

EQUIVALENCE ON HIROSHIMA'S TYPE INEQUALITIES FOR POSITIVE SEMIDEFINITE BLOCK MATRICES

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Abstract. In this paper, we prove that some Hiroshima's type inequalities for positive semidefinite block matrices are equivalent. These interesting results are due to Hiroshima [Phys. Rev. Lett. **91**, (2003), 057902], Lin and Wolkowicz [Linear Multilinear Algebra **60**, 11–12 (2012), 1365–1368], Turkmen, Paksoy and Zhang [Linear Algebra Appl. **437**, 6 (2012), 1305–1316], Zhang and Xu [J. Math. Inequal. **14**, 4 (2020), 1383–1388], respectively.

1. Introduction

Let $M_{m,n}$ be the space of all complex matrices of size $m \times n$ with $M_n = M_{n,n}$. For $A \in M_n$, the vector of eigenvalues of A is denoted by $\lambda(A) = (\lambda_1(A), \lambda_2(A), \dots, \lambda_n(A))$ and A^* is the conjugate transpose of A . In this paper, we use $A \oplus B$ to represent the block matrix $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$. Now, we recall the definition of majorization. Given a real vector $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, we rearrange its components as $x_{[1]} \geq x_{[2]} \geq \dots \geq x_{[n]}$. For $x = (x_1, x_2, \dots, x_n), y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$, if

$$\sum_{i=1}^k x_{[i]} \leq \sum_{i=1}^k y_{[i]}, k = 1, 2, \dots, n,$$

then we say that x is weakly majorized by y and denote $x \prec_w y$. If $x \prec_w y$ and $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$ holds, then we say that x is majorized by y and denote $x \prec y$.

The study of eigenvalue majorization inequalities plays an important role in matrix analysis. A fundamental result due to Schur [1, 5, 8, 12], which stated that the diagonal entries of a Hermitian matrix A are majorized by its eigenvalues, i.e.,

$$\text{diag}(A) \prec \lambda(A). \tag{1.1}$$

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This result was extended by Ky Fan [2] to block Hermitian matrices, *i.e.*, let $H = \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{22} \end{bmatrix}$ be a block Hermitian matrix with $A_{11}, A_{22} \in M_n$. Then

$$\lambda(A_{11} \oplus A_{22}) \prec \lambda(H). \tag{1.2}$$

For the case of 2×2 block-partitioned positive semidefinite matrices, we list some related interesting results. Let $H = \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{22} \end{bmatrix}$ be a positive semidefinite block matrix with $A_{11}, A_{22} \in M_n$. Lin and Wolkowicz [7] proved that if $A_{12} \in M_n$ is Hermitian, then

$$\lambda(H) \prec \lambda((A_{11} + A_{22}) \oplus 0). \tag{1.3}$$

Turkmen, Paksoy and Zhang [9] proved a parallel result: If $A_{12} \in M_n$ is Skew-Hermitian, then

$$\lambda(H) \prec \lambda((A_{11} + A_{22}) \oplus 0). \tag{1.4}$$

Recently, Zhang [13] gave a general result:

$$\begin{aligned} \lambda(H) \prec & \frac{1}{2} \lambda([A_{11} + A_{22} + i(zA_{12}^* - \bar{z}A_{12})] \oplus 0) \\ & + \frac{1}{2} \lambda([A_{11} + A_{22} + i(\bar{z}A_{12} - zA_{12}^*)] \oplus 0), \end{aligned} \tag{1.5}$$

in which $i^2 = -1$ and $|z| = 1$. We notice that (1.3) and (1.4) are special cases of (1.5) by setting different values of z .

For the case of $s \times s$ block-partitioned positive semidefinite matrices with $s \geq 2$, there are some parallel results to (1.3) and (1.4). Let $H = [A_{ij}] \in M_{sn}$ be an $s \times s$ block-partitioned positive semidefinite matrix with $A_{ij} \in M_n$ for $i, j = 1, \dots, s$. In 2003, Hiroshima [4] proved a very attractive result, *i.e.*, if $A_{ij} \in M_n$ are Hermitian matrices for all $i \neq j$, then

$$\lambda(H) \prec \lambda \left(\left(\sum_{i=1}^s A_{ii} \right) \oplus 0 \right), \tag{1.6}$$

which is extremely valuable in quantum physics. Motivated by the inequality (1.4), Zhang and Xu [11] generalized (1.4) to the following: If $A_{ij} \in M_n$ are Skew-Hermitian for all $i \neq j$, then

$$\lambda(H) \prec \lambda \left(\left(\sum_{i=1}^s A_{ii} \right) \oplus 0 \right). \tag{1.7}$$

Another elegant proof of (1.7) is due to Zhang [10].

Although Hiroshima’s result (1.6) has useful applications in quantum physics (see, e.g., [3, 6]), it seems not widely known in the field of matrix analysis. Indeed, independent of the Hiroshima paper, Lin and Wolkowicz derived (1.3) as the special case of (1.6). In fact, (1.3) and (1.6) are equivalent. In this paper, we aim to show that all the above results are actually equivalent.

2. Main results

The following lemma is well-known.

LEMMA 2.1. [12, p. 32] *Let $A \in M_{m,n}$ with $m \geq n$. Then*

$$\lambda(AA^*) = \lambda(A^*A \oplus 0).$$

THEOREM 2.2. *The following statements are equivalent:*

(a) [7, Theorem 1.1] *Let $H = \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{22} \end{bmatrix}$ be a positive semidefinite block matrix with $A_{11}, A_{22} \in M_n$. If $A_{12} \in M_n$ is Hermitian, then*

$$\lambda(H) \prec \lambda((A_{11} + A_{22}) \oplus 0).$$

(b) [7, Corollary 2.2] *Let $A_1, A_2 \in M_n$ with $A_1^*A_2 = A_2^*A_1$. Then*

$$\lambda(A_1A_1^* + A_2A_2^*) \prec \lambda(A_1^*A_1 + A_2^*A_2).$$

(c) [4] *Let $A_1, A_2, \dots, A_s \in M_n$ ($s \geq 2$) with $A_i^*A_j = A_j^*A_i$ ($1 \leq i < j \leq s$). Then*

$$\lambda\left(\sum_{i=1}^s A_iA_i^*\right) \prec \lambda\left(\sum_{i=1}^s A_i^*A_i\right).$$

(d) [4] *Let $H = [A_{ij}] \in M_{sn}$ be an $s \times s$ block-partitioned positive semidefinite matrix with $A_{ij} \in M_n$ for $i, j = 1, \dots, s$, ($s \geq 2$). If $A_{ij} \in M_n$ are Hermitian matrices for all $i \neq j$, then*

$$\lambda(H) \prec \lambda\left(\left(\sum_{i=1}^s A_{ii}\right) \oplus 0\right).$$

Proof. (a) \Rightarrow (b) See [7, Theorem 1.1].

(b) \Rightarrow (c). We use induction on the number of matrices. For $s = 2$ the assertion is due to (b). Now let $s \geq 2$ and assume that the statement (c) holds for the number s of matrices.

Setting

$$H_1 = \begin{bmatrix} A_1 & 0 \\ 0 & A_1 \end{bmatrix}, \quad H_2 = \begin{bmatrix} A_2 & 0 \\ 0 & A_2 \end{bmatrix}, \dots, \quad H_{s-1} = \begin{bmatrix} A_{s-1} & 0 \\ 0 & A_{s-1} \end{bmatrix}$$

and

$$H_s = \begin{bmatrix} A_s & A_{s+1} \\ A_{s+1} & -A_s \end{bmatrix}.$$

Next we only need to show that $H_i^*H_j = H_j^*H_i$ ($1 \leq i < j \leq s$). We can divide it into two cases.

Case 1: $1 \leq i < j \leq s - 1$. It is easy to check that

$$H_i^* H_j = H_j^* H_i.$$

Case 2: $1 \leq i \leq s - 1, j = s$. By a straight calculation, we also have

$$H_i^* H_s = H_s^* H_i.$$

By the induction hypothesis, we get

$$\lambda \left(\sum_{k=1}^s H_k H_k^* \right) \prec \lambda \left(\sum_{k=1}^s H_k^* H_k \right).$$

That is

$$\lambda \left(\begin{bmatrix} \sum_{k=1}^{s+1} A_k A_k^* & A_s A_{s+1}^* - A_{s+1} A_s^* \\ A_{s+1} A_s^* - A_s A_{s+1}^* & \sum_{k=1}^{s+1} A_k A_k^* \end{bmatrix} \right) \prec \lambda \left(\begin{bmatrix} \sum_{k=1}^{s+1} A_k^* A_k & 0 \\ 0 & \sum_{k=1}^{s+1} A_k^* A_k \end{bmatrix} \right). \quad (2.3)$$

Applying (1.2), we have

$$\lambda \left(\begin{bmatrix} \sum_{k=1}^{s+1} A_k A_k^* & 0 \\ 0 & \sum_{k=1}^{s+1} A_k A_k^* \end{bmatrix} \right) \prec \lambda \left(\begin{bmatrix} \sum_{k=1}^{s+1} A_k A_k^* & A_s A_{s+1}^* - A_{s+1} A_s^* \\ A_{s+1} A_s^* - A_s A_{s+1}^* & \sum_{k=1}^{s+1} A_k A_k^* \end{bmatrix} \right). \quad (2.4)$$

Combining (2.3) and (2.4) gives

$$\lambda \left(\begin{bmatrix} \sum_{k=1}^{s+1} A_k A_k^* & 0 \\ 0 & \sum_{k=1}^{s+1} A_k A_k^* \end{bmatrix} \right) \prec \lambda \left(\begin{bmatrix} \sum_{k=1}^{s+1} A_k^* A_k & 0 \\ 0 & \sum_{k=1}^{s+1} A_k^* A_k \end{bmatrix} \right),$$

i.e.,

$$\lambda \left(\sum_{k=1}^{s+1} A_k A_k^* \right) \prec \lambda \left(\sum_{k=1}^{s+1} A_k^* A_k \right).$$

(c) \Rightarrow (d). Let $H = P^* P$, where $P = [A_1, A_2, \dots, A_s] \in M_{n, sm}$ with $A_i \in M_n$ for $1 \leq i \leq s$. By Lemma 2.1 and the statement (c), we have

$$\begin{aligned} \lambda(H) &= \lambda(P^* P) = \lambda(P P^* \oplus 0) = \lambda \left(\left(\sum_{i=1}^s A_i A_i^* \right) \oplus 0 \right) \\ &\prec \lambda \left(\left(\sum_{i=1}^s A_i^* A_i \right) \oplus 0 \right), \end{aligned}$$

i.e.,

$$\lambda(H) \prec \lambda \left(\left(\sum_{i=1}^s A_{ii} \right) \oplus 0 \right).$$

(d) \Rightarrow (a) Taking $s = 2$. \square

THEOREM 2.5. *The following statements are equivalent:*

- (e) [9, Corollary 5] *Let $H = \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{22} \end{bmatrix}$ be a positive semidefinite block matrix with $A_{11}, A_{22} \in M_n$. If $A_{12} \in M_n$ is Skew-Hermitian, then*

$$\lambda(H) \prec \lambda((A_{11} + A_{22}) \oplus 0).$$

- (f) [9, Corollary 4] *Let $A_1, A_2 \in M_n$ with $A_1^*A_2 = -A_2^*A_1$. Then*

$$\lambda(A_1A_1^* + A_2A_2^*) \prec \lambda(A_1^*A_1 + A_2^*A_2).$$

- (g) [11, Corollary 6] *Let $A_1, A_2, \dots, A_s \in M_n$ ($s \geq 2$) with $A_i^*A_j = -A_j^*A_i$ ($1 \leq i < j \leq s$). Then*

$$\lambda\left(\sum_{i=1}^s A_iA_i^*\right) \prec \lambda\left(\sum_{i=1}^s A_i^*A_i\right).$$

- (h) [11, Theorem 1] *Let $H = [A_{ij}] \in M_{sn}$ be an $s \times s$ block-partitioned positive semidefinite matrix with $A_{ij} \in M_n$ for $i, j = 1, \dots, s$, ($s \geq 2$). If $A_{ij} \in M_n$ are Skew-Hermitian matrices for all $i \neq j$, then*

$$\lambda(H) \prec \lambda\left(\left(\sum_{i=1}^s A_{ii}\right) \oplus 0\right).$$

Proof. The proof for Theorem 2.5 is similar to that for Theorem 2.2. We only show that (f) \Rightarrow (g). These are left as exercises for the reader.

We use induction on the number of matrices. For $s = 2$ the assertion is due to (f). Now let $s \geq 2$ and assume that the (g) holds for the number s of matrices. Setting

$$H_1 = \begin{bmatrix} A_1 & 0 \\ 0 & A_1 \end{bmatrix}, \quad H_2 = \begin{bmatrix} A_2 & 0 \\ 0 & A_2 \end{bmatrix}, \dots, \quad H_{s-1} = \begin{bmatrix} A_{s-1} & 0 \\ 0 & A_{s-1} \end{bmatrix}$$

and

$$H_s = \begin{bmatrix} -A_s & -A_{s+1} \\ -A_{s+1} & -A_s \end{bmatrix}.$$

For $1 \leq i < j \leq s-1$, it is clear that

$$H_i^*H_j = -H_j^*H_i.$$

For $1 \leq i \leq s-1$ and $j = s$, we also have

$$H_i^*H_s = -H_s^*H_i.$$

By the induction hypothesis, we obtain

$$\lambda\left(\sum_{k=1}^s H_kH_k^*\right) \prec \lambda\left(\sum_{k=1}^s H_k^*H_k\right).$$

That is

$$\lambda \left(\begin{bmatrix} \sum_{k=1}^{s+1} A_k A_k^* & A_s A_{s+1}^* + A_{s+1} A_s^* \\ A_{s+1} A_s^* + A_s A_{s+1}^* & \sum_{k=1}^{s+1} A_k A_k^* \end{bmatrix} \right) \prec \lambda \left(\begin{bmatrix} \sum_{k=1}^{s+1} A_k^* A_k & 0 \\ 0 & \sum_{k=1}^{s+1} A_k^* A_k \end{bmatrix} \right). \quad (2.6)$$

Applying (1.2), we have

$$\lambda \left(\begin{bmatrix} \sum_{k=1}^{s+1} A_k A_k^* & 0 \\ 0 & \sum_{k=1}^{s+1} A_k A_k^* \end{bmatrix} \right) \prec \lambda \left(\begin{bmatrix} \sum_{k=1}^{s+1} A_k A_k^* & A_s A_{s+1}^* + A_{s+1} A_s^* \\ A_{s+1} A_s^* + A_s A_{s+1}^* & \sum_{k=1}^{s+1} A_k A_k^* \end{bmatrix} \right). \quad (2.7)$$

Combining (2.6) and (2.7) gives

$$\lambda \left(\begin{bmatrix} \sum_{k=1}^{s+1} A_k A_k^* & 0 \\ 0 & \sum_{k=1}^{s+1} A_k A_k^* \end{bmatrix} \right) \prec \lambda \left(\begin{bmatrix} \sum_{k=1}^{s+1} A_k^* A_k & 0 \\ 0 & \sum_{k=1}^{s+1} A_k^* A_k \end{bmatrix} \right),$$

i.e.,

$$\lambda \left(\sum_{k=1}^{s+1} A_k A_k^* \right) \prec \lambda \left(\sum_{k=1}^{s+1} A_k^* A_k \right). \quad \square$$

REMARK 2.8. From the proof of [9, Corollary 4], we can see that the statement (b) in Theorem 2.2 and the statement (f) in Theorem 2.5 are equivalent. Therefore we can conclude that all the statements of Theorem 2.2 and Theorem 2.5 are equivalent.

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