right| < \beta\pi < \frac{\pi}{2}, \quad (z \in \mathbb{D}),$$

where  $\beta = 0.319161\dots$ , is the unique root of of the equation

$$\beta\pi + \tan^{-1}(2\beta) = \frac{\pi}{2}. \tag{3.8}$$

Notice here the well-known Noshiro-Warschawski theorem and some related results. The Noshiro-Warschawski theorem [1, 6], says that if  $f \in \mathcal{H}$  satisfies

$$\Re \{ e^{i\alpha} f'(z) \} > 0, \quad (z \in \mathbb{D}) \tag{3.9}$$

for some real  $\alpha$ , then  $f(z)$  is univalent in  $\mathbb{D}$ . Ozaki [5], generalized the above theorem for  $f \in \mathcal{A}_p$ : if

$$\Re \{ e^{i\alpha} f^{(p)}(z) \} > 0, \quad (z \in \mathbb{D}) \tag{3.10}$$

for some real  $\alpha$ , then  $f(z)$  is at most  $p$ -valent in  $\mathbb{D}$ . Also in [3, 454] it was shown that if  $f \in \mathcal{A}_p$ ,  $p \geq 2$ , and

$$|\arg \{ f^{(p)}(z) \}| < \frac{3\pi}{4} \quad (z \in \mathbb{D}), \tag{3.11}$$

then  $f$  is at most  $p$ -valent in  $\mathbb{D}$ .

**THEOREM 3.4.** *If  $f(z) \in \mathcal{A}_p$  and*

$$\forall s \in \{0, 1, \dots, p\} \quad \frac{f^{(p-s)}(z)}{z^s} \neq 0, \quad (z \in \mathbb{D})$$

and

$$\left| \arg \left\{ f^{(p)}(z) \right\} \right| < \alpha_0\pi, \quad (z \in \mathbb{D}), \tag{3.12}$$

for some  $\alpha_0 \in (0, 1]$ , then

$$\forall s \in \{1, \dots, p\} \quad \left| \arg \left\{ \frac{f^{(p-s)}(z)}{z^s} \right\} \right| < \alpha_s\pi, \quad (z \in \mathbb{D}), \tag{3.13}$$

where  $\{\alpha_n\}$ ,  $n = 0, 1, 2, \dots$ , is the number sequence such that

$$\alpha_{n+1}\pi + \tan^{-1} \left( \frac{2\alpha_{n+1}}{n+1} \right) = \alpha_n\pi. \tag{3.14}$$

*Proof.* If we put  $q(z) = f^{(p)}(z)$ ,  $\alpha = \alpha_0$ ,  $\beta = \alpha_1$  in Theorem 3.4 then we have:

$$\left| \arg \left\{ \frac{f^{(p-1)}(z)}{z} \right\} \right| < \alpha_1\pi, \quad (z \in \mathbb{D}), \tag{3.15}$$

where  $\alpha_1$  is the unique root of of the equation (3.14) with  $n = 0$  namely

$$\alpha_1 \pi + \tan^{-1}(2\alpha_1) = \alpha_0 \pi.$$

This proves Theorem 3.4 for the case  $s = 1$ .

For the case  $s = 2$ , let us put

$$g_2(z) = \frac{2!}{p!z^2} f^{(p-2)}(z), \quad g_2(0) = 1, \quad (z \in \mathbb{D}),$$

then it follows that  $f^{(p-2)}(z) = p!z^2 g_2(z)/2!$  and

$$f^{(p-1)}(z) = \frac{p!}{2!}(2zg_2(z) + z^2 g_2'(z))$$

and so

$$\frac{f^{(p-1)}(z)}{p!z} = g_2(z) + \frac{1}{2}z g_2'(z). \quad (3.16)$$

If there exists a point  $z_2 \in \mathbb{D}$ , such that

$$|\arg\{g_2(z)\}| < \alpha_2 \pi, \quad (|z| < |z_2|)$$

and

$$|\arg\{g_2(z_2)\}| = \alpha_2 \pi,$$

then from Lemma 2.2, we have

$$\frac{z_2 g_2'(z_2)}{g_2(z_2)} = 2ik\alpha_2$$

where

$$k \geq \frac{1}{2} \left( a + \frac{1}{a} \right) \quad \text{when } \arg\{p(z_2)\} = \pi\alpha_2$$

and

$$k \leq -\frac{1}{2} \left( a + \frac{1}{a} \right) \quad \text{when } \arg\{p(z_2)\} = -\pi\alpha_2,$$

where

$$\{p(z_2)\}^{1/(2\alpha_2)} = \pm ia, \quad \text{and } a > 0.$$

Therefore, applying (3.16), we have

$$\begin{aligned} \left| \arg \left\{ \frac{f^{(p-1)}(z_2)}{z_2} \right\} \right| &= \left| \arg \left\{ g_2(z_2) \left( 1 + \frac{z_2 g_2'(z_2)}{2g_2(z_2)} \right) \right\} \right| \\ &= \left| \arg\{g_2(z_2)\} + \arg \left\{ 1 + \frac{z_2 g_2'(z_2)}{2g_2(z_2)} \right\} \right| \\ &= \left| \alpha_2 \pi + \arg \left\{ 1 + \frac{z_2 g_2'(z_2)}{2g_2(z_2)} \right\} \right| \\ &= |\alpha_2 \pi + \arg\{1 + i\alpha_2 k\}| \\ &\geq \alpha_2 \pi + \tan^{-1}(\alpha_2) \\ &= \alpha_1 \pi \end{aligned}$$

because of (3.14) with  $n = 1$ . On the other hand this contradicts (3.15). This shows that

$$\left| \arg \left\{ \frac{f^{(p-2)}(z)}{z^2} \right\} \right| < \alpha_2 \pi, \quad (z \in \mathbb{D}). \quad (3.17)$$

this completes the proof of Theorem 3.4 for the case  $s = 2$ .

For the case  $s = 3$ , let us put

$$g_3(z) = \frac{3!}{p!z^3} f^{(p-3)}(z), \quad g_3(0) = 1, \quad (z \in \mathbb{D}),$$

it follows that

$$\begin{aligned} \frac{f^{(p-2)}(z)}{p!z^2} &= g_3(z) + \frac{1}{3} z g_3'(z) \\ &= g_3(z) \left( 1 + \frac{1}{3} \frac{z g_3'(z)}{g_3(z)} \right). \end{aligned} \quad (3.18)$$

If there exists a point  $z_3 \in \mathbb{D}$ , such that

$$|\arg \{g_3(z)\}| < \alpha_3 \pi, \quad (|z| < |z_3|)$$

and

$$|\arg \{g_3(z_3)\}| = \alpha_3 \pi,$$

then from Lemma 2.2, we have

$$\frac{z_3 g_3'(z_3)}{g_3(z_3)} = 2ik\alpha_3$$

where

$$k \geq \frac{1}{2} \left( a + \frac{1}{a} \right) \quad \text{when } \arg \{p(z_3)\} = \pi\alpha_3$$

and

$$k \leq -\frac{1}{2} \left( a + \frac{1}{a} \right) \quad \text{when } \arg \{p(z_3)\} = -\pi\alpha_3,$$

where

$$\{p(z_3)\}^{1/(2\alpha_3)} = \pm ia, \quad \text{and } a > 0.$$

Therefore, applying (3.18), we have

$$\begin{aligned} \left| \arg \left\{ \frac{f^{(p-2)}(z_3)}{z_3^2} \right\} \right| &= \left| \arg \left\{ g_3(z_3) \left( 1 + \frac{z_3 g_3'(z_3)}{3g_3(z_3)} \right) \right\} \right| \\ &= \left| \arg \{g_3(z_3)\} + \arg \left\{ 1 + \frac{z_3 g_3'(z_3)}{3g_3(z_3)} \right\} \right| \\ &= \left| \alpha_3 \pi + \arg \left\{ 1 + \frac{z_3 g_3'(z_3)}{3g_3(z_3)} \right\} \right| \\ &= |\alpha_3 \pi + \arg \{1 + 2i\alpha_2 k/3\}| \\ &\geq \alpha_3 \pi + \tan^{-1}(2\alpha_3/3) \\ &= \alpha_2 \pi \end{aligned}$$

because of (3.14) with  $n = 2$ . On the other hand this contradicts (3.17). This shows that

$$\left| \arg \left\{ \frac{f^{(p-2)}(z)}{z^2} \right\} \right| < \alpha_3 \pi, \quad (z \in \mathbb{D}). \quad (3.19)$$

This completes the proof of Theorem 3.4 for the case  $s = 3$ . The proof runs in the same way for  $s = 4, 5, \dots, p$ .  $\square$

**COROLLARY 3.5.** *If  $f(z) \in \mathcal{A}_p$ ,  $p \geq 10$  and*

$$\forall s \in \{0, 1, \dots, 10\} \quad \frac{f^{(p-s)}(z)}{z^s} \neq 0, \quad (z \in \mathbb{D})$$

and

$$\left| \arg \left\{ f^{(p)}(z) \right\} \right| < \pi, \quad (z \in \mathbb{D}),$$

then

$$\left| \arg \left\{ \frac{zf^{(p-9)}(z)}{f^{(p-10)}(z)} \right\} \right| < \frac{\pi}{2}$$

or  $f^{(p-10)}(z)$  is  $(p-10)$ -valently starlike.

*Proof.* The initial values of the sequence (3.14) with  $\alpha_0 = 1$  are

$$\begin{aligned} \alpha_1 &= 0.697887\dots & \alpha_2 &= 0.540226\dots & \alpha_3 &= 0.447868\dots & \alpha_4 &= 0.0.387023\dots \\ \alpha_5 &= 0.343553\dots & \alpha_6 &= 0.310703\dots & \alpha_7 &= 0.284854\dots & \alpha_8 &= 0.263885\dots \\ \alpha_9 &= 0.246468\dots & \alpha_{10} &= 0.231726\dots, \end{aligned}$$

so  $\alpha_9 + \alpha_{10} < 1/2$ . On the other hand, from Theorem 3.4, we have

$$\begin{aligned} \left| \arg \left\{ \frac{zf^{(p-9)}(z)}{f^{(p-10)}(z)} \right\} \right| &= \left| \arg \left\{ \frac{\frac{f^{(p-9)}(z)}{z^{p-9}}}{\frac{f^{(p-10)}(z)}{z^{p-10}}} \right\} \right| \\ &\leq \left| \arg \left\{ \frac{f^{(p-9)}(z)}{z^{p-9}} \right\} \right| + \left| \arg \left\{ \frac{f^{(p-10)}(z)}{z^{p-10}} \right\} \right| \\ &\leq (\alpha_9 + \alpha_{10})\pi \\ &< \frac{\pi}{2}. \quad \square \end{aligned}$$

**COROLLARY 3.6.** *If  $f(z) \in \mathcal{A}_p$ ,  $p \geq 5$  and*

$$\forall s \in \{0, 1, \dots, 5\} \quad \frac{f^{(p-s)}(z)}{z^s} \neq 0, \quad (z \in \mathbb{D})$$

and

$$\left| \arg \left\{ f^{(p)}(z) \right\} \right| < \frac{\pi}{2}, \quad (z \in \mathbb{D}),$$

then

$$\left| \arg \left\{ \frac{zf^{(p-4)}(z)}{f^{(p-5)}(z)} \right\} \right| < \frac{\pi}{2}$$

or  $f^{(p-5)}(z)$  is  $(p-10)$ -valently starlike.

*Proof.* The initial values of the sequence (3.14) with  $\alpha_0 = 1/2$  are

$$\begin{aligned} \alpha_1 &= 0.382449\dots & \alpha_2 &= 0.317129\dots & \alpha_3 &= 0.273819\dots \\ \alpha_4 &= 0.0.242979\dots & \alpha_5 &= 0.219709\dots, \end{aligned}$$

so  $\alpha_4 + \alpha_5 < 1/2$ . On the other hand, from Theorem 3.4, we have

$$\begin{aligned} \left| \arg \left\{ \frac{zf^{(p-4)}(z)}{f^{(p-5)}(z)} \right\} \right| &= \left| \arg \left\{ \frac{\frac{f^{(p-4)}(z)}{z^{p-4}}}{\frac{f^{(p-5)}(z)}{z^{p-5}}} \right\} \right| \\ &\leq \left| \arg \left\{ \frac{f^{(p-4)}(z)}{z^{p-4}} \right\} \right| + \left| \arg \left\{ \frac{f^{(p-5)}(z)}{z^{p-5}} \right\} \right| \\ &\leq (\alpha_4 + \alpha_5)\pi \\ &< \frac{\pi}{2}. \quad \square \end{aligned}$$

COROLLARY 3.7. If  $f(z) \in \mathcal{A}_p$ ,  $p \geq 10$  and

$$\forall s \in \{0, 1, \dots, 10\} \quad \frac{f^{(p-s)}(z)}{z^s} \neq 0, \quad (z \in \mathbb{D})$$

and

$$\left| \arg \left\{ f^{(p)}(z) \right\} \right| < \pi, \quad (z \in \mathbb{D}), \quad (3.20)$$

then

$$\left| \arg \left\{ \frac{zf'(z)}{f(z)} \right\} \right| < \frac{\pi}{2}$$

or  $f(z)$  is  $p$ -valently starlike.

*Proof.* By immediately applying Lemma 2.1 in Corollary 3.7, we get (3.20).  $\square$

Corollary 3.7 may be rewritten in the following form

COROLLARY 3.8. If  $f(z) \in \mathcal{A}_p$ ,  $p \geq 10$  and

$$\forall s \in \{0, 1, \dots, 10\} \quad \frac{f^{(p-s)}(z)}{z^s} \neq 0, \quad (z \in \mathbb{D})$$

and  $\arg \left\{ f^{(p)}(z) \right\}$  does not take its values in  $(-\infty, 0]$ , then  $f(z)$  is  $p$ -valently starlike.

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