

ON MULTIPLICATIVE SUM ZAGREB INDEX OF TREES WITH FIXED DOMINATION NUMBER

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Abstract. The multiplicative sum Zagreb index of a graph G is the product of the sums of degrees of pairs of adjacent vertices. In this paper, the maximal and minimal multiplicative sum Zagreb indices of trees with fixed domination number are presented. Furthermore, the corresponding extremal trees are identified.

1. Introduction

In chemical graph theory and mathematical chemistry, a topological index is a numerical parameter that can be applied in describing the properties or activities of organic compounds, and it plays a substantial role in pharmacology, materials science and chemistry, etc. (see [12], [13], [23]). One of the most studied topological indices is the Zagreb indices. They first appeared within certain approximate expressions for the total π -electron energy [14]. For a graph G , the first Zagreb index M_1 and the second Zagreb index M_2 are defined as:

$$M_1(G) = \sum_{v \in V(G)} d(v)^2, \quad M_2(G) = \sum_{uv \in E(G)} d(u)d(v),$$

where $d(v)$ is the degree of vertex v in G .

These two classical topological indices (M_1 and M_2) and their modified versions have been used to study ZE-isomerism, heterosystems, complexity and chirality of molecule, etc. [6], [19], [20]. Among the modified versions, the multiplicative Zagreb indices, namely, the first and second multiplicative Zagreb indices (denoted by Π_1 and Π_2) [24] and multiplicative sum Zagreb index (denoted by Π_1^*) [10] have attracted considerable attention from researchers. The indices Π_1 and Π_2 are defined as below:

$$\Pi_1(G) = \prod_{u \in V(G)} d(u)^2, \quad \Pi_2(G) = \prod_{uv \in E(G)} d(u)d(v) = \prod_{u \in V(G)} d(u)^{d(u)},$$

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while the index Π_1^* is defined as:

$$\Pi_1^*(G) = \prod_{uv \in E(G)} (d_G(u) + d_G(v)).$$

In [10], Eliasi et al. obtained the minimum multiplicative sum Zagreb index of connected graphs. Xu and Das [26] presented the minimum and maximum multiplicative sum Zagreb indices of trees, unicyclic graphs and bicyclic graphs. The authors of this paper [8] determined the maximal multiplicative sum Zagreb indices with given number of cut vertices/cut edges/vertex connectivity/edge connectivity of graphs. For other mathematical investigations of multiplicative sum Zagreb index, the readers can refer to [1], [5], [16], [22], [25].

In this work, we only consider the connected graphs without multiple edges and loops. A graph $G = (V(G), E(G))$ consists of a set of vertices $V(G)$ and a set of edges $E(G)$. We denote the set of all neighbors of vertex x in G by $N_G(x)$ ($N(x)$ for short), and denote the number of vertices with degree i by n_i . Let $G - uv$ be the graph obtained from G by deleting the edge $uv \in E(G)$. The subgraphs of G obtained by deleting the vertex x ($x \in V(G)$) as well as its incident edges is denoted by $G - x$. Let S_n and P_n be the n -vertex star and the n -vertex path, respectively.

A dominating set D of a graph G is a subset of $V(G)$ such that each vertex in $V(G) \setminus D$ is adjacent to at least one vertex in D . The minimum number of $|D|$ in G is called the domination number, which is denoted by $\gamma(G)$. It is evident that $\gamma(T) = 1$ for a tree T of order n if and only if $T \cong S_n$. That, as we all know, for an n -vertex graph G , $\gamma(G) \leq \frac{n}{2}$ [21]. Fink et al. [11] determined the n -vertex graphs G with $\gamma(G) = \frac{n}{2}$. Let $\mathbf{T}_{n,\gamma}$ be the set of all n -vertex trees with domination number γ . One can see [3] for other terminologies and notations.

At present, studying the properties of topological indices of graphs with given different graph parameters is an important task. Furthermore, it is meaningful to study the topological indices of trees with given parameters since the molecular graph of alkanes is a tree. And many researchers have paid attention to the relation of topological indices and domination number recently [2], [4], [7], [15], [17], [18]. Therefore, here we provide the maximum and minimum multiplicative sum Zagreb indices of trees with fixed domination number, and identify the corresponding extremal trees.

2. Some lemmas

In this section, we present some lemmas which are used to prove the forthcoming section results. The proofs of unreferenced lemmas can be found in the appendix.

LEMMA 2.1. [9] *The function $h_1(x) = \frac{x+c}{x}$ is strictly decreasing with respect to $x \geq 1$, where x is a real number and $c \geq 1$ is an integer.*

LEMMA 2.2. [9] *The function $h_2(x) = \frac{(x+c)^{x+c}}{x^x}$ is strictly increasing with respect to $x \geq 1$, where x is a real number and $c \geq 1$ is an integer.*

LEMMA 2.3. *Let*

$$h_3(r) = (r+2) \left(\frac{r+1}{r} \right)^{r-2},$$

where $r \geq 2$ is a real number. Then $h_3(r)$ is increasing for r .

LEMMA 2.4. *Let*

$$l_1(n, \gamma) = \frac{(n-\gamma)^{n-2\gamma}(n-2\gamma+3)^{n-2\gamma}(n-2\gamma+4)}{(n-\gamma+1)^{n-3\gamma+2}(n-2\gamma+2)^{n-2\gamma}(n-\gamma+2)^{\gamma-1}},$$

where $n \geq 6$ is a finite positive integer and $3 \leq \gamma \leq \frac{n}{2}$. Then $l_1(n, \gamma) < 1$.

LEMMA 2.5. *Let*

$$g(s, s_1) = \frac{(s+2)^{s-s_1}}{s^{s_1-1}(s+1)^{s-2s_1}},$$

where $s_1 \geq 2$, $s \geq 4$ are real numbers and $s_1 < s$. Then $g(s, s_1)$ is increasing for s and s_1 , respectively.

LEMMA 2.6. *Let*

$$l_2(n, \gamma) = \frac{(n-\gamma)^{2n-5\gamma+3}(n-\gamma+1)^{4\gamma-n-4}}{(n-\gamma-1)^{n-2\gamma}(n-\gamma+2)^{\gamma-1}},$$

where $n \geq 8$ is a finite positive integer and $3 \leq \gamma \leq \frac{n}{2}$. Then $l_2(n, \gamma) < 1$.

LEMMA 2.7. [11] *A tree T on n vertices has $\gamma(T) = \frac{n}{2}$ if and only if each vertex with degree greater than 1 of T is adjacent to exactly one pendant vertex.*

LEMMA 2.8. *Let*

$$g(s) = \frac{(s+2)^{s-1}}{s(s+1)^{s-3}},$$

where $s \geq 2$ is a real number. Then $g(s)$ is increasing for s .

LEMMA 2.9. *Let*

$$l_3(n, \gamma) = \frac{(n-\gamma)^{2n-3\gamma-2}}{(n-\gamma-2)(n-\gamma-1)^{n-\gamma-5}(n-\gamma+1)^{n-3\gamma+3}(n-\gamma+2)^{\gamma-1}},$$

where $n \geq 7$ is a finite positive integer and $3 \leq \gamma \leq \frac{n}{2}$. Then $l_3(n, \gamma) < 1$.

LEMMA 2.10. *Let*

$$\varphi(n) = \frac{n^{n-1}}{3^{n-1} \left(\frac{5}{4}\right)^{3(n-3)}},$$

where $n \geq 5$ is an integer. Then $\varphi(n) > 1$.

LEMMA 2.11. *Let*

$$\phi(x) = \left(\frac{x+2}{x+1} \right)^{x-1},$$

where $x \geq 3$ is a real number. Then $\phi(x)$ is increasing for x .

3. Maximal multiplicative sum Zagreb index of trees with fixed domination number

Let $\mathcal{T}_{n,\gamma}$ be the trees obtained from $S_{n-\gamma+1}$ by attaching a pendant edge to its $\gamma-1$ pendant vertices. Notice that if $T \in \mathbf{T}_{n,\gamma}$ with maximum degree $n-\gamma$, then $T \cong \mathcal{T}_{n,\gamma}$. Let

$$f_1(n, \gamma) = \Pi_1^*(\mathcal{T}_{n,\gamma}) = 3^{\gamma-1} (n-\gamma+1)^{n-2\gamma+1} (n-\gamma+2)^{\gamma-1}.$$

THEOREM 3.1. *Let $T \in \mathbf{T}_{n,\gamma}$. Then*

$$\Pi_1^*(T) \leq f_1(n, \gamma).$$

The equality occurs if and only if $T \cong \mathcal{T}_{n,\gamma}$.

Proof. If $n=3$, $\Pi_1^*(P_3) = 3^2 = f_1(3, 1)$. If $n=4$, $\Pi_1^*(P_4) = 4 \cdot 3^2 = f_1(4, 2)$ and $\Pi_1^*(S_4) = 4^3 = f_1(4, 1)$. Now, suppose $n \geq 5$ and the result holds for any trees of order $n-1$. Denoted by $u_1 u_2 \cdots u_{d+1}$ a diameter in T . If $d=2$, then $T \cong S_n$, $\gamma(S_n) = 1$ and $\Pi_1^*(S_n) = n^{n-1} = f_1(n, 1)$, the result is verified. So in what follows, we suppose that $d \geq 3$ and $\gamma(T) \geq 2$. Let us denote $d(u_2) = r \geq 2$, $N(u_2) = \{u_1, u_3, x_1, x_2, \dots, x_{r-2}\}$ and $d(u_3) = s \geq 2$, $N(u_3) = \{u_2, u_4, y_1, y_2, \dots, y_{s-2}\}$. Set $T_1 = T - \{u_1\}$. We discuss in two cases.

Case 1. $\gamma(T_1) = \gamma(T)$.

By induction hypothesis and Lemmas 2.1 and 2.3, we deduce that

$$\begin{aligned} \Pi_1^*(T) &= \Pi_1^*(T_1) \cdot \frac{(r+1)(r+s)}{r+s-1} \cdot \prod_{i=1}^{r-2} \frac{r+1}{r} \\ &\leq f_1(n-1, \gamma) \cdot \frac{(r+1)(r+2)}{r+1} \cdot \left(\frac{r+1}{r} \right)^{r-2} \\ &= f_1(n, \gamma) \cdot \frac{(n-\gamma)^{n-2\gamma}(n-\gamma+1)^{\gamma-1}}{(n-\gamma+1)^{n-2\gamma+1}(n-\gamma+2)^{\gamma-1}} \cdot (r+2) \left(\frac{r+1}{r} \right)^{r-2} \\ &\leq f_1(n, \gamma) \cdot \frac{(n-\gamma)^{n-2\gamma}(n-\gamma+1)^{\gamma-1}}{(n-\gamma+1)^{n-2\gamma+1}(n-\gamma+2)^{\gamma-1}} \cdot (n-2\gamma+4) \left(\frac{n-2\gamma+3}{n-2\gamma+2} \right)^{n-2\gamma} \\ &= f_1(n, \gamma) \cdot \frac{(n-\gamma)^{n-2\gamma}(n-2\gamma+3)^{n-2\gamma}(n-2\gamma+4)}{(n-\gamma+1)^{n-3\gamma+2}(n-2\gamma+2)^{n-2\gamma}(n-\gamma+2)^{\gamma-1}} \end{aligned}$$

since $\gamma \leq \frac{n-(r-2)}{2}$, that is $r \leq n-2\gamma+2$.

If $\gamma = 2$, $\Pi_1^*(T) \leq f_1(n, \gamma)$. The equalities occur if and only if $s = 2$ and $r = n - 2$. This implies $T \cong \mathcal{T}_{n,2}$. If $\gamma \geq 3$, by lemma 2.4, one has $\Pi_1^*(T) < f_1(n, \gamma)$.

Case 2. $\gamma(T_1) = \gamma(T) - 1$.

In this case, we have $r = 2$, otherwise u_2 belongs to each minimum dominating set and implies $\gamma(T_1) = \gamma(T)$. If $s = n - \gamma$, then $T \cong \mathcal{T}_{n,\gamma}$, and the theorem holds. So in what follows, we assume that $s \leq n - \gamma - 1$. By Case 1, we suppose that $d(y_i) \leq 2$, $i \in \{1, 2, \dots, s - 2\}$. If u_4 is a pendant vertex or a support vertex with $d(u_4) = 2$, then $T \cong \mathcal{T}_{n,\gamma}$. In other cases, without loss of generality, we suppose that $d(y_1) = \dots = d(y_{s_1}) = 1$, $d(y_{s_1+1}) = \dots = d(y_{s_1+s_2}) = 2$, where $s_1 + s_2 = s - 2$.

Case 2.1. $s_1 \geq 2$.

Set $T_2 = T - \{y_1\}$. Then $\gamma(T_2) = \gamma(T)$. By induction hypothesis and Lemma 2.1, we arrive at

$$\begin{aligned} \Pi_1^*(T) &= \Pi_1^*(T_2) \cdot \frac{(s+1)(s+2)(s+d(u_4))}{(s+1)(s+d(u_4)-1)} \cdot \left(\prod_{i=2}^{s_1} \frac{s+1}{s}\right) \cdot \left(\prod_{j=1}^{s_2} \frac{s+2}{s+1}\right) \\ &\leq f_1(n-1, \gamma) \cdot \frac{(s+2)(s+2)}{s+1} \cdot \left(\frac{s+1}{s}\right)^{s_1-1} \cdot \left(\frac{s+2}{s+1}\right)^{s-s_1-2} \\ &= f_1(n, \gamma) \cdot \frac{(n-\gamma)^{n-2\gamma}(n-\gamma+1)^{\gamma-1}}{(n-\gamma+1)^{n-2\gamma+1}(n-\gamma+2)^{\gamma-1}} \cdot \frac{(s+2)^{s-s_1}}{s^{s_1-1}(s+1)^{s-2s_1}}. \end{aligned}$$

It is easy to see that $\gamma \leq \frac{n-(s_1-1)}{2}$, that is $s_1 \leq n - 2\gamma + 1$. Furthermore, since $s \leq n - \gamma - 1$ and $\gamma \geq 3$, $n \geq 8$ in this case, by Lemmas 2.5 and 2.6, we derive

$$\begin{aligned} \Pi_1^*(T) &\leq f_1(n, \gamma) \cdot \frac{(n-\gamma)^{n-2\gamma}(n-\gamma+1)^{\gamma-1}}{(n-\gamma+1)^{n-2\gamma+1}(n-\gamma+2)^{\gamma-1}} \cdot \frac{(n-\gamma+1)^{\gamma-2}}{(n-\gamma-1)^{n-2\gamma}(n-\gamma)^{3\gamma-n-3}} \\ &= f_1(n, \gamma) \cdot \frac{(n-\gamma)^{2n-5\gamma+3}(n-\gamma+1)^{4\gamma-n-4}}{(n-\gamma-1)^{n-2\gamma}(n-\gamma+2)^{\gamma-1}} \\ &< f_1(n, \gamma). \end{aligned}$$

Case 2.2. $s_1 \leq 1$.

In this case, one can see that $\gamma \geq 3$, $n \geq 7$, and $n - \gamma \geq 4$.

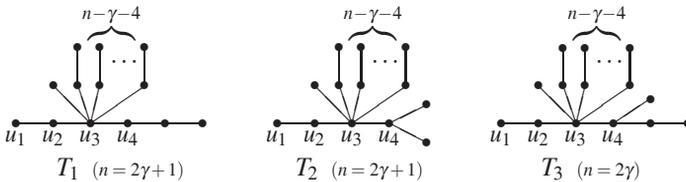


Figure 1. The graphs T_1 , T_2 and T_3 .

Case 2.2.1. $s = n - \gamma - 1$.

If $s_1 = 0$, we have $n \geq 6 + 2(n - \gamma - 3)$, that is $n \leq 2\gamma$. Since $n \geq 2\gamma$, it follows that $n = 2\gamma$. By Lemma 2.7, u_3 should be adjacent to exactly one pendant vertex, a contradiction. Hence, $s_1 = 1$. Now, we can get that $n \geq 7 + 2(n - \gamma - 4)$, that is $n \leq 2\gamma + 1$. It is straightforward to check that $T \in \{T_1, T_2, T_3\}$ (see Figure 1). If $n = 2\gamma + 1$,

we can see that $\Pi_1^*(T_1) < \Pi_1^*(T_2)$, and for $\gamma \geq 3$, $\frac{\Pi_1^*(T_2)}{f_1(2\gamma+1,\gamma)} = \frac{4^2 \cdot 3^{\gamma-2}(\gamma+1)(\gamma+3)(\gamma+2)^{\gamma-2}}{3^{\gamma-1}(\gamma+2)^2(\gamma+3)^{\gamma-1}} = \frac{4^2}{3(\gamma+3)} \cdot \frac{\gamma+1}{\gamma+2} \cdot \left(\frac{\gamma+2}{\gamma+3}\right)^{\gamma-3} < 1$. For $n = 2\gamma$ and $\gamma \geq 3$, $\frac{\Pi_1^*(T_3)}{f_1(2\gamma,\gamma)} = \frac{4 \cdot 5 \cdot 3^{\gamma-2}\gamma(\gamma+2)(\gamma+1)^{\gamma-3}}{3^{\gamma-1}(\gamma+1)(\gamma+2)^{\gamma-1}} = \frac{4 \cdot 5\gamma}{3(\gamma+1)(\gamma+2)} \cdot \left(\frac{\gamma+1}{\gamma+2}\right)^{\gamma-3} < 1$ ($\frac{4 \cdot 5\gamma}{3(\gamma+1)(\gamma+2)} \leq 1$ since $\frac{\gamma}{(\gamma+1)(\gamma+2)}$ is decreasing for $\gamma \geq 3$). The Theorem holds.

Case 2.2.1. $s < n - \gamma - 1$.

Set $T_3 = T - \{u_1, u_2\}$. Then $T_3 \in \mathbf{T}_{n-2,\gamma-1}$. Since $s \leq n - \gamma - 2$ and $\gamma \geq 3, n \geq 7$, by Lemmas 2.1, 2.8, 2.9 and induction hypothesis, we deduce that

$$\begin{aligned} \Pi_1^*(T) &= \Pi_1^*(T_3) \cdot \frac{(1+2)(s+2)(s+d(u_4))(s+d(y_1))}{(s+d(u_4)-1)(s+d(y_1)-1)} \cdot \left(\prod_{i=2}^{s-2} \frac{s+2}{s+1}\right) \\ &\leq f_1(n-2,\gamma-1) \cdot \frac{3(s+2)(s+2)(s+1)}{(s+1)s} \cdot \left(\frac{s+2}{s+1}\right)^{s-3} \\ &= f_1(n-2,\gamma-1) \cdot \frac{3(s+2)^{s-1}}{s(s+1)^{s-3}} \\ &\leq f_1(n,\gamma) \cdot \frac{(n-\gamma)^{n-2\gamma+1}(n-\gamma+1)^{\gamma-2}}{(n-\gamma+1)^{n-2\gamma+1}(n-\gamma+2)^{\gamma-1}} \cdot \frac{(n-\gamma)^{n-\gamma-3}}{(n-\gamma-2)(n-\gamma-1)^{n-\gamma-5}} \\ &= f_1(n,\gamma) \cdot \frac{(n-\gamma)^{2n-3\gamma-2}}{(n-\gamma-2)(n-\gamma-1)^{n-\gamma-5}(n-\gamma+1)^{n-3\gamma+3}(n-\gamma+2)^{\gamma-1}} \\ &< f_1(n,\gamma). \quad \square \end{aligned}$$

4. Minimal multiplicative sum Zagreb index of trees with fixed domination number

We define a class of trees \mathcal{T} by recursion. We consider the path on $3a$ vertices for any integer $a \geq 1$ in \mathcal{T} and construct new trees in the class by the following two ways.

(1) If $T \in \mathcal{T}$ satisfies that there is $u \in V(T)$ (denote $N(u) = \{v_1, v_2\}$) such that $d(v_1) = d(v_2) = 2$ and u belongs to a minimal dominating set of T , and choose any path $P = z_1z_2 \cdots z_{3b+1}$ ($b \geq 1$ is an integer), then the trees $T' = (V(T'), E(T'))$ with $V(T') = V(P) \cup V(T)$ and $E(T') = E(P) \cup E(T) \cup \{uz_1\}$, belongs to \mathcal{T} .

(2) If $T \in \mathcal{T}$, w is a pendant vertex of T and choose any path $P = z_1z_2 \cdots z_{3b}$ ($b \geq 1$ is an integer), then $T'' = (V(T''), E(T''))$ with $V(T'') = V(P) \cup V(T)$ and $E(T'') = E(P) \cup E(T) \cup \{wz_1\}$, belongs to \mathcal{T} .

Let $\mathcal{T}_{n,\gamma}$ be the set of all n -vertex trees $T \in \mathcal{T}$ with domination number γ . Let

$$f_2(n,\gamma) = 3^{n-3\gamma+2} \cdot 4^{3(4\gamma-n-1)} \cdot 5^{3(n-3\gamma)}.$$

LEMMA 4.1. Let $T \in \mathcal{T}_{n,\gamma}$. Then

$$\Pi_1^*(T) = f_2(n,\gamma).$$

Proof. Since $T \in \mathcal{T}_{n,\gamma}$, then

$$\begin{cases} n_1 + n_2 + n_3 = n, \\ n_1 + 2n_2 + 3n_3 = 2(n - 1), \\ n_3 + \frac{n - n_3 - 3n_3}{3} = \gamma. \end{cases}$$

Thus $n_1 = n - 3\gamma + 2$, $n_2 = 6\gamma - n - 2$ and $n_3 = n - 3\gamma$. By the definitions of multiplicative sum Zagreb index and \mathcal{T} , we obtain

$$\begin{aligned} \Pi_1^*(T) &= (3 + 2)^{3n_3} \cdot (1 + 2)^{n_1} \cdot (2 + 2)^{(n-1-3n_3-n_1)} \\ &= 5^{3(n-3\gamma)} \cdot 3^{(n-3\gamma+2)} \cdot 4^{n-1-(4n-12\gamma+2)} \\ &= 3^{n-3\gamma+2} \cdot 4^{3(4\gamma-n-1)} \cdot 5^{3(n-3\gamma)} \\ &= f_2(n, \gamma). \end{aligned}$$

This finishes the proof. \square

THEOREM 4.2. *Let $T \in \mathbf{T}_{n,\gamma}$. Then*

$$\Pi_1^*(T) \geq f_2(n, \gamma).$$

The equality occurs if and only if $T \in \mathcal{T}_{n,\gamma}$.

Proof. If $n = 3$, $\Pi_1^*(P_3) = 3^2 = f_2(3, 1)$. If $n = 4$, $\Pi_1^*(P_4) = 4 \cdot 3^2 > f_2(4, 2) = \frac{4^9}{5^6}$ and $\Pi_1^*(S_4) = 4^3 > f_2(4, 1) = (\frac{15}{4})^3$. Suppose now that $n \geq 5$ and the result holds for any trees of order $n - 1$. Denoted by $u_1 u_2 \cdots u_{d+1}$ a diameter in T . If $d = 2$, $T \cong S_n$ and $\gamma(S_n) = 1$. By Lemma 2.10, for $n \geq 5$, $\frac{\Pi_1^*(S_n)}{f_2(n, 1)} = \frac{n^{n-1}}{3^{n-1}(\frac{5}{4})^{3(n-3)}} > 1$ and the theorem is verified. So in what follows, we assume that $d \geq 3$. For convenience, we denote $d(u_2) = r$, $N(u_2) = \{u_1, u_3, x_1, x_2, \dots, x_{r-2}\}$ and $d(u_3) = s$, $N(u_3) = \{u_2, u_4, y_1, y_2, \dots, y_{s-2}\}$, where $r, s \geq 2$. Next, we consider two possible cases.

Case 1. $d(u_2) = r \geq 3$.

Note that $d(x_i) = 1$ for each $i \in \{1, 2, \dots, r - 2\}$. Set $T_1 = T - u_1$. Then $T_1 \in \mathbf{T}_{n-1,\gamma}$. By the definition of Sombor index and induction hypothesis, it follows that

$$\Pi_1^*(T) = \Pi_1^*(T_1) \cdot \frac{(r + 1)(r + 1)^{r-2}(r + s)}{r^{r-2}(r + s - 1)}.$$

Case 1.1. $r \geq 4$.

By Lemma 2.2, we get

$$\begin{aligned} \Pi_1^*(T) &> \Pi_1^*(T_1) \cdot \frac{(r + 1)^{r-1}}{r^{r-2}} \\ &\geq f_2(n - 1, \gamma) \cdot \frac{(r + 1)^{r+1}}{r^r} \cdot \frac{r^2}{(r + 1)^2} \\ &\geq f_2(n, \gamma) \cdot \frac{4^3}{3 \cdot 5^3} \cdot \frac{5^3}{4^2} \\ &> f_2(n, \gamma). \end{aligned}$$

Case 1.2. $r = 3$ and $s \leq 8$.

By Lemma 2.1, we derive

$$\begin{aligned} \Pi_1^*(T) &\geq f_2(n, \gamma) \cdot \frac{4^3}{3 \cdot 5^3} \cdot \frac{4^2}{3} \cdot \frac{s+3}{s+2} \\ &\geq f_2(n, \gamma) \cdot \frac{1024}{1125} \cdot \frac{11}{10} \\ &= f_2(n, \gamma) \cdot \frac{11264}{11250} > f_2(n, \gamma). \end{aligned}$$

Case 1.3. $r = 3$ and $s \geq 9$.

Set $T'_1 = T - \{u_1, u_2, x_1\}$. In this case, there is a domination set D with $|D| = \gamma$ in T such that $u_2 \in D$, and $u_3 \in D$ or $u_3 \in N[D \setminus \{u_2\}]$, then $\gamma(T'_1) = \gamma(T) - 1$ and $T'_1 \in \mathbf{T}_{n-3, \gamma-1}$. By the induction hypothesis, it follows that

$$\begin{aligned} \Pi_1^*(T) &= \Pi_1^*(T'_1) \cdot \frac{4^2(s+3)(s+d(u_4))}{(s+d(u_4)-1)} \cdot \prod_{i=1}^{s-2} \frac{s+d(y_i)}{s+d(y_i)-1} \\ &> f_2(n-3, \gamma-1) \cdot 4^2(s+3) \\ &\geq f_2(n, \gamma) \cdot \frac{1}{4^3} \cdot 4^2 \cdot 12 \\ &> f_2(n, \gamma). \end{aligned}$$

Case 2. $d(u_2) = 2$ for any diameter $u_1 u_2 \cdots u_{d+1}$.

Case 2.1. $d(u_3) = s \geq 3$.

Denote $d(u_4) = t$. By Case 1, we suppose $d(y_i) \leq 2$, $i = 1, 2, \dots, s-2$. Set $T_2 = T - \{u_1, u_2\}$. Since there exists a dominating set D with $|D| = \gamma$ in T such that $u_2 \in D$, and $u_3 \in D$ or $u_3 \in N[D \setminus \{u_2\}]$, then $\gamma(T_2) = \gamma(T) - 1$ and $T_2 \in \mathbf{T}_{n-2, \gamma-1}$. By induction hypothesis and Lemmas 2.1, 2.2, we deduce that

$$\begin{aligned} \Pi_1^*(T) &= \Pi_1^*(T_2) \cdot \frac{3(s+2)(s+d(u_4))}{(s+d(u_4)-1)} \cdot \prod_{i=1}^{s-2} \frac{s+d(y_i)}{s+d(y_i)-1} \\ &> f_2(n-2, \gamma-1) \cdot 3(s+2) \cdot \left(\frac{s+2}{s+1}\right)^{s-2} \\ &= f_2(n-2, \gamma-1) \cdot 3 \cdot \frac{(s+2)^{s+2}}{(s+1)^{s+1}} \cdot \frac{(s+1)^3}{(s+2)^3} \\ &\geq f_2(n, \gamma) \cdot \frac{3 \cdot 5^3}{4^6} \cdot \frac{3 \cdot 5^2}{4} \\ &= f_2(n, \gamma) \cdot \frac{28125}{16384} > f_2(n, \gamma). \end{aligned}$$

Case 2.2. $d(u_3) = 2$.

Denote $N(u_4) = \{u_3, u_5, w_1, w_2, \dots, w_{t-2}\}$ and $d(u_5) = k$. For $i \in \{1, 2, \dots, t-2\}$, if there are $w', w'' \in V(T)$ such that $w' \in N(w_i)$ and $w'' \in N(w')$, then $w'' w' w_i u_4 u_5 \cdots, u_{d+1}$ is a diameter of T ; if w_i is a support vertex with $d(w_i) \geq 3$, similarly to Case

1, we can prove that $\Pi_1^*(T) > f_2(n, \gamma)$. So by the above cases, we may assume that $d(w_i) \leq 2, i = 1, 2, \dots, t-2$.

Case 2.2.1. $d(u_4) = t \geq 3$.

Set $T_3 = T - \{u_1, u_2, u_3\}$. Then $T_3 \in \mathbf{T}_{n-3, \gamma-1}$. By induction hypothesis and Lemmas 2.1, 2.2, we have

$$\begin{aligned} \Pi_1^*(T) &= \Pi_1^*(T_3) \cdot \frac{4 \cdot 3 \cdot (t+2)(t+k)}{(t+k-1)} \cdot \prod_{i=1}^{t-2} \frac{t+d(w_i)}{t+d(w_i)-1} \\ &> f_2(n-3, \gamma-1) \cdot 12(t+2) \cdot \left(\frac{t+2}{t+1}\right)^{t-2} \\ &= f_2(n-3, \gamma-1) \cdot 12 \cdot \frac{(t+2)^{t+2}}{(t+1)^{t+1}} \cdot \frac{(t+1)^3}{(t+2)^3} \\ &\geq f_2(n, \gamma) \cdot \frac{1}{4^3} \cdot \frac{12 \cdot 5^2}{4} \\ &= f_2(n, \gamma) \cdot \frac{75}{64} > f_2(n, \gamma). \end{aligned}$$

Case 2.2.2. $d(u_4) = t = 2$.

Case 2.2.2.1. There is a minimum dominating set D with $u_4 \in D$ in T .

Set $T_4 = T - \{u_1, u_2\}$. Now we can get that

$$\begin{aligned} \Pi_1^*(T) &= \Pi_1^*(T_4) \cdot \frac{4 \cdot 4 \cdot 3}{3} \\ &\geq 16f_2(n-2, \gamma-1) \\ &= 16f_2(n, \gamma) \cdot \frac{3 \cdot 5^3}{4^6} \\ &= f_2(n, \gamma) \cdot \frac{375}{256} > f_2(n, \gamma). \end{aligned}$$

Case 2.2.2.2. $u_4 \notin D$ for each minimum dominating set D in T .

Denote $N(u_5) = \{u_4, v_1, v_2, \dots, v_{k-1}\}$.

If $d(u_5) = k = 2$, set $T_5 = T - \{u_1, u_2, u_3\}$. Then $T_5 \in \mathbf{T}_{n-3, \gamma-1}$. Thus

$$\begin{aligned} \Pi_1^*(T) &= \Pi_1^*(T_5) \cdot \frac{4^3 \cdot 3}{3} \\ &\geq f_2(n-3, \gamma-1) \cdot 4^3 \\ &= f_2(n, \gamma). \end{aligned}$$

With the equality holds only if $T_5 \in \mathcal{F}_{n-3, \gamma-1}$, which implies that $T \in \mathcal{F}_{n, \gamma}$.

If $d(u_5) = k \geq 3$ and for each $i \in \{1, 2, \dots, k-1\}$, $d(v_i) \leq 2$, set $T_6 = T - \{u_1, u_2, u_3, u_4\}$. Then $T_6 \in \mathbf{T}_{n-4, \gamma-1}$. By induction hypothesis and Lemma 2.11, one

has

$$\begin{aligned} \Pi_1^*(T) &= \Pi_1^*(T_6) \cdot 3 \cdot 4^2 \cdot (k+2) \cdot \prod_{i=1}^{k-1} \frac{k+d(v_i)}{k+d(v_i)-1} \\ &\geq f_2(n-4, \gamma-1) \cdot 3 \cdot 4^2 \cdot (k+2) \cdot \left(\frac{k+2}{k+1}\right)^{k-1} \\ &\geq f_2(n-4, \gamma-1) \cdot 3 \cdot 4^2 \cdot 5 \cdot \frac{5^2}{4^2} \\ &= f_2(n, \gamma). \end{aligned}$$

With the equalities if and only if $T_6 \in \mathcal{T}_{n-4, \gamma-1}$ and $d(u_5) = k = 3$, $d(v_1) = d(v_2) = 2$, which implies that $T \in \mathcal{T}_{n, \gamma}$.

If $d(u_5) = k \geq 3$ and for $p = \max\{d(v_1), d(v_2), \dots, d(v_{k-1})\} \geq 3$, without loss of generality, we assume that $d(v_1) = p$. Denote $N(v_1) = \{u_5, z_1, z_2, \dots, z_{p-1}\}$. By the above cases, we suppose that $d(z_i) \leq 2$ for $i \in \{1, 2, \dots, p-1\}$. We discuss two disjoint components T_7 and T_8 in $T - u_5v_1$, containing the vertex u_5 and v_1 , respectively. By induction hypothesis and Lemmas 2.1, 2.11, we deduce that

$$\begin{aligned} \Pi_1^*(T) &= \Pi_1^*(T_7) \cdot \Pi_1^*(T_8) \cdot \frac{(k+2)(p+k)}{k+1} \cdot \prod_{i=2}^{k-1} \frac{k+d(v_i)}{k+d(v_i)-1} \cdot \prod_{j=1}^{p-1} \frac{p+d(z_j)}{p+d(z_j)-1} \\ &= f_2(n, \gamma) \cdot \frac{3^2}{4^3} \cdot \frac{(k+2)(p+k)}{k+1} \cdot \prod_{i=2}^{k-1} \frac{k+d(v_i)}{k+d(v_i)-1} \cdot \prod_{j=1}^{p-1} \frac{p+d(z_j)}{p+d(z_j)-1} \\ &> f_2(n, \gamma) \cdot \frac{3^2}{4^3} \cdot \frac{(k+2)(k+3)}{k+1} \cdot \left(\frac{p+2}{p+1}\right)^{p-1} \\ &\geq f_2(n, \gamma) \cdot \frac{3^2}{4^3} \cdot \frac{5 \cdot 6}{4} \cdot \frac{5^2}{4^2} \\ &= f_2(n, \gamma) \cdot \frac{6750}{4096} > f_2(n, \gamma). \end{aligned}$$

The proof is completed. \square

Appendix

Proof of Lemma 2.3. It is easy to see that $h_3(r) = \frac{(r+1)^{r+1}}{r^r} \cdot \frac{r^2(r+2)}{(r+1)^3}$. Let $h_4(r) = \frac{r^2(r+2)}{(r+1)^3}$. Then for $r \geq 2$,

$$\frac{d \ln h_4(r)}{dr} = \frac{2}{r} + \frac{1}{r+2} - \frac{3}{r+1} = \frac{r+4}{r(r+1)(r+2)} > 0.$$

Hence $h_4(r)$ is increasing for r . Moreover, by Lemma 2.2, we deduce that $h_3(r)$ is increasing for r . \square

Proof of Lemma 2.4. It is routine to check that $\ln l_1(n, \gamma) = (n - 2\gamma) \ln(n - \gamma) + (n - 2\gamma) \ln(n - 2\gamma + 3) + \ln(n - 2\gamma + 4) - (n - 3\gamma + 2) \ln(n - \gamma + 1) - (n - 2\gamma) \ln(n - 2\gamma + 2) - (\gamma - 1) \ln(n - \gamma + 2)$. Thus,

$$\begin{aligned} & \frac{\partial \ln l_1(n, \gamma)}{\partial n} \\ &= \ln \frac{(n - \gamma)(n - 2\gamma + 3)}{(n - \gamma + 1)(n - 2\gamma + 2)} + \left(\frac{2\gamma - 1}{n - \gamma + 1} - \frac{\gamma}{n - \gamma} - \frac{\gamma - 1}{n - \gamma + 2} \right) \\ & \quad + \left(\frac{2}{n - 2\gamma + 2} + \frac{1}{n - 2\gamma + 4} - \frac{3}{n - 2\gamma + 3} \right) \\ &= \ln \frac{\frac{n - 2\gamma + 3}{n - 2\gamma + 2} \frac{n - \gamma + 1}{n - \gamma}}{1} + \frac{g_1(n, \gamma)}{(n - \gamma)(n - \gamma + 1)(n - \gamma + 2)(n - 2\gamma + 2)(n - 2\gamma + 3)(n - 2\gamma + 4)}, \end{aligned}$$

where $g_1(n, \gamma) = 3n^2\gamma^2 - 3n^2\gamma - 6n^2 - 11n\gamma^3 + 33n\gamma^2 - 16n\gamma - 12n + 10\gamma^4 - 48\gamma^3 + 74\gamma^2 - 36\gamma$. Since $n \geq 2\gamma$, then for $\gamma \geq 3$,

$$\begin{aligned} \frac{\partial g_1(n, \gamma)}{\partial n} &= 6n(\gamma^2 - \gamma - 2) - 11\gamma^3 + 33\gamma^2 - 16\gamma - 12 \\ &\geq 12\gamma(\gamma^2 - \gamma - 2) - 11\gamma^3 + 33\gamma^2 - 16\gamma - 12 \\ &= \gamma^3 + 21\gamma^2 - 40\gamma - 12 \\ &= \gamma^3 - 12 + \gamma(21\gamma - 40) > 0. \end{aligned}$$

So $g_1(n, \gamma) \geq g_1(2\gamma, \gamma) = 6\gamma(\gamma - 2)(\gamma + 5) > 0$. Furthermore, for $\gamma \geq 3$, $n - \gamma > n - 2\gamma + 2$, by Lemma 2.1, we have $\ln \frac{\frac{n - 2\gamma + 3}{n - 2\gamma + 2} \frac{n - \gamma + 1}{n - \gamma}}{1} > 0$. Consequently, $\frac{\partial \ln l_1(n, \gamma)}{\partial n} > 0$ and $l_1(n, \gamma)$ is increasing for n .

Notice that

$$\begin{aligned} & \frac{\partial \ln l_1(n, \gamma)}{\partial \gamma} \\ &= \ln \frac{(n - \gamma + 1)^3(n - 2\gamma + 2)^2}{(n - \gamma + 2)(n - \gamma)^2(n - 2\gamma + 3)^2} - \left(\frac{2\gamma - 1}{n - \gamma + 1} - \frac{\gamma}{n - \gamma} - \frac{\gamma - 1}{n - \gamma + 2} \right) \\ & \quad - 2 \left(\frac{2}{n - 2\gamma + 2} + \frac{1}{n - 2\gamma + 4} - \frac{3}{n - 2\gamma + 3} \right) \\ &= \ln \frac{(n - \gamma + 1)^3(n - 2\gamma + 2)^2}{(n - \gamma + 2)(n - \gamma)^2(n - 2\gamma + 3)^2} - \left(\frac{2}{n - 2\gamma + 2} + \frac{1}{n - 2\gamma + 4} - \frac{3}{n - 2\gamma + 3} \right) \\ & \quad - \left[\left(\frac{2\gamma - 1}{n - \gamma + 1} - \frac{\gamma}{n - \gamma} - \frac{\gamma - 1}{n - \gamma + 2} \right) + \left(\frac{2}{n - 2\gamma + 2} + \frac{1}{n - 2\gamma + 4} - \frac{3}{n - 2\gamma + 3} \right) \right] \\ &= \ln \left[\frac{n - \gamma + 1}{n - \gamma + 2} \cdot \left(\frac{\frac{n - \gamma + 1}{n - \gamma}}{\frac{n - 2\gamma + 3}{n - 2\gamma + 2}} \right)^2 \right] - \frac{n - 2\gamma + 6}{(n - 2\gamma + 2)(n - 2\gamma + 3)(n - 2\gamma + 4)} \\ & \quad - \frac{g_1(n, \gamma)}{(n - \gamma)(n - \gamma + 1)(n - \gamma + 2)(n - 2\gamma + 2)(n - 2\gamma + 3)(n - 2\gamma + 4)}. \end{aligned}$$

For $\gamma \geq 3$, by Lemma 2.1, we have $\ln \left[\frac{n-\gamma+1}{n-\gamma+2} \cdot \left(\frac{\frac{n-\gamma+1}{n-\gamma}}{\frac{n-2\gamma+3}{n-2\gamma+2}} \right)^2 \right] < 0$. Consequently, $\frac{\partial \ln l_1(n, \gamma)}{\partial \gamma} < 0$ and $l_1(n, \gamma)$ is decreasing for γ . So we conclude that

$$l_1(n, \gamma) \leq l_1(n, 3) = \frac{(n-3)^{2n-12}}{(n-2)^{n-8}(n-4)^{n-6}(n-1)^2}.$$

Since

$$\begin{aligned} \lim_{n \rightarrow +\infty} l_1(n, 3) &= \lim_{n \rightarrow +\infty} \left[\left(1 - \frac{1}{n-2} \right)^{-(n-2)} \right]^{-1} \cdot \left(1 + \frac{1}{n-4} \right)^{n-4} \cdot \frac{(n-2)^6(n-4)^2}{(n-3)^6(n-1)^2} \\ &= e^{-1} \cdot e \cdot 1 = 1, \end{aligned}$$

then $l_1 = 1$ is the horizontal asymptote of $l_1(n, 3)$. Furthermore, since $0 < l_1(6, 3) = \frac{16}{25} < 1$, $l_1(n, 3)$ is increasing for n and n is a finite positive integer, we deduce that $l_1(n, 3) < 1$ and $l_1(n, \gamma) \leq l_1(n, 3) < 1$. \square

Proof of Lemma 2.5. Observe that $\ln g(s, s_1) = (s - s_1) \ln(s + 2) - (s_1 - 1) \ln s - (s - 2s_1) \ln(s + 1)$. So

$$\frac{\partial \ln g(s, s_1)}{\partial s} = \ln \frac{s+2}{s+1} + \frac{3s-2s_1+2}{s(s+1)(s+2)} > 0$$

and

$$\frac{\partial \ln g(s, s_1)}{\partial s_1} = \ln \frac{(s+1)^2}{s(s+2)} > 0.$$

Therefore, $g(s, s_1)$ is increasing for s and s_1 , respectively. \square

Proof of Lemma 2.6. Note that $\ln l_2(n, \gamma) = (2n - 5\gamma + 3) \ln(n - \gamma) - (n - 4\gamma + 4) \ln(n - \gamma + 1) - (n - 2\gamma) \ln(n - \gamma - 1) - (\gamma - 1) \ln(n - \gamma + 2)$. Thus,

$$\begin{aligned} &\frac{\partial \ln l_2(n, \gamma)}{\partial n} \\ &= \ln \frac{(n-\gamma)^2}{(n-\gamma-1)(n-\gamma+1)} + (\gamma-1) \left(\frac{3}{n-\gamma+1} + \frac{1}{n-\gamma-1} - \frac{3}{n-\gamma} - \frac{1}{n-\gamma+2} \right) \\ &= \ln \frac{(n-\gamma)^2}{(n-\gamma)^2-1} + \frac{6(\gamma-1)}{(n-\gamma)(n-\gamma-1)(n-\gamma+1)(n-\gamma+2)} > 0. \end{aligned}$$

Hence $l_2(n, \gamma)$ is increasing for n .

Furthermore,

$$\begin{aligned} &\frac{\partial \ln l_2(n, \gamma)}{\partial \gamma} \\ &= \ln \frac{(n-\gamma+1)^4(n-\gamma-1)^2}{(n-\gamma)^5(n-\gamma+2)} - (\gamma-1) \left(\frac{3}{n-\gamma+1} + \frac{1}{n-\gamma-1} - \frac{3}{n-\gamma} - \frac{1}{n-\gamma+2} \right) \\ &= \ln \frac{(n-\gamma+1)^4(n-\gamma-1)^2}{(n-\gamma)^5(n-\gamma+2)} - \frac{6(\gamma-1)}{(n-\gamma)(n-\gamma-1)(n-\gamma+1)(n-\gamma+2)}. \end{aligned}$$

For $3 \leq \gamma \leq \frac{n}{2}$, one can easily check that $(n - \gamma)^5(n - \gamma + 2) - (n - \gamma + 1)^4(n - \gamma - 1)^2 = n^4 - 4n^3\gamma + 4n^3 + 6n^2\gamma^2 - 12n^2\gamma + n^2 - 4n\gamma^3 + 12n\gamma^2 - 2n\gamma - 2n + \gamma^4 - 4\gamma^3 + \gamma^2 + 2\gamma - 1 = n^2(n - 2\gamma)^2 + 2n\gamma^2(n - 2\gamma) + 4n(n - \gamma)(n - 2\gamma) + 2n\gamma(\gamma - 1) + 2n(\gamma^2 - 1) + n^2 + \gamma^4 - 4\gamma^3 + \gamma^2 + 2\gamma - 1 > 0$ ($n^2 + \gamma^4 - 4\gamma^3 + \gamma^2 + 2\gamma - 1 \geq 4\gamma^2 + \gamma^4 - 4\gamma^3 + \gamma^2 + 2\gamma - 1 = \gamma^2(\gamma - 2)^2 + \gamma^2 + 2\gamma - 1 > 0$). Thus $\ln \frac{(n-\gamma+1)^4(n-\gamma-1)^2}{(n-\gamma)^5(n-\gamma+2)} < 0$. Consequently, $\frac{\partial \ln l_2(n, \gamma)}{\partial \gamma} < 0$ and $l_2(n, \gamma)$ is decreasing for γ . So it can be concluded that

$$l_2(n, \gamma) \leq l_2(n, 3) = \frac{(n - 3)^{2n-12}}{(n - 2)^{n-8}(n - 4)^{n-6}(n - 1)^2}.$$

The rest of the proof is similar to lemma 2.4. \square

Proof of Lemma 2.8. One can write $g(s) = \frac{(s+2)^{s+2}}{(s+1)^{s+1}} \cdot \frac{(s+1)^4}{s(s+2)^3}$. Let $h(s) = \ln \frac{(s+1)^4}{s(s+2)^3} = 4\ln(s+1) - \ln s - 3\ln(s+2)$. Then for $s \geq 2$,

$$\frac{dh(s)}{ds} = \frac{4}{s+1} - \frac{1}{s} - \frac{3}{s+2} = \frac{2(s-1)}{s(s+1)(s+2)} > 0.$$

Therefore, $h(s)$ is increasing for s . Moreover, by Lemma 2.2, $g(s)$ is increasing for s . \square

Proof of Lemma 2.9. Considering that $\ln l_3(n, \gamma) = (2n - 3\gamma - 2)\ln(n - \gamma) - \ln(n - \gamma - 2) - (n - \gamma - 5)\ln(n - \gamma - 1) - (n - 3\gamma + 3)\ln(n - \gamma + 1) - (\gamma - 1)\ln(n - \gamma + 2)$. Thus,

$$\begin{aligned} & \frac{\partial \ln l_3(n, \gamma)}{\partial n} \\ &= \ln \frac{(n - \gamma)^2}{(n - \gamma - 1)(n - \gamma + 1)} + \left(\frac{2\gamma - 2}{n - \gamma + 1} + \frac{4}{n - \gamma - 1} - \frac{\gamma + 2}{n - \gamma} - \frac{\gamma - 1}{n - \gamma + 2} - \frac{1}{n - \gamma - 2} \right) \\ &= \ln \frac{(n - \gamma)^2}{(n - \gamma)^2 - 1} + \frac{g_2(n, \gamma)}{(n - \gamma - 2)(n - \gamma - 1)(n - \gamma)(n - \gamma + 1)(n - \gamma + 2)}, \end{aligned}$$

where $g_2(n, \gamma) = 2n^3 - 8n^2\gamma + 2n^2 + 10n\gamma^2 + 2n\gamma - 20n - 4\gamma^3 - 4\gamma^2 + 16\gamma - 8$. Since $n \geq 2\gamma$, then for $\gamma \geq 4$,

$$\begin{aligned} \frac{\partial g_2(n, \gamma)}{\partial n} &= 6n^2 - 16n\gamma + 4n + 10\gamma^2 + 2\gamma - 20 \\ &= 2(n - \gamma)(3n - 5\gamma) + 4n + 2\gamma - 20 > 0. \end{aligned}$$

So $g_2(n, \gamma) \geq g_2(2\gamma, \gamma) = 8(\gamma^2 - 3\gamma - 1) = 8[(\gamma - 4)(\gamma + 1) + 3] > 0$.

For $\gamma = 3$,

$$\left. \frac{\partial \ln l_3(n, \gamma)}{\partial n} \right|_{\gamma=3} = \ln \frac{(n - 3)^2}{(n - 3)^2 - 1} + \frac{2n^3 - 22n^2 + 76n - 104}{(n - 5)(n - 4)(n - 3)(n - 2)(n - 1)}.$$

Since $2n^3 - 22n^2 + 76n - 104 = 2n(n-7)(n-4) + 20n - 104 > 0$ for $n \geq 7$, we have $\frac{\partial \ln l_3(n, \gamma)}{\partial n} \Big|_{\gamma=3} > 0$.

Consequently, $\frac{\partial \ln l_3(n, \gamma)}{\partial n} > 0$ and $l_3(n, \gamma)$ is increasing for n .

If $\gamma = 3$,

$$l_3(n, 3) = \frac{(n-3)^{2n-11}}{(n-2)^{n-6}(n-4)^{n-8}(n-1)^2(n-5)}.$$

Since

$$\begin{aligned} \lim_{n \rightarrow +\infty} l_3(n, 3) &= \lim_{n \rightarrow +\infty} \left[\left(1 - \frac{1}{n-2}\right)^{-(n-2)} \right]^{-1} \cdot \left(1 + \frac{1}{n-4}\right)^{n-4} \cdot \frac{(n-2)^4(n-4)^4}{(n-3)^5(n-1)^2(n-5)} \\ &= e^{-1} \cdot e \cdot 1 = 1, \end{aligned}$$

then $l_3 = 1$ is the horizontal asymptote of $l_3(n, 3)$. Furthermore, since $0 < l_3(7, 3) = \frac{24}{45} < 1$, $l_3(n, 3)$ is increasing for n and n is a finite positive integer, we deduce that $l_3(n, 3) < 1$.

If $\gamma \geq 4$, one can obtain

$$\begin{aligned} &\frac{\partial \ln l_3(n, \gamma)}{\partial \gamma} \\ &= \ln \frac{(n-\gamma-1)(n-\gamma+1)^3}{(n-\gamma+2)(n-\gamma)^3} - \left(\frac{2\gamma-2}{n-\gamma+1} + \frac{4}{n-\gamma-1} - \frac{\gamma+2}{n-\gamma} - \frac{\gamma-1}{n-\gamma+2} - \frac{1}{n-\gamma-2} \right) \\ &= \ln \frac{(n-\gamma-1)(n-\gamma+1)^3}{(n-\gamma+2)(n-\gamma)^3} - \frac{g_2(n, \gamma)}{(n-\gamma-2)(n-\gamma-1)(n-\gamma)(n-\gamma+1)(n-\gamma+2)}. \end{aligned}$$

For $\gamma \geq 4$, $(n-\gamma+2)(n-\gamma)^3 - (n-\gamma-1)(n-\gamma+1)^3 = 2n - 2\gamma + 1 > 0$, we have $\ln \frac{(n-\gamma-1)(n-\gamma+1)^3}{(n-\gamma+2)(n-\gamma)^3} < 0$. As a consequence, $\frac{\partial \ln l_3(n, \gamma)}{\partial \gamma} < 0$ and $l_3(n, \gamma)$ is decreasing for $\gamma \geq 4$. So we know that

$$l_3(n, \gamma) \leq l_3(n, 4) = \frac{(n-4)^{2n-14}}{(n-6)(n-3)^{n-9}(n-5)^{n-9}(n-2)^3}.$$

Since

$$\begin{aligned} \lim_{n \rightarrow +\infty} l_3(n, 4) &= \lim_{n \rightarrow +\infty} \left[\left(1 - \frac{1}{n-3}\right)^{-(n-3)} \right]^{-1} \cdot \left(1 + \frac{1}{n-5}\right)^{n-5} \cdot \frac{(n-3)^6(n-5)^4}{(n-6)(n-4)^6(n-2)^3} \\ &= e^{-1} \cdot e \cdot 1 = 1, \end{aligned}$$

then $l_1 = 1$ is the horizontal asymptote of $l_3(n, 4)$. Furthermore, since $0 < l_3(8, 4) = \frac{5}{9} < 1$, $l_3(n, 4)$ is increasing for n and n is a finite positive integer, we deduce that $l_3(n, 4) < 1$ and $l_3(n, \gamma) \leq l_3(n, 4) < 1$. \square

Proof of Lemma 2.10. It is clear that $\ln \varphi(n) = (n-1)\ln n - (n-1)\ln 3 + 3(n-3)\ln 4 - 3(n-3)\ln 5$. Hence

$$\begin{aligned} \frac{d \ln \varphi(n)}{dn} &= \ln n + \frac{n-1}{n} + 3 \ln 4 - \ln 3 - 3 \ln 5 \\ &\geq \ln 5 + \frac{4}{5} + 3 \ln 4 - \ln 3 - 3 \ln 5 \\ &= \ln \frac{320 \cdot e^{0.8}}{375} \approx 0.6414 > 0. \end{aligned}$$

Therefore, $\varphi(n)$ is increasing for n , and $\varphi(n) \geq \varphi(5) = \frac{5^4 4^6}{3^4 5^6} = \frac{4096}{2025} > 1$. \square

Proof of Lemma 2.11. We first prove that for $t > 0$, $\ln(1+t) > \frac{t}{1+t}$. Let $\psi(t) = \ln(1+t) - \frac{t}{1+t}$, where $t > 0$. Then

$$\frac{d\psi(t)}{dt} = \frac{1}{t+1} - \frac{1}{(t+1)^2} > 0.$$

Thus for $t > 0$, $\psi(t) > \psi(0) = 0$, that is $\ln(1+t) > \frac{t}{1+t}$.

Noting that $\ln \phi(x) = (x-1)[\ln(x+2) - \ln(x+1)]$. Hence, by the above inequality we proved, for $x \geq 3$, one has

$$\begin{aligned} \frac{d\phi(x)}{dx} &= \ln\left(1 + \frac{1}{x+1}\right) + \frac{x-1}{x+2} - \frac{x-1}{x+1} \\ &> \frac{\frac{1}{x+1}}{1 + \frac{1}{x+1}} + \frac{x-1}{x+2} - \frac{x-1}{x+1} \\ &= \frac{x}{x+2} - \frac{x-1}{x+1} = \frac{2}{(x+1)(x+2)} > 0. \end{aligned}$$

Therefore, $\phi(x)$ is increasing for x . \square

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