

SOME NEW NUMERICAL RADIUS AND HILBERT–SCHMIDT NUMERICAL RADIUS INEQUALITIES FOR HILBERT SPACE OPERATORS

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Abstract. In this article, we give new upper and lower bounds of numerical radius and Hilbert-Schmidt numerical radius inequalities for Hilbert space operators. In particular, we show that if $X \in C_2$ with the Cartesian decomposition $X = A + iB$, then

$$\frac{1}{4} \| |X|^2 + |X^*|^2 \|_2 \leq \frac{1}{\sqrt{2}} \omega_2 \left(\begin{bmatrix} 0 & A^{2T} \\ B^2 & 0 \end{bmatrix} \right) \leq \omega_2^2(X).$$

This is an analog of Kittaneh in [Studia Math. 168 (2005): 73-80].

1. Introduction

Let $B(\mathcal{H})$ denote the C^* -algebra of all bounded linear operators on a complex Hilbert space \mathcal{H} with an inner product $\langle \cdot, \cdot \rangle$. If, in addition, \mathcal{H} is separable, we say $A \in B(\mathcal{H})$ belong to the Hilbert-Schmidt class C_2 if $\|A\|_2 = (tr A^*A)^{\frac{1}{2}} = \left(\sum_{j=1}^{\infty} s_j^2(A) \right)^{\frac{1}{2}}$ is finite, where $s_1(A) \geq s_2(A) \dots$ are the singular values of A . Throughout this article, we assume $A \in B(\mathcal{H})$ is compact whenever $A \in C_2$. For $A \in B(\mathcal{H})$, let $\|A\|$ denote the usual operator norm of A . The numerical range of A is defined by $W(A) = \{ \langle Ax, x \rangle : x \in \mathcal{H}, \|x\| = 1 \}$. The numerical radius of A is defined by $\omega(A) = \sup\{ |\lambda| : \lambda \in W(A) \}$. We note that if $A \in B(\mathcal{H})$ and if f is a non-negative increasing function on $[0, \infty)$, then $\|f(|A|)\| = f(\|A\|)$.

It is well known that $\omega(\cdot)$ defines a norm on $B(\mathcal{H})$. In fact, for any $A \in B(\mathcal{H})$,

$$\frac{1}{2} \|A\| \leq \omega(A) \leq \|A\|, \tag{1}$$

which indicates the usual operator norm and the numerical radius are equivalent.

Recently, Abu-Omar and Kittaneh defined the Hilbert-Schmidt numerical radius as follows:

$$\omega_2(A) = \sup_{\theta \in \mathbb{R}} \| \Re(e^{i\theta} A) \|_2.$$

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Another equality to characterize the Hilbert-Schmidt numerical radius was proved that [5]:

$$\omega_2(A) = \sqrt{\frac{1}{2}\|A\|_2^2 + \frac{1}{2}|\operatorname{tr}A^2|}. \quad (2)$$

As an analog of (1), we also have the following inequalities for the Hilbert-Schmidt numerical radius:

$$\frac{1}{\sqrt{2}}\|A\|_2 \leq \omega_2(A) \leq \|A\|_2. \quad (3)$$

If the norm $\|\cdot\|_2$ is replaced by any norm $\|\cdot\|_N$ on $B(\mathcal{H})$ for a separable Hilbert space \mathcal{H} , we call $\omega_N(\cdot)$ the generalized numerical radius. For recent studies on Hilbert-Schmidt numerical radius and the generalized numerical radius, we refer to [1, 2, 13, 23].

Before proceeding, we give the definition of geometrically convex. First we note that all functions in this article satisfy the following condition unless otherwise specified: J is a sub interval of $(0, \infty)$ and $f: J \rightarrow (0, \infty)$. We call f is geometrically convex if $f(a^{1-t}b^t) \leq f^{1-t}(a)f^t(b)$ for $t \in [0, 1]$. Next we introduce the definition of operator convex. A real-valued continuous function f on an interval J is said to be operator convex if $f((1-t)A+tB) \leq (1-t)f(A)+tf(B)$ for all self-adjoint operators $A, B \in B(\mathcal{H})$ whose spectra are contained in J . Recent studies on numerical radius inequalities involving geometrically convex and operator convex functions can be found in [18, 19].

Kittaneh [14] had shown the following inequalities which improved the inequalities in (1) by using several norm inequalities and ingenious techniques:

$$\frac{1}{4}\| |A|^2 + |A^*|^2 \| \leq \omega^2(A) \leq \frac{1}{2}\| |A|^2 + |A^*|^2 \|, \quad A \in B(\mathcal{H}). \quad (4)$$

Later in [3] Abu-Omar and Kittaneh proved that if $A \in B(\mathcal{H})$, then

$$\omega^2(A) \leq \frac{1}{4}\| |A|^2 + |A^*|^2 \| + \frac{1}{2}\omega(A^2). \quad (5)$$

In [12] Hassain, Omidvar and Moradi gave integral type numerical radius inequalities as follows: if $A \in B(\mathcal{H})$ and f is a non-negative increasing operator convex function on $[0, \infty)$, then

$$\begin{aligned} f(\omega^2(A)) &\leq \left\| \int_0^1 f\left((1-t)\frac{|A|^2 + |A^*|^2}{2} + t\omega(A^2)I\right) dt \right\| \\ &\leq \frac{1}{4}\|f(|A|^2) + f(|A^*|^2)\| + \frac{1}{2}f(\omega(A^2)). \end{aligned} \quad (6)$$

Some related integral type numerical radius inequalities can be found in [17, 20, 21].

In [5] Aldalabih and Kittaneh obtained Hilbert-Schmidt numerical radius inequalities on $\mathcal{H} \oplus \mathcal{H}$ as follows: if $A \in C_2$, then

$$\frac{\max(\omega_2(A+B), \omega_2(A-B))}{\sqrt{2}} \leq \omega_2 \left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \right) \leq \frac{\omega_2(A+B) + \omega_2(A-B)}{\sqrt{2}}. \tag{7}$$

In [11] Hajmohamadi and Lashkaripour obtained an Hilbert-Schmidt numerical radius inequality as follows: if $A, B, X \in C_2$, then

$$\omega_2(BX^*A^* + AXB^*) \leq 2\|A\|_2\|B\|_2 \frac{\omega_2(X+X^*) + \omega_2(X-X^*)}{\sqrt{2}}. \tag{8}$$

They also obtained the following inequality: if $A, B, C, D \in C_2$ such that B, C be self-adjoint, then

$$\omega_2 \left(\begin{bmatrix} A & B \\ C & D \end{bmatrix} \right) \geq \frac{1}{2} \max(\omega_2(A+D), \omega_2(B+C)). \tag{9}$$

We note that there is a gap in the proof of (8) and (9). For identity operator I and block operator matrices $\begin{bmatrix} 0 & I \\ I & I \end{bmatrix}$, their proof is based on $\|I\|_2 = 1$ and $\left\| \begin{bmatrix} 0 & I \\ I & I \end{bmatrix} \right\|_2 = \sqrt{2}$, which is not true in general.

In this paper, we first give an analog of (4) for Hilbert-Schmidt numerical radius, then we give corrections and refinements of (8) and (9). Moreover, we give a different proof of (6) along with presenting more numerical radius inequalities which are refinements of (4).

2. Inequalities for Hilbert-Schmidt numerical radius

First we give an Hilbert-Schmidt numerical radius inequality that is an analog of (4).

THEOREM 2.1. *Let $X \in C_2$ with the Cartesian decomposition $X = A + iB$. Then*

$$\frac{1}{4} \| |X|^2 + |X^*|^2 \|_2 \leq \frac{1}{\sqrt{2}} \omega_2 \left(\begin{bmatrix} 0 & A^2 \\ B^2 & 0 \end{bmatrix} \right) \leq \omega_2^2(X).$$

Proof. Note that if $T \in C_2$ is self-adjoint, then $\omega_2(T) = \|T\|_2$. This is because $\omega_2(T) = \sqrt{\frac{1}{2}\|T\|_2^2 + \frac{1}{2}|trT^2|} = \sqrt{\frac{1}{2}\|T\|_2^2 + \frac{1}{2}\|T\|_2^2} = \|T\|_2$. Since $X = A + iB$ is the Cartesian decomposition of X , then $\frac{|X|^2 + |X^*|^2}{2} = A^2 + B^2$. Hence

$$\begin{aligned} \frac{1}{4} \| |X|^2 + |X^*|^2 \|_2 &= \frac{1}{2} \|A^2 + B^2\|_2 \leq \frac{1}{\sqrt{2}} \omega_2 \left(\begin{bmatrix} 0 & A^2 \\ B^2 & 0 \end{bmatrix} \right) \quad (\text{by (7)}) \\ &= \frac{1}{\sqrt{2}} \omega_2 \left(\begin{bmatrix} 0 & A^2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ B^2 & 0 \end{bmatrix} \right) \end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{\sqrt{2}} \left(\omega_2 \left(\begin{bmatrix} 0 & A^2 \\ 0 & 0 \end{bmatrix} \right) + \omega_2 \left(\begin{bmatrix} 0 & 0 \\ B^2 & 0 \end{bmatrix} \right) \right) \\
&= \frac{1}{\sqrt{2}} \left(\frac{1}{\sqrt{2}} \|A^2\|_2 + \frac{1}{\sqrt{2}} \|B^2\|_2 \right) \\
&= \frac{1}{2} (\|A^2\|_2 + \|B^2\|_2) \leq \frac{1}{2} (\|A\|_2^2 + \|B\|_2^2) \\
&= \frac{1}{2} \|X\|_2^2 \leq \frac{1}{2} 2\omega_2^2(X) = \omega_2^2(X). \quad \square
\end{aligned}$$

THEOREM 2.2. *Let $A, B \in C_2$. Then*

$$\omega_2^2 \left(\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \right) \geq \omega_2(A^*B).$$

Proof. Compute

$$\begin{aligned}
\omega_2^2 \left(\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \right) &\geq \frac{1}{2} \left\| \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \right\|_2^2 \quad (\text{by (3)}) = \frac{1}{2} (\|A\|_2^2 + \|B\|_2^2) \\
&= \frac{1}{2} (\|A\|_2 \|A^*\|_2 + \|B\|_2 \|B^*\|_2) \geq \frac{1}{2} (\|AA^*\|_2 + \|BB^*\|_2) \\
&\geq \frac{1}{2} \|AA^* + BB^*\|_2 \geq \|A^*B\|_2 \\
&\geq \omega_2(A^*B),
\end{aligned}$$

completing the proof. \square

THEOREM 2.3. *Let $A, X \in C_2$. Then*

$$\omega_2(AXA^*) \leq \|A\|^2 \omega_2(X).$$

Proof. Since for any two operators $S, T \in C_2$, $s_j(ST) \leq \min(\|S\|s_j(T), \|T\|s_j(S))$, we have

$$\begin{aligned}
s_j(\Re(e^{i\theta}AXA^*)) &= s_j(A\Re(e^{i\theta}X)A^*) \quad j = 1, 2, \dots \\
&\leq \|A\|^2 s_j(\Re(e^{i\theta}X)).
\end{aligned}$$

Hence

$$\sum_{j=1}^k s_j(\Re(e^{i\theta}AXA^*)) \leq \|A\|^2 \sum_{j=1}^k s_j(\Re(e^{i\theta}X)), \quad k = 1, 2, \dots$$

By Fan Dominance Theorem and the unitarily invariant of $\|\cdot\|_2$, we obtain

$$\|\Re(e^{i\theta}AXA^*)\|_2 \leq \|A\|^2 \|\Re(e^{i\theta}X)\|_2.$$

Taking the supremum over θ , we have

$$\omega_2(AXA^*) \leq \|A\|^2 \omega_2(X). \quad \square$$

We give one example for Theorem 2.3. Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $X = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Then

$$\omega_2(AXA^*) = 1 \leq \sqrt{2} = \|A\|^2 \omega_2(X).$$

By Theorem 2.3 and the proof of Theorem 2.3 in [11] we obtain the following result.

THEOREM 2.4. *Let $A, B, X \in C_2$. Then*

$$\omega_2(BX^*A^* + AXB^*) \leq 2\|A\|\|B\| \frac{\omega_2(X + X^*) + \omega_2(X - X^*)}{\sqrt{2}}.$$

REMARK 2.5. Since $\|\cdot\| \leq \|\cdot\|_2$, Theorem 2.4 is a refinement of Theorem 2.3 in [11].

Since the technique used in the next theorem is similar to Theorem 2.3, we omit the proof.

THEOREM 2.6. *Let $A, X \in C_2$ such that A be self-adjoint. Then*

$$\omega_2(AX + XA) \leq 2\|A\|\omega_2(X).$$

By Theorem 2.6 and the proof of Theorem 3.2 in [11] we derive the following result.

THEOREM 2.7. *Let $A, B, C, D \in C_2$ such that B, C be self-adjoint. Then*

$$\omega_2 \left(\begin{bmatrix} A & B \\ C & D \end{bmatrix} \right) \geq \frac{1}{\sqrt{2}} \max(\omega_2(A + D), \omega_2(B + C)).$$

REMARK 2.8. Theorem 2.7 is a refinement of Theorem 3.2 in [11].

The following theorem was involved in [6], we include a proof here for completeness.

THEOREM 2.9. *Let $A, B, C, D \in C_2$. Then*

$$\omega_2 \left(\begin{bmatrix} A & B \\ C & D \end{bmatrix} \right) \leq \sqrt{\omega_2^2(A) + \frac{1}{2}\|B\|_2^2} + \sqrt{\omega_2^2(D) + \frac{1}{2}\|C\|_2^2}.$$

Proof. Compute

$$\begin{aligned}
 \omega_2 \left(\begin{bmatrix} A & B \\ C & D \end{bmatrix} \right) &= \omega_2 \left(\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ C & D \end{bmatrix} \right) \\
 &\leq \omega_2 \left(\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \right) + \omega_2 \left(\begin{bmatrix} 0 & 0 \\ C & D \end{bmatrix} \right) \\
 &= \omega_2 \left(\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \right) + \omega_2 \left(U^* \begin{bmatrix} D & C \\ 0 & 0 \end{bmatrix} U \right) \quad \left(U = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \right) \\
 &= \omega_2 \left(\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \right) + \omega_2 \left(\begin{bmatrix} D & C \\ 0 & 0 \end{bmatrix} \right) \\
 &= \sqrt{\omega_2^2(A) + \frac{1}{2}\|B\|_2^2} + \sqrt{\omega_2^2(D) + \frac{1}{2}\|C\|_2^2}. \quad \square
 \end{aligned}$$

REMARK 2.10. Let $A = D = 0$ in Theorem 2.9, then

$$\begin{aligned}
 \sqrt{\omega_2^2(A) + \frac{1}{2}\|B\|_2^2} + \sqrt{\omega_2^2(D) + \frac{1}{2}\|C\|_2^2} &= \sqrt{\frac{1}{2}\|B\|_2^2} + \sqrt{\frac{1}{2}\|C\|_2^2} \\
 &\leq \sqrt{\|B\|_2^2 + \|C\|_2^2} \\
 &= \sqrt{\omega_2^2(A) + \omega_2^2(D) + \|B\|_2^2 + \|C\|_2^2}.
 \end{aligned}$$

Thus under this condition, Theorem 2.9 is a refinement of Theorem 2 in [5].

3. Inequalities for numerical radius

Recently, many scholars have paid much attention to applications of geometrically convex functions to numerical radius operator inequalities, we refer the reader to [22] for a sample of such study.

Next we give some lemmas which will be necessary to prove our main results.

LEMMA 3.1. (see [9]) Let $a, b, e \in \mathcal{H}$ with $\|e\| = 1$. Then

$$|\langle a, e \rangle \langle e, b \rangle| \leq \frac{1}{2}(\|a\|\|b\| + |\langle a, b \rangle|).$$

LEMMA 3.2. (see [7]) Let $a, b, e \in \mathcal{H}$ with $\|e\| = 1$ and $t \in [0, 1]$. Then

$$|\langle a, e \rangle \langle e, b \rangle|^2 \leq \frac{1+t}{4}\|a\|^2\|b\|^2 + \frac{3-t}{4}\|a\|\|b\||\langle a, b \rangle|.$$

LEMMA 3.3. (see [10]) If $f: J \rightarrow \mathbb{R}$ is an operator convex function on the interval J , then for any self-adjoint operator X and Y with spectra in J , we have

$$f\left(\frac{X+Y}{2}\right) \leq \int_0^1 f((1-t)X + tY) dt \leq \frac{f(X) + f(Y)}{2}.$$

If f is non-negative, one can obtain

$$\left\| f\left(\frac{X+Y}{2}\right) \right\| \leq \left\| \int_0^1 f((1-t)X+tY) dt \right\| \leq \left\| \frac{f(X)+f(Y)}{2} \right\|.$$

THEOREM 3.4. *Let $A \in B(\mathcal{H})$ and f be an increasing geometrically convex function. If in addition f is convex, then for any $t \in [0, 1]$,*

$$f(\omega^4(A)) \leq \frac{5+t}{16} \|f(|A|^4) + f(|A^*|^4)\| + \frac{3-t}{8} f(\omega^2(A^2)).$$

Proof. From Lemma 3.2 we know

$$|\langle a, e \rangle \langle e, b \rangle|^2 \leq \frac{1+t}{4} \|a\|^2 \|b\|^2 + \frac{3-t}{4} \|a\| \|b\| |\langle a, b \rangle|, \tag{10}$$

where $a, b, e \in \mathcal{H}$ and $\|e\| = 1$. For any unit vector $x \in \mathcal{H}$, put $e = x$, and replace a, b by Ax, A^*x , respectively, in (10) we have

$$\begin{aligned} & f(|\langle Ax, x \rangle|^4) \\ & \leq f\left(\frac{1+t}{4} \|Ax\|^2 \|A^*x\|^2 + \frac{3-t}{4} \|Ax\| \|A^*x\| |\langle Ax, A^*x \rangle|\right) \\ & = f\left(\frac{1+t}{4} \langle |A|^2x, x \rangle \langle |A^*|^2x, x \rangle + \frac{3-t}{4} \sqrt{\langle |A|^2x, x \rangle \langle |A^*|^2x, x \rangle} |\langle A^2x, x \rangle|\right) \\ & \leq \frac{1+t}{4} f(\langle |A|^2x, x \rangle \langle |A^*|^2x, x \rangle) + \frac{3-t}{4} f(\sqrt{\langle |A|^2x, x \rangle \langle |A^*|^2x, x \rangle} |\langle A^2x, x \rangle|) \\ & \leq \frac{1+t}{4} f\left(\frac{\langle |A|^2x, x \rangle^2 + \langle |A^*|^2x, x \rangle^2}{2}\right) + \frac{3-t}{4} f(\sqrt{\langle |A|^2x, x \rangle \langle |A^*|^2x, x \rangle} |\langle A^2x, x \rangle|) \\ & \leq \frac{1+t}{4} f\left(\frac{\langle |A|^4x, x \rangle + \langle |A^*|^4x, x \rangle}{2}\right) + \frac{3-t}{4} f(\sqrt{\langle |A|^2x, x \rangle \langle |A^*|^2x, x \rangle} |\langle A^2x, x \rangle|) \\ & \leq \frac{1+t}{4} f\left(\frac{\langle |A|^4x, x \rangle + \langle |A^*|^4x, x \rangle}{2}\right) + \frac{3-t}{4} f^{\frac{1}{2}}(\langle |A|^2x, x \rangle \langle |A^*|^2x, x \rangle) f^{\frac{1}{2}}(|\langle A^2x, x \rangle|^2) \\ & \leq \frac{1+t}{4} f\left(\frac{\langle |A|^4x, x \rangle + \langle |A^*|^4x, x \rangle}{2}\right) + \frac{3-t}{4} \frac{f(\langle |A|^2x, x \rangle \langle |A^*|^2x, x \rangle) + f(|\langle A^2x, x \rangle|^2)}{2} \\ & \leq \frac{1+t}{4} f\left(\frac{\langle |A|^4x, x \rangle + \langle |A^*|^4x, x \rangle}{2}\right) + \frac{3-t}{4} \frac{f\left(\frac{\langle |A|^4x, x \rangle + \langle |A^*|^4x, x \rangle}{2}\right) + f(\omega^2(A^2))}{2} \\ & \leq \frac{1+t}{4} \frac{f(\langle |A|^4x, x \rangle) + f(\langle |A^*|^4x, x \rangle)}{2} + \frac{3-t}{4} \frac{\frac{f(\langle |A|^4x, x \rangle) + f(\langle |A^*|^4x, x \rangle)}{2} + f(\omega^2(A^2))}{2} \\ & \leq \frac{1+t}{4} \frac{\langle (f(|A|^4) + f(|A^*|^4))x, x \rangle}{2} + \frac{3-t}{4} \frac{\frac{\langle (f(|A|^4) + f(|A^*|^4))x, x \rangle}{2} + f(\omega^2(A^2))}{2} \\ & \leq \frac{1+t}{4} \frac{\|f(|A|^4) + f(|A^*|^4)\|}{2} + \frac{3-t}{4} \frac{\frac{\|f(|A|^4) + f(|A^*|^4)\|}{2} + f(\omega^2(A^2))}{2} \\ & = \frac{5+t}{16} \|f(|A|^4) + f(|A^*|^4)\| + \frac{3-t}{8} f(\omega^2(A^2)). \end{aligned}$$

By taking the supremum over unit vector x , we obtain

$$f(\omega^4(A)) \leq \frac{5+t}{16} \|f(|A|^4) + f(|A^*|^4)\| + \frac{3-t}{8} f(\omega^2(A^2)). \quad \square$$

REMARK 3.5. Letting $f(s) = s$ in Theorem 3.4, we have

$$\begin{aligned} \omega^4(A) &\leq \frac{5+t}{16} \| |A|^4 + |A^*|^4 \| + \frac{3-t}{8} \omega^2(A^2). \\ \omega^4(A) &\leq \frac{5+t}{16} \| |A|^4 + |A^*|^4 \| + \frac{3-t}{8} \omega^2(A^2) \\ &\leq \frac{5+t}{16} \| |A|^4 + |A^*|^4 \| + \frac{3-t}{8} \frac{1}{2} \| |A|^4 + |A^*|^4 \| \\ &= \frac{1}{2} \| |A|^4 + |A^*|^4 \|. \end{aligned}$$

THEOREM 3.6. Let $A, B, X \in B(\mathcal{H})$ such that A, B be positive. Then

$$\omega^2((A\sharp B)X) \leq \frac{1}{4} \|A^2 + (X^*BX)^2\| + \frac{1}{2} \omega(X^*BXA),$$

where $A\sharp B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}}A^{\frac{1}{2}}$ is the geometric mean of A and B .

Proof. By the property of geometric mean and Lemma 3.1, we have

$$\begin{aligned} |\langle (A\sharp B)Xx, x \rangle|^2 &= \left| \left\langle A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}}A^{\frac{1}{2}}Xx, x \right\rangle \right|^2 \\ &= \left| \left\langle (A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}}A^{\frac{1}{2}}Xx, A^{\frac{1}{2}}x \right\rangle \right|^2 \\ &\leq \| (A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}}A^{\frac{1}{2}}Xx \|^2 \| A^{\frac{1}{2}}x \|^2 \\ &= \langle Ax, x \rangle \langle X^*BXx, x \rangle \\ &\leq \frac{1}{2} \|Ax\| \|X^*BXx\| + \frac{1}{2} \langle Ax, X^*BXx \rangle \\ &= \frac{1}{2} \sqrt{\langle A^2x, x \rangle \langle (X^*BX)^2x, x \rangle} + \frac{1}{2} \langle X^*BXAx, x \rangle \\ &\leq \frac{1}{4} \langle (A^2 + (X^*BX)^2)x, x \rangle + \frac{1}{2} \langle X^*BXAx, x \rangle. \end{aligned}$$

By taking the supremum over unit vector x , we obtain

$$\omega^2((A\sharp B)X) \leq \frac{1}{4} \|A^2 + (X^*BX)^2\| + \frac{1}{2} \omega(X^*BXA),$$

completing the proof. \square

Let $X = I$ in Theorem 3.6, we have the following corollary.

COROLLARY 3.7. *Let $A, B \in B(\mathcal{H})$ be positive. Then*

$$\omega^2(A\sharp B) \leq \frac{1}{4}\|A^2 + B^2\| + \frac{1}{2}\omega(BA).$$

THEOREM 3.8. *Let $A \in B(\mathcal{H})$. Then*

$$\omega^4(A) \leq \frac{t}{4} \left(\frac{1}{2} \| |A|^2 + |A^*|^2 \| + \omega(A^2) \right)^2 + \frac{1-t}{2} \omega^2(A) \left(\frac{1}{2} \| |A|^2 + |A^*|^2 \| + \omega(A^2) \right),$$

where $t \in [0, 1]$.

Proof. By Lemma 3.1, we have

$$\begin{aligned} |\langle a, e \rangle \langle e, b \rangle|^2 &\leq t |\langle a, e \rangle \langle e, b \rangle|^2 + (1-t) |\langle a, e \rangle \langle e, b \rangle|^2 \\ &\leq \frac{t}{4} (\|a\| \|b\| + |\langle a, b \rangle|)^2 + \frac{1-t}{2} |\langle a, e \rangle \langle e, b \rangle| (\|a\| \|b\| + |\langle a, b \rangle|). \end{aligned} \tag{11}$$

For every unit vector $x \in \mathcal{H}$, put $e = x$, and replace a, b by Ax, A^*x , respectively, in (11) we have

$$\begin{aligned} |\langle Ax, x \rangle|^4 &= |\langle Ax, x \rangle \langle x, A^*x \rangle|^2 \\ &\leq \frac{t}{4} (\|Ax\| \|A^*x\| + |\langle A^2x, x \rangle|)^2 + \frac{1-t}{2} |\langle Ax, x \rangle|^2 (\|Ax\| \|A^*x\| + |\langle A^2x, x \rangle|) \\ &= \frac{t}{4} (\sqrt{\langle |A|^2x, x \rangle \langle |A^*|^2x, x \rangle} + |\langle A^2x, x \rangle|)^2 \\ &\quad + \frac{1-t}{2} |\langle Ax, x \rangle|^2 (\sqrt{\langle |A|^2x, x \rangle \langle |A^*|^2x, x \rangle} + |\langle A^2x, x \rangle|) \\ &\leq \frac{t}{4} \left(\frac{1}{2} \langle (|A|^2 + |A^*|^2)x, x \rangle + |\langle A^2x, x \rangle| \right)^2 \\ &\quad + \frac{1-t}{2} |\langle Ax, x \rangle|^2 \left(\frac{1}{2} \langle (|A|^2 + |A^*|^2)x, x \rangle + |\langle A^2x, x \rangle| \right). \end{aligned}$$

By taking the supremum over unit vector x , we obtain

$$\omega^4(A) \leq \frac{t}{4} \left(\frac{1}{2} \| |A|^2 + |A^*|^2 \| + \omega(A^2) \right)^2 + \frac{1-t}{2} \omega^2(A) \left(\frac{1}{2} \| |A|^2 + |A^*|^2 \| + \omega(A^2) \right),$$

as required. \square

REMARK 3.9. Note that

$$\begin{aligned} &\omega^4(A) \\ &\leq \frac{t}{4} \left(\frac{1}{2} \| |A|^2 + |A^*|^2 \| + \omega(A^2) \right)^2 + \frac{1-t}{2} \omega^2(A) \left(\frac{1}{2} \| |A|^2 + |A^*|^2 \| + \omega(A^2) \right) \\ &\leq \frac{t}{4} \left(\frac{1}{2} \| |A|^2 + |A^*|^2 \| + \frac{1}{2} \| |A|^2 + |A^*|^2 \| \right)^2 \\ &\quad + \frac{1-t}{2} \frac{1}{2} \| |A|^2 + |A^*|^2 \| \left(\frac{1}{2} \| |A|^2 + |A^*|^2 \| + \frac{1}{2} \| |A|^2 + |A^*|^2 \| \right) \\ &= \frac{1}{4} \| |A|^2 + |A^*|^2 \|^2, \end{aligned}$$

thus Theorem 3.8 is a refinement of (4).

Next we give an alternative proof for (6), which we hope may provide new perspectives toward the integral type refinements of the numerical radius inequalities.

THEOREM 3.10. *Let $A \in B(\mathcal{H})$ and let f be a non-negative increasing operator convex function on $[0, \infty)$. Then*

$$\begin{aligned} f(\omega^2(A)) &\leq \left\| \int_0^1 f\left((1-t)\frac{|A|^2 + |A^*|^2}{2} + t\omega(A^2)I\right) dt \right\| \\ &\leq \frac{1}{4} \|f(|A|^2) + f(|A^*|^2)\| + \frac{1}{2} f(\omega(A^2)). \end{aligned}$$

Proof. From [15] we know that for two positive operator $X, Y \in B(\mathcal{H})$, $\|X + Y\| = \|X\| + \|Y\|$ if and only if $\|XY\| = \|X\|\|Y\|$. Since $\omega^2(A) \leq \frac{1}{4}\| |A|^2 + |A^*|^2 \| + \frac{1}{2}\omega(A^2)$, we have

$$\left\| \frac{|A|^2 + |A^*|^2}{2} \cdot \omega(A^2)I \right\| = \left\| \frac{|A|^2 + |A^*|^2}{2} \right\| \omega(A^2).$$

Thus we get

$$\left\| \frac{|A|^2 + |A^*|^2}{2} + \omega(A^2)I \right\| = \left\| \frac{|A|^2 + |A^*|^2}{2} \right\| + \omega(A^2).$$

Thus by Lemma 3.3, we have

$$\begin{aligned} f(\omega^2(A)) &\leq f\left(\frac{1}{2} \left\| \frac{|A|^2 + |A^*|^2}{2} + \omega(A^2)I \right\|\right) \\ &= \left\| f\left(\frac{\frac{|A|^2 + |A^*|^2}{2} + \omega(A^2)I}{2}\right) \right\| \\ &\leq \left\| \int_0^1 f\left((1-t)\frac{|A|^2 + |A^*|^2}{2} + t\omega(A^2)I\right) dt \right\| \\ &\leq \frac{1}{2} \left\| f\left(\frac{|A|^2 + |A^*|^2}{2}\right) + f(\omega(A^2)I) \right\| \\ &= \frac{1}{2} \left\| f\left(\frac{|A|^2 + |A^*|^2}{2}\right) \right\| + \frac{1}{2} f(\omega(A^2)) \\ &\leq \frac{1}{4} \|f(|A|^2) + f(|A^*|^2)\| + \frac{1}{2} f(\omega(A^2)). \quad \square \end{aligned}$$

Along the same line as in Theorem 3.10, one can obtain the following result.

THEOREM 3.11. *Let $A, B \in B(\mathcal{H})$ and let f be a non-negative increasing operator convex function on $[0, \infty)$. Then*

$$\begin{aligned} f(\omega^2(B^*A)) &\leq \left\| \int_0^1 f\left((1-t)\frac{|A|^4 + |B|^4}{2} + t\omega(|B|^2|A^2)I\right) dt \right\| \\ &\leq \frac{1}{4} \|f(|A|^4) + f(|B|^4)\| + \frac{1}{2} f(\omega(|B|^2|A^2)). \end{aligned}$$

THEOREM 3.12. *Let $A \in B(\mathcal{H})$ with the Cartesian decomposition $A = B + iC$ such that B, C be positive and let f be a non-negative increasing operator convex function on $[0, \infty)$. Then*

$$\begin{aligned} \left\| f\left(\frac{|A|^2 + |A^*|^2}{4}\right) \right\| &\leq \left\| \int_0^1 f\left(\frac{1-t}{2}|A|^2 + \frac{t}{2}|A^*|^2\right) dt \right\| \\ &\leq f(\omega^2(A)). \end{aligned}$$

Proof. Since $A = B + iC$ is the Cartesian decomposition of A and B, C are positive, it follows from [16] that $\frac{\|A\|^2}{2} \leq \omega^2(A)$. Therefore by Lemma 3.3, we get

$$\begin{aligned} \left\| f\left(\frac{|A|^2 + |A^*|^2}{4}\right) \right\| &= \left\| f\left(\frac{\frac{|A|^2}{2} + \frac{|A^*|^2}{2}}{2}\right) \right\| \\ &\leq \left\| \int_0^1 f\left(\frac{1-t}{2}|A|^2 + \frac{t}{2}|A^*|^2\right) dt \right\| \\ &\leq \frac{1}{2} \left\| f\left(\frac{|A|^2}{2}\right) + f\left(\frac{|A^*|^2}{2}\right) \right\| \\ &\leq \frac{1}{2} \left\| f\left(\frac{|A|^2}{2}\right) \right\| + \frac{1}{2} \left\| f\left(\frac{|A^*|^2}{2}\right) \right\| \\ &= \frac{1}{2} f\left(\left\| \frac{|A|^2}{2} \right\| \right) + \frac{1}{2} f\left(\left\| \frac{|A^*|^2}{2} \right\| \right) \\ &= f\left(\frac{\|A\|^2}{2}\right) \\ &\leq f(\omega^2(A)). \quad \square \end{aligned}$$

We note that under certain situations, Theorem 3.12 is stronger than Theorem 2.2 in [17] and Theorem 4 in [4]. Let $A = B + ixB$ such that $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $x > 0$, $C = xB$ and $f(s) = s^2$. Then

$$\begin{aligned} \left\| f\left(\frac{|A|^2 + |A^*|^2}{4}\right) \right\| &\leq \left\| \int_0^1 f\left(\frac{1-t}{2}|A|^2 + \frac{t}{2}|A^*|^2\right) dt \right\| \\ &= \left(\frac{1+x^2}{2}\right)^2 \leq \frac{1+x^4}{2} \\ &= \frac{1}{2} \|f(B^2) + f(C^2)\| \leq f(\omega^2(A)). \end{aligned}$$

COROLLARY 3.13. *Let $A \in B(\mathcal{H})$ with the Cartesian decomposition $A = B + iC$ such that B, C be positive and $r \in [1, 2]$. Then*

$$\begin{aligned} \left\| \left(\frac{|A|^2 + |A^*|^2}{4}\right)^r \right\| &\leq \left\| \int_0^1 \left(\frac{1-t}{2}|A|^2 + \frac{t}{2}|A^*|^2\right)^r dt \right\| \\ &\leq (\omega^2(A))^r. \end{aligned}$$

REMARK 3.14. In Corollary 3.13, by letting $r = 2$, we have

$$\begin{aligned} \left\| \frac{|A|^2 + |A^*|^2}{4} \right\| &\leq \left\| \int_0^1 \left(\frac{1-t}{2} |A|^2 + \frac{t}{2} |A^*|^2 \right)^2 dt \right\|^{\frac{1}{2}} \\ &\leq \omega^2(A), \end{aligned}$$

which is a refinement of (4).

In [8], Bhatia and Kittaneh presented an important norm inequality for two positive operators below: Let $A, B \in B(\mathcal{H})$ be positive and z be any complex number. Then

$$\|A + zB\| \leq \|A + |z|B\|. \quad (12)$$

Next, we shall give a refinement of (12).

THEOREM 3.15. Let $A, B \in B(\mathcal{H})$ be positive and let f be a non-negative increasing operator convex function on $[0, \infty)$. Then for any complex number z , we have

$$\begin{aligned} f(\|A + zB\|) &\leq \left\| \int_0^1 f\left((1-t)(A + |z|B) + t\|A + zB\|I\right) dt \right\| \\ &\leq \|f(A + |z|B)\|. \end{aligned}$$

Proof. Since $\|A + zB\| \leq \|A + |z|B\|$, from the proof of Theorem 3.10 we have

$$\|A + |z|B + \|A + zB\|I\| = \|A + |z|B\| + \|A + zB\|.$$

Thus we get

$$\|A + zB\| \leq \frac{1}{2} \|A + |z|B + \|A + zB\|I\|.$$

Thus by Lemma 3.3, we have

$$\begin{aligned} f(\|A + zB\|) &\leq f\left(\frac{1}{2} \|A + |z|B + \|A + zB\|I\|\right) \\ &= \left\| f\left(\frac{A + |z|B + \|A + zB\|I}{2}\right) \right\| \\ &\leq \left\| \int_0^1 f\left((1-t)(A + |z|B) + t\|A + zB\|I\right) dt \right\| \\ &\leq \frac{1}{2} \|f(A + |z|B) + f(\|A + zB\|)I\| \\ &= \frac{1}{2} \|f(A + |z|B)\| + \frac{1}{2} f(\|A + zB\|) \\ &\leq \frac{1}{2} \|f(A + |z|B)\| + \frac{1}{2} f(\|A + |z|B\|) \\ &= \frac{1}{2} \|f(A + |z|B)\| + \frac{1}{2} \|f(A + |z|B)\| \\ &= \|f(A + |z|B)\|. \quad \square \end{aligned}$$

COROLLARY 3.16. *Let $A, B \in B(\mathcal{H})$ be positive and $r \in [1, 2]$. Then for any complex number z , we have*

$$\begin{aligned} \|A + zB\|^r &\leq \left\| \int_0^1 \left((1-t)(A + |z|B) + t\|A + zB\|I \right)^r dt \right\| \\ &\leq \| (A + |z|B)^r \|. \end{aligned}$$

REMARK 3.17. In Corollary 3.16, by letting $r = 2$, we have

$$\begin{aligned} \|A + zB\| &\leq \left\| \int_0^1 \left((1-t)(A + |z|B) + t\|A + zB\|I \right)^2 dt \right\|^{\frac{1}{2}} \\ &\leq \|A + |z|B\|, \end{aligned}$$

which is a refinement of (12).

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