

SHARP BOUND OF m -LINEAR n -DIMENSIONAL p -ADIC HAUSDORFF OPERATORS ON p -ADIC MORREY SPACES

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Abstract. In this paper, we study the sharp bound for m -linear n -dimensional p -adic Hausdorff operators on central and noncentral p -adic Morrey spaces with power weights, and we also give the sharp bounds of p -adic Hausdorff operators on central and noncentral p -adic Morrey spaces. Moreover, this is a generalization of the previous results.

1. Introduction and main results

For the past few years, people who study mathematical physics are more and more concerned with p -adic field, because p -adic is widely used in many mathematics and physics (cf. [1], [3], [11], [21] and [22]). For this reason, the harmonic analysis on p -adic field has attracted some attention (cf. [2], [9], [13], [17] and [18]).

As a prime number p , let \mathbb{Q}_p be the field of p -adic numbers, it's defined as a completely rational number field \mathbb{Q} with a non-Archimedean p -adic norm. This norm is following that: $|0|_p = 0$. If any non-zero rational number x can be expressed as $x = p^{\frac{\gamma}{n}}$, where m and n are integers that are not divisible by p , and γ is an integer, then $|x|_p = p^{-\gamma}$. It is not difficult to prove that the norm satisfies the following properties:

$$|xy|_p = |x|_p |y|_p \quad \text{and} \quad |x+y|_p \leq \max\{|x|_p, |y|_p\}.$$

From the second property, there is $|x+y|_p = \max\{|x|_p, |y|_p\}$ when $|x|_p \neq |y|_p$. From the standard p -adic analysis [22], we see that any non-zero p -adic number $x \in \mathbb{Q}_p$ can be uniquely represented as the standard series:

$$x = p^\gamma \sum_{j=0}^{\infty} a_j p^j, \quad \gamma = \gamma(x) \in \mathbb{Z}, \quad (1.1)$$

where a_j are integers, and $0 \leq a_j \leq p-1$, $a_0 \neq 0$. We calculate that $|a_j p^j|_p = p^{-\gamma}$, so the series (1.1) converges in the p -adic norm.

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This space \mathbb{Q}_p^n is formed by points $x = (x_1, x_2, \dots, x_n)$, where $x_j \in \mathbb{Q}_p$, $j = 1, 2, \dots, n$. The p -adic norm on \mathbb{Q}_p^n is

$$|x|_p := \max_{1 \leq j \leq n} |x_j|_p, \quad x \in \mathbb{Q}_p^n.$$

Denote the ball whose center at $a \in \mathbb{Q}_p^n$ radius p^γ is $B_\gamma(a) = \{x \in \mathbb{Q}_p^n : |x - a|_p \leq p^\gamma\}$, the sphere with center at $a \in \mathbb{Q}_p^n$ radius p^γ is defined as $S_\gamma(a) := \{x \in \mathbb{Q}_p^n : |x - a|_p = p^\gamma\}$. It is clear that $S_\gamma(a) = B_\gamma(a) \setminus B_{\gamma-1}(a)$, and

$$B_\gamma(a) = \bigcup_{k \leq \gamma} S_k(a).$$

We set $B_\gamma(0) = B_\gamma$ and $S_\gamma(0) = S_\gamma$.

\mathbb{Q}_p^n is a locally compact commutative group under addition, it follows from the standard analysis that there exists a unique Harr measure dx on \mathbb{Q}_p^n , which is a translation invariant. We normalize the measure dx so that $|B_0(0)|_H = 1$, where $|E|_H$ denotes the Harr measure of a measurable subset E of \mathbb{Q}_p^n .

From this theory of integral, it is easy to obtain that $|S_\gamma(a)|_H = p^m(1 - p^{-n})$ and $|B_\gamma(a)|_H = p^{\gamma m}$ for any $a \in \mathbb{Q}_p^n$. For a more complete introduction to p -adic field, see [13, 22] and the references therein.

Many researchers introduce a lot of cutting edge knowledge about Fourier analysis (cf. [8], [12] and [20]), which has enriched our knowledge and broadened our horizon. In this paper, we study some properties and applications about Hausdorff operators (cf. [25] and [24]), and we generalize some known results to the p -adic field.

In [4] and [6], we realize many theorems and properties of Morrey spaces, and some professors think about the problem of boundedness of operators on function spaces (cf. [5], [10], [14], [16], [19] and [23]). These papers provide some ideas for us to solve this problem.

In [15], the authors study the sharp bound of p -adic Hardy-Littlewood-pólya operators on p -adic Morrey spaces, and they prove that the sharp bound is

$$\int_{\mathbb{Q}_p^n} \cdots \int_{\mathbb{Q}_p^n} K(y_1, \dots, y_m) \prod_{i=1}^m |y_i|_p^{-n\lambda_i + \frac{\beta_i}{q} - \alpha(\lambda_i + \frac{1}{q_i})} dy_1 \cdots dy_m,$$

In [25], Zhang, Wei and Yan study the Hausdorff operators on Morrey spaces. The n -dimensional Hausdorff operator is defined by

$$\mathcal{H}_\Phi f(x) = \int_{\mathbb{R}^n} \frac{\Phi(y)}{|y|^n} f\left(\frac{x}{|y|}\right) dy,$$

and the sharp bound is

$$C := \int_{\mathbb{R}^n} \frac{\Phi(y)}{|y|^{n+(\alpha+n)\lambda}} dy < \infty.$$

Inspired by [15] and [25], we study the sharp bound of m -linear n -dimensional p -adic Hausdorff operators on p -adic Morrey spaces, then we extend their results to the p -adic field.

DEFINITION 1.1. The n -dimensional p -adic Hausdorff operators is defined by

$$\mathcal{H}_\Phi f(x) = \int_{\mathbb{Q}_p^n} \frac{\Phi(y)}{|y|_p^n} f\left(\frac{x}{|y|_p}\right) dy, \quad x \in \mathbb{Q}_p^n$$

and the m -linear n -dimensional p -adic Hausdorff operators is defined by

$$\mathcal{H}_\Phi^m f(x_1, \dots, x_m) = \int_{\mathbb{Q}_p^n} \cdots \int_{\mathbb{Q}_p^n} \frac{\Phi(y_1, \dots, y_m)}{|y_1|_p^n \cdots |y_m|_p^n} f_1\left(\frac{x}{|y_1|_p}\right) \cdots f_m\left(\frac{x}{|y_m|_p}\right) dy_1 \cdots dy_m,$$

where $x = (x_1, \dots, x_m) \in \mathbb{Q}_p^n \times \cdots \times \mathbb{Q}_p^n$.

DEFINITION 1.2. Let $1 \leq q < \infty$ and $\frac{1}{q} < \lambda < 0$, then the weighted p -adic Morrey space $L^{q,\lambda}(\mathbb{Q}_p^n, w_1, w_2)$ is the set of all $f \in L_{loc}^q(\mathbb{Q}_p^n)$ for which the norm

$$\|f\|_{L^{q,\lambda}(\mathbb{Q}_p^n, w_1, w_2)} = \sup_{\gamma \in \mathbb{Z}, a \in \mathbb{Q}_p^n} \left(\int_{B_\gamma(a)} w_1(x) dx \right)^{-\left(\lambda + \frac{1}{q}\right)} \left(\int_{B_\gamma(a)} |f(x)|^q w_2(x) dx \right)^{\frac{1}{q}} < +\infty.$$

If B_γ replaces with $B_\gamma(a)$ in the above definition, $L^{q,\lambda}(\mathbb{Q}_p^n, w_1, w_2)$ becomes the weighted p -adic central Morrey spaces $B^{q,\lambda}(\mathbb{Q}_p^n, w_1, w_2)$. In this paper, we particularly research the norm

$$\|f\|_{L^{q,\lambda}(\mathbb{Q}_p^n, |x|_p^\alpha, |x|_p^\beta)} = \sup_{\gamma \in \mathbb{Z}, a \in \mathbb{Q}_p^n} \left(\int_{B_\gamma(a)} |x|_p^\alpha dx \right)^{-\left(\lambda + \frac{1}{q}\right)} \left(\int_{B_\gamma(a)} |f(x)|^q |x|_p^\beta dx \right)^{\frac{1}{q}}$$

and

$$\|f\|_{B^{q,\lambda}(\mathbb{Q}_p^n, |x|_p^\alpha, |x|_p^\beta)} = \sup_{\gamma \in \mathbb{Z}} \left(\int_{B_\gamma} |x|_p^\alpha dx \right)^{-\left(\lambda + \frac{1}{q}\right)} \left(\int_{B_\gamma} |f(x)|^q |x|_p^\beta dx \right)^{\frac{1}{q}}.$$

In this paper, we expound our results in the setting of $L^{q,\lambda}(\mathbb{Q}_p^n, |x|_p^\alpha, |x|_p^\beta)$ and $B^{q,\lambda}(\mathbb{Q}_p^n, |x|_p^\alpha, |x|_p^\beta)$, and we obtain the operator norm of m -linear p -adic operators as well as the one for the Hausdorff operator \mathcal{H}_Φ . Next, we will prove the following results.

THEOREM 1.3. Let $1 \leq q < \infty$, $-1/q \leq \lambda < 0$, and $\alpha > 0$, if

$$C_{sharp} = \int_{\mathbb{Q}_p^n} \Phi(y) |y|_p^{n(\lambda-1) - \frac{\beta}{q} + \alpha(\lambda + \frac{1}{q})} dy < \infty,$$

then \mathcal{H}_Φ is bounded from $B^{q,\lambda}(\mathbb{Q}_p^n, |x|_p^\alpha, |x|_p^\beta)$ to $B^{q,\lambda}(\mathbb{Q}_p^n, |x|_p^\alpha, |x|_p^\beta)$ with its operator norm not more than C_{sharp} . Moreover,

$$\|\mathcal{H}_\Phi\|_{B^{q,\lambda}(\mathbb{Q}_p^n, |x|_p^\alpha, |x|_p^\beta) \rightarrow B^{q,\lambda}(\mathbb{Q}_p^n, |x|_p^\alpha, |x|_p^\beta)} = C_{sharp}.$$

COROLLARY 1.4. Assume that q, λ, α and C_{sharp} as in Theorem 1.3. Then \mathcal{H}_Φ is bounded from $L^{q,\lambda}(\mathbb{Q}_p^n, |x|_p^\alpha, |x|_p^\beta)$ to $L^{q,\lambda}(\mathbb{Q}_p^n, |x|_p^\alpha, |x|_p^\beta)$ with its operator norm not more than C_{sharp} . Moreover,

$$\|\mathcal{H}_\Phi\|_{L^{q,\lambda}(\mathbb{Q}_p^n, |x|_p^\alpha, |x|_p^\beta) \rightarrow L^{q,\lambda}(\mathbb{Q}_p^n, |x|_p^\alpha, |x|_p^\beta)} = C_{sharp}.$$

THEOREM 1.5. Let $1 < q < q_i < \infty$, $1/q = 1/q_1 + \dots + 1/q_m$, $\beta = \beta_1 + \dots + \beta_m$, $\lambda = \lambda_1 + \dots + \lambda_m$, $-1/q_i < \lambda_i < 0$ and $\alpha > 0$, with $i = 1, \dots, m$.

(i) If

$$C_m = \int_{\mathbb{Q}_p^n} \dots \int_{\mathbb{Q}_p^n} \Phi(y_1, \dots, y_m) \prod_{i=1}^m |y_i|_p^{n(\lambda_i-1) - \frac{\beta_i}{q} + \alpha(\lambda_i + \frac{1}{q_i})} dy_1 \dots dy_m < \infty, \tag{1.2}$$

then \mathcal{H}_Φ^m is bounded from $B^{q_1, \lambda_1}(\mathbb{Q}_p^n, |x|_p^\alpha, |x|_p^{\frac{q_1 \beta_1}{q}}) \times \dots \times B^{q_m, \lambda_m}(\mathbb{Q}_p^n, |x|_p^\alpha, |x|_p^{\frac{q_m \beta_m}{q}})$ to $B^{q, \lambda}(\mathbb{Q}_p^n, |x|_p^\alpha, |x|_p^\beta)$ with its operators norm not more than C_m .

(ii) Assume that $\lambda q = \lambda_1 q_1 = \dots = \lambda_m q_m$. In this case, condition (1.2) is also necessary for the boundedness of $\mathcal{H}_\Phi^m : B^{q_1, \lambda_1}(\mathbb{Q}_p^n, |x|_p^\alpha, |x|_p^{\frac{q_1 \beta_1}{q}}) \times \dots \times B^{q_m, \lambda_m}(\mathbb{Q}_p^n, |x|_p^\alpha, |x|_p^{\frac{q_m \beta_m}{q}}) \rightarrow B^{q, \lambda}(\mathbb{Q}_p^n, |x|_p^\alpha, |x|_p^\beta)$. Moreover,

$$\|\mathcal{H}_\Phi^m\|_{B^{q_1, \lambda_1}(\mathbb{Q}_p^n, |x|_p^\alpha, |x|_p^{\frac{q_1 \beta_1}{q}}) \times \dots \times B^{q_m, \lambda_m}(\mathbb{Q}_p^n, |x|_p^\alpha, |x|_p^{\frac{q_m \beta_m}{q}}) \rightarrow B^{q, \lambda}(\mathbb{Q}_p^n, |x|_p^\alpha, |x|_p^\beta)} = C_m.$$

COROLLARY 1.6. Let $\alpha > 0$, $1 < q < q_i < \infty$, $1/q = 1/q_1 + \dots + 1/q_m$, $\beta = \beta_1 + \dots + \beta_m$, $\lambda = \lambda_1 + \dots + \lambda_m$ and $-1/q_i < \lambda_i < 0$ with $i = 1, \dots, m$. Then \mathcal{H}_Φ^m is bounded from $L^{q_1, \lambda_1}(\mathbb{Q}_p^n, |x|_p^\alpha, |x|_p^{\frac{q_1 \beta_1}{q}}) \times \dots \times L^{q_m, \lambda_m}(\mathbb{Q}_p^n, |x|_p^\alpha, |x|_p^{\frac{q_m \beta_m}{q}})$ to $L^{q, \lambda}(\mathbb{Q}_p^n, |x|_p^\alpha, |x|_p^\beta)$ with its operators norm no more than C_m , where C_m is defined as in (1.2). Furthermore, assume that $\lambda q = \lambda_1 q_1 = \dots = \lambda_m q_m$. In this case, condition (1.2) is also necessary for the boundedness of $\mathcal{H}_\Phi^m : L^{q_1, \lambda_1}(\mathbb{Q}_p^n, |x|_p^\alpha, |x|_p^{\frac{q_1 \beta_1}{q}}) \times \dots \times L^{q_m, \lambda_m}(\mathbb{Q}_p^n, |x|_p^\alpha, |x|_p^{\frac{q_m \beta_m}{q}}) \rightarrow L^{q, \lambda}(\mathbb{Q}_p^n, |x|_p^\alpha, |x|_p^\beta)$. Moreover,

$$\|\mathcal{H}_\Phi^m\|_{L^{q_1, \lambda_1}(\mathbb{Q}_p^n, |x|_p^\alpha, |x|_p^{\frac{q_1 \beta_1}{q}}) \times \dots \times L^{q_m, \lambda_m}(\mathbb{Q}_p^n, |x|_p^\alpha, |x|_p^{\frac{q_m \beta_m}{q}}) \rightarrow L^{q, \lambda}(\mathbb{Q}_p^n, |x|_p^\alpha, |x|_p^\beta)} = C_m.$$

2. Proof of the main results

Proof of Theorem 1.3. Using the generalized Minkowski’s inequality and the scaling law, we have

$$\begin{aligned} \|\mathcal{H}\Phi f\|_{B^{q,\lambda}(\mathbb{Q}_p^n, |x|^\alpha, |x|^\beta)} &= \left\| \int_{\mathbb{Q}_p^n} \frac{\Phi(y)}{|y|_p^n} f(|y|_p^{-1} \cdot) dy \right\|_{B^{q,\lambda}(\mathbb{Q}_p^n, |x|^\alpha, |x|^\beta)} \\ &\leq \int_{\mathbb{Q}_p^n} \frac{\Phi(y)}{|y|_p^n} \|f(|y|_p^{-1} \cdot)\|_{B^{q,\lambda}(\mathbb{Q}_p^n, |x|^\alpha, |x|^\beta)} dy. \end{aligned}$$

According to Lemma 2 in [15], we obtain

$$\|f(|y|_p^{-1} \cdot)\|_{B^{q,\lambda}(\mathbb{Q}_p^n, |x|^\alpha, |x|^\beta)} = |y|_p^{n\lambda - \frac{\beta}{q} + \alpha(\lambda + \frac{1}{q})} \|f\|_{B^{q,\lambda}(\mathbb{Q}_p^n, |x|^\alpha, |x|^\beta)}.$$

It gives that

$$\begin{aligned} \|\mathcal{H}\Phi f\|_{B^{q,\lambda}(\mathbb{Q}_p^n, |x|^\alpha, |x|^\beta)} &\leq \int_{\mathbb{Q}_p^n} \frac{\Phi(y)}{|y|_p^n} |y|_p^{n\lambda - \frac{\beta}{q} + \alpha(\lambda + \frac{1}{q})} dy \cdot \|f\|_{B^{q,\lambda}(\mathbb{Q}_p^n, |x|^\alpha, |x|^\beta)} \\ &= C_{\text{sharp}} \cdot \|f\|_{B^{q,\lambda}(\mathbb{Q}_p^n, |x|^\alpha, |x|^\beta)}. \end{aligned}$$

Next, we will prove C_{sharp} is the sharp constant. Define

$$f(x) = |x|_p^{n\lambda - \frac{\beta}{q} + \alpha(\lambda + \frac{1}{q})}.$$

By a simple compute, we obtain

$$\begin{aligned} \|f\|_{B^{q,\lambda}(\mathbb{Q}_p^n, |x|^\alpha, |x|^\beta)} &= \sup_{\gamma \in \mathbb{Z}} \left(\int_{B_\gamma} |x|_p^\alpha dx \right)^{-(\lambda + \frac{1}{q})} \left(\int_{B_\gamma} |f(x)|^q |x|_p^\beta dx \right)^{\frac{1}{q}} \\ &= \sup_{\gamma \in \mathbb{Z}} \left(\int_{B_\gamma} |x|_p^\alpha dx \right)^{-(\lambda + \frac{1}{q})} \left(\int_{B_\gamma} |x|_p^{nq\lambda - \beta + \alpha(q\lambda + 1)} |x|_p^\beta dx \right)^{\frac{1}{q}} \\ &= \sup_{\gamma \in \mathbb{Z}} \left(\int_{B_\gamma} |x|_p^\alpha dx \right)^{-(\lambda + \frac{1}{q})} \left(\int_{B_\gamma} |x|_p^{nq\lambda + \alpha(q\lambda + 1)} dx \right)^{\frac{1}{q}} \\ &= \sup_{\gamma \in \mathbb{Z}} \left(\sum_{k=-\infty}^{\gamma} p^{\alpha k} \int_{S_k} dx \right)^{-(\lambda + \frac{1}{q})} \left(\sum_{k=-\infty}^{\gamma} p^{(nq\lambda + \alpha(q\lambda + 1))k} \int_{S_k} dx \right)^{\frac{1}{q}} \\ &= \sup_{\gamma \in \mathbb{Z}} \left(\int_{S_\gamma} dx \right)^{-\lambda} \left(\sum_{k=-\infty}^{\gamma} p^{\alpha k} \right)^{-(\lambda + \frac{1}{q})} \left(\sum_{k=-\infty}^{\gamma} p^{(nq\lambda + \alpha(q\lambda + 1))k} \right)^{\frac{1}{q}}. \end{aligned}$$

Note that $\int_{S_k} dx = 1 - p^{-n}$, $\sum_{k=-\infty}^{\gamma} p^{\alpha k} = \frac{p^{\alpha\gamma}}{1 - p^{-\alpha}}$ and

$$\sum_{k=-\infty}^{\gamma} p^{(nq\lambda + \alpha(q\lambda + 1))k} = \frac{p^{\gamma(nq\lambda + \alpha(q\lambda + 1))}}{1 - p^{-(nq\lambda + \alpha(q\lambda + 1))}}. \tag{2.3}$$

Thus, we have

$$\|f\|_{B^{q,\lambda}(\mathbb{Q}_p^n, |x|_p^\alpha, |x|_p^\beta)} = \frac{(1 - p^{-n})^{-\lambda}}{(1 - p^{-\alpha})^{-(\lambda + \frac{1}{q})} (1 - p^{-(nq\lambda + \alpha(q\lambda + 1))})^{\frac{1}{q}}}.$$

It implies that $f(x) \in B^{q,\lambda}(\mathbb{Q}_p^n, |x|_p^\alpha, |x|_p^\beta)$.

By the definition of \mathcal{H}_Φ , we calculate that

$$\begin{aligned} & \|\mathcal{H}_\Phi f\|_{B^{q,\lambda}(\mathbb{Q}_p^n, |x|_p^\alpha, |x|_p^\beta)} \\ &= \sup_{\gamma \in \mathbb{Z}} \left(\int_{B_\gamma} |x|_p^\alpha dx \right)^{-(\lambda + \frac{1}{q})} \left(\int_{B_\gamma} |\mathcal{H}_\Phi f(x)|^q |x|_p^\beta dx \right)^{\frac{1}{q}} \\ &= \sup_{\gamma \in \mathbb{Z}} \left(\int_{B_\gamma} |x|_p^\alpha dx \right)^{-(\lambda + \frac{1}{q})} \left(\int_{B_\gamma} \left| \int_{Q_p^n} \frac{\Phi(y)}{|y|_p^n} f(|y|_p^{-1}x) dy \right|^q |x|_p^\beta dx \right)^{\frac{1}{q}} \\ &= \sup_{\gamma \in \mathbb{Z}} \left(\int_{B_\gamma} |x|_p^\alpha dx \right)^{-(\lambda + \frac{1}{q})} \left(\int_{B_\gamma} \left| \int_{Q_p^n} |y|_p^{-n} \Phi(y) |y|_p^{-1} |x|_p^{n\lambda - \frac{\beta}{q} + \alpha(\lambda + \frac{1}{q})} dy \right|^q |x|_p^\beta dx \right)^{\frac{1}{q}} \\ &= \sup_{\gamma \in \mathbb{Z}} \left(\int_{B_\gamma} |x|_p^\alpha dx \right)^{-(\lambda + \frac{1}{q})} \\ & \quad \times \left(\int_{B_\gamma} \left| \int_{Q_p^n} \Phi(y) |y|_p^{n(\lambda - 1) - \frac{\beta}{q} + \alpha(\lambda + \frac{1}{q})} |x|_p^{n\lambda - \frac{\beta}{q} + \alpha(\lambda + \frac{1}{q})} dy \right|^q |x|_p^\beta dx \right)^{\frac{1}{q}} \\ &= C_{\text{sharp}} \sup_{\gamma \in \mathbb{Z}} \left(\int_{B_\gamma} |x|_p^\alpha dx \right)^{-(\lambda + \frac{1}{q})} \left(\int_{B_\gamma} |x|_p^{nq\lambda - \beta + \alpha(q\lambda + 1)} |x|_p^\beta dx \right)^{\frac{1}{q}}. \end{aligned}$$

Using the above argument, we have

$$\sup_{\gamma \in \mathbb{Z}} \left(\int_{B_\gamma} |x|_p^\alpha dx \right)^{-(\lambda + \frac{1}{q})} \left(\int_{B_\gamma} |x|_p^{nq\lambda - \beta + \alpha(q\lambda + 1)} |x|_p^\beta dx \right)^{\frac{1}{q}} = \|f\|_{B^{q,\lambda}(\mathbb{Q}_p^n, |x|_p^\alpha, |x|_p^\beta)}.$$

Therefore, we show that

$$C_{\text{sharp}} \leq \|\mathcal{H}_\Phi f\|_{B^{q,\lambda}(\mathbb{Q}_p^n, |x|_p^\alpha, |x|_p^\beta) \rightarrow B^{q,\lambda}(\mathbb{Q}_p^n, |x|_p^\alpha, |x|_p^\beta)} < \infty.$$

This finishes the proof of Theorem 1.3. \square

Proof of Corollary 1.4. Similar to prove Theorem 1.3, it is easy to obtain that

$$\|\mathcal{H}_\Phi f\|_{L^{q,\lambda}(\mathbb{Q}_p^n, |x|_p^\alpha, |x|_p^\beta) \rightarrow L^{q,\lambda}(\mathbb{Q}_p^n, |x|_p^\alpha, |x|_p^\beta)} \leq C_{\text{sharp}}.$$

Let $f(x) = |x|_p^{n\lambda - \frac{\beta}{q} + \alpha(\lambda + \frac{1}{q})}$, then for any $B_\gamma(a) = B(a, p^\gamma)$, we need to show that $f(x) \in L^{q,\lambda}(\mathbb{Q}_p^n, |x|_p^\alpha, |x|_p^\beta)$. Considering the following two cases.

(i) If $|a|_p > p^\gamma$ and $x \in B_\gamma(a)$, then $|x|_p = \max\{|x-a|_p, |a|_p\} = |a|_p > p^\gamma$, so we have

$$\begin{aligned} & \|f\|_{L^{q,\lambda}(\mathbb{Q}_p^n, |x|_p^\alpha, |x|_p^\beta)} \\ &= \sup_{a \in \mathbb{Q}_p^n, \gamma \in \mathbb{Z}} \left(\int_{B_\gamma(a)} |x|_p^\alpha dx \right)^{-(\lambda + \frac{1}{q})} \left(\int_{B_\gamma(a)} |x|_p^{nq\lambda - \beta + \alpha(\lambda q + 1)} |x|_p^\beta dx \right)^{\frac{1}{q}} \\ &= \sup_{a \in \mathbb{Q}_p^n, \gamma \in \mathbb{Z}} |a|_p^{n\lambda} |B_\gamma(a)|_H^{-\lambda} \\ &= \sup_{a \in \mathbb{Q}_p^n, \gamma \in \mathbb{Z}} |a|_p^{n\lambda} p^{-n\lambda\gamma} \leq 1. \end{aligned}$$

(ii) If $|a|_p \leq p^\gamma$ and $x \in B_\gamma(a)$, then $|x|_p = \max\{|x-a|_p, |a|_p\} \leq p^\gamma$. Therefore $x \in B_\gamma$. Recall that two balls in \mathbb{Q}_p^n are either disjoint or one is contained in the other (see [2]). So we have $B_\gamma(a) = B_\gamma$, thus

$$\begin{aligned} & \|f\|_{L^{q,\lambda}(\mathbb{Q}_p^n, |x|_p^\alpha, |x|_p^\beta)} \\ &= \sup_{a \in \mathbb{Q}_p^n, \gamma \in \mathbb{Z}} \left(\int_{B_\gamma(a)} |x|_p^\alpha dx \right)^{-(\lambda + \frac{1}{q})} \left(\int_{B_\gamma(a)} |x|_p^{nq\lambda - \beta + \alpha(\lambda q + 1)} |x|_p^\beta dx \right)^{\frac{1}{q}} \\ &= \sup_{\gamma \in \mathbb{Z}} \left(\int_{B_\gamma} |x|_p^\alpha dx \right)^{-(\lambda + \frac{1}{q})} \left(\int_{B_\gamma} |x|_p^{nq\lambda + \alpha(\lambda q + 1)} dx \right)^{\frac{1}{q}}. \end{aligned}$$

According to the proof of Theorem 1.3, we have that for $|a|_p \leq p^\gamma$ and $x \in B_\gamma(a)$

$$\|f\|_{L^{q,\lambda}(\mathbb{Q}_p^n, |x|_p^\alpha, |x|_p^\beta)} < \infty.$$

This gives that $f(x) \in L^{q,\lambda}(\mathbb{Q}_p^n, |x|_p^\alpha, |x|_p^\beta)$. By the similar argument to prove Theorem 1.3, we have

$$C_{\text{sharp}} \leq \| \mathcal{H}_\Phi f \|_{L^{q,\lambda}(\mathbb{Q}_p^n, |x|_p^\alpha, |x|_p^\beta) \rightarrow L^{q,\lambda}(\mathbb{Q}_p^n, |x|_p^\alpha, |x|_p^\beta)} < \infty.$$

Then we finish the proof of Corollary 1.4. \square

Proof of Theorem 1.5. By using Minkowski's inequality, we have

$$\begin{aligned} & \| \mathcal{H}_\Phi^m(f_1, \dots, f_m) \|_{B^{q,\lambda}(\mathbb{Q}_p^n, |x|_p^\alpha, |x|_p^\beta)} \\ &= \left\| \int_{\mathbb{Q}_p^n} \dots \int_{\mathbb{Q}_p^n} \frac{\Phi(y_1, \dots, y_m)}{|y_1|_p^n \dots |y_m|_p^n} \prod_{i=1}^m f_i(|y_i|_p^{-1}) dy_1 \dots dy_m \right\|_{B^{q,\lambda}(\mathbb{Q}_p^n, |x|_p^\alpha, |x|_p^\beta)} \\ &\leq \int_{\mathbb{Q}_p^n} \dots \int_{\mathbb{Q}_p^n} \frac{\Phi(y_1, \dots, y_m)}{|y_1|_p^n \dots |y_m|_p^n} \prod_{i=1}^m \|f_i(|y_i|_p^{-1})\|_{B^{q_i, \lambda_i}(\mathbb{Q}_p^n, |x|_p^\alpha, |x|_p^{\frac{q_i \beta_i}{q}})} dy_1 \dots dy_m. \end{aligned}$$

According to Lemma 2 in [15], we have

$$\|f_i(|y_i|_p^{-1} \cdot)\|_{B^{q_i \cdot \lambda_i}(\mathbb{Q}_p^n, |x|_p^\alpha, |x|_p^{\frac{q_i \beta_i}{q}})} = |y_i|_p^{n\lambda_i - \frac{\beta_i}{q} + \alpha(\lambda_i + \frac{1}{q_i})} \|f_i\|_{B^{q_i \cdot \lambda_i}(\mathbb{Q}_p^n, |x|_p^\alpha, |x|_p^{\frac{q_i \beta_i}{q}})}. \tag{2.4}$$

Inequality (2.4) gives that

$$\begin{aligned} & \| \mathcal{H}_\Phi^m f(x_1, \dots, x_m) \|_{B^{q, \lambda}(\mathbb{Q}_p^n, |x|_p^\alpha, |x|_p^\beta)} \\ & \leq \int_{\mathbb{Q}_p^n} \cdots \int_{\mathbb{Q}_p^n} \Phi(y_1, \dots, y_m) \prod_{i=1}^m |y_i|_p^{n\lambda_i - \frac{\beta_i}{q} + \alpha(\lambda_i + \frac{1}{q_i})} \|f_i\|_{B^{q_i \cdot \lambda_i}(\mathbb{Q}_p^n, |x|_p^\alpha, |x|_p^{\frac{q_i \beta_i}{q}})} dy_1 \cdots dy_m \\ & = C_m \prod_{i=1}^m \|f_i\|_{B^{q_i \cdot \lambda_i}(\mathbb{Q}_p^n, |x|_p^\alpha, |x|_p^{\frac{q_i \beta_i}{q}})}. \end{aligned}$$

For the necessity, when $\lambda q = \lambda_1 q_1 = \cdots = \lambda_m q_m$ and $\alpha > 0$, let $f_i(x) = |x|_p^{n\lambda_i - \frac{\beta_i}{q} + \alpha(\lambda_i + \frac{1}{q_i})}$ ($i = 1, \dots, m$). Then, by a simple compute and equality (2.3), we have

$$\begin{aligned} & \|f_i\|_{B^{q_i \cdot \lambda_i}(\mathbb{Q}_p^n, |x|_p^\alpha, |x|_p^{\frac{q_i \lambda_i}{q}})} \\ & = \sup_{\gamma \in \mathbb{Z}} \left(\int_{B_\gamma} |x|_p^\alpha dx \right)^{-(\lambda_i + \frac{1}{q_i})} \left(\int_{B_\gamma} |x|_p^{nq_i \lambda_i - \frac{q_i \lambda_i}{q} + \alpha(q_i \lambda_i + 1)} |x|_p^{\frac{q_i \lambda_i}{q}} dx \right)^{\frac{1}{q_i}} \\ & = \sup_{\gamma \in \mathbb{Z}} \left(\int_{B_\gamma} |x|_p^\alpha dx \right)^{-(\lambda_i + \frac{1}{q_i})} \left(\int_{B_\gamma} |x|_p^{nq_i \lambda_i + \alpha(q_i \lambda_i + 1)} dx \right)^{\frac{1}{q_i}} \\ & = \sup_{\gamma \in \mathbb{Z}} \left(\sum_{k=-\infty}^{\gamma} p^{\alpha k} \int_{S_k} dx \right)^{-(\lambda_i + \frac{1}{q_i})} \left(\sum_{k=-\infty}^{\gamma} p^{(nq_i \lambda_i + \alpha(q_i \lambda_i + 1))k} \int_{S_k} dx \right)^{\frac{1}{q_i}} \\ & = \sup_{\gamma \in \mathbb{Z}} \left(\int_{S_k} dx \right)^{-\lambda_i} \left(\sum_{k=-\infty}^{\gamma} p^{\alpha k} \right)^{-(\lambda_i + \frac{1}{q_i})} \left(\sum_{k=-\infty}^{\gamma} p^{(nq_i \lambda_i + \alpha(q_i \lambda_i + 1))k} \right)^{\frac{1}{q_i}} \\ & = \frac{(1 - p^{-n})^{-\lambda_i}}{(1 - p^{-\alpha})^{-(\lambda_i + \frac{1}{q_i})} (1 - p^{-(nq_i \lambda_i + \alpha(q_i \lambda_i + 1))})^{\frac{1}{q_i}}}. \end{aligned} \tag{2.5}$$

It infer that $f_i \in B^{q_i \cdot \lambda_i}(\mathbb{Q}_p^n, |x|_p^\alpha, |x|_p^{\frac{q_i \lambda_i}{q}})$ with ($i = 1, \dots, m$). Note that

$$\begin{aligned} & \mathcal{H}_\Phi^m(f_1, \dots, f_m)(x) \\ & = \int_{\mathbb{Q}_p^n} \cdots \int_{\mathbb{Q}_p^n} \frac{\Phi(y_1, \dots, y_m)}{|y_1|_p^n \cdots |y_m|_p^n} \prod_{i=1}^m |y_i|_p^{-1} |x|_p^{n\lambda_i - \frac{\beta_i}{q} + \alpha(\lambda_i + \frac{1}{q_i})} dy_1 \cdots dy_m \\ & = \int_{\mathbb{Q}_p^n} \cdots \int_{\mathbb{Q}_p^n} \frac{\Phi(y_1, \dots, y_m)}{|y_1|_p^n \cdots |y_m|_p^n} \prod_{i=1}^m |y_i|_p^{n\lambda_i - \frac{\beta_i}{q} + \alpha(\lambda_i + \frac{1}{q_i})} |x|_p^{n\lambda_i - \frac{\beta_i}{q} + \alpha(\lambda_i + \frac{1}{q_i})} dy_1 \cdots dy_m \end{aligned}$$

$$\begin{aligned}
 &= \prod_{i=1}^m |x|_p^{n\lambda_i - \frac{\beta_i}{q} + \alpha(\lambda_i + \frac{1}{q_i})} \int_{\mathbb{Q}_p^n} \cdots \int_{\mathbb{Q}_p^n} \Phi(y_1, \dots, y_m) \prod_{i=1}^m |y_i|_p^{n(\lambda_i - 1) - \frac{\beta_i}{q} + \alpha(\lambda_i + \frac{1}{q_i})} dy_1 \cdots dy_m \\
 &= C_m \prod_{i=1}^m |x|_p^{n\lambda_i - \frac{\beta_i}{q} + \alpha(\lambda_i + \frac{1}{q_i})}.
 \end{aligned}$$

By the definition of the central Morrrey space $B^{q,\lambda}(\mathbb{Q}_p^n, |x|_p^\alpha, |x|_p^\beta)$, we have

$$\begin{aligned}
 &\| \mathcal{H}_\Phi^m(f_1, \dots, f_m) \|_{B^{q,\lambda}(\mathbb{Q}_p^n, |x|_p^\alpha, |x|_p^\beta)} \\
 &= \sup_{\gamma \in \mathbb{Z}} \left(\int_{B_\gamma} |x|_p^\alpha dx \right)^{-(\lambda + \frac{1}{q})} \left(\int_{B_\gamma} |\mathcal{H}_\Phi^m(f_1, \dots, f_m)(x)|^q |x|_p^\beta dx \right)^{\frac{1}{q}} \\
 &= C_m \sup_{\gamma \in \mathbb{Z}} \left(\int_{B_\gamma} |x|_p^\alpha dx \right)^{-(\lambda + \frac{1}{q})} \left(\int_{B_\gamma} \left(\prod_{i=1}^m |x|_p^{n\lambda_i - \frac{\beta_i}{q} + \alpha(\lambda_i + \frac{1}{q_i})} \right)^q |x|_p^\beta dx \right)^{\frac{1}{q}} \\
 &= C_m \sup_{\gamma \in \mathbb{Z}} \left(\int_{B_\gamma} |x|_p^\alpha dx \right)^{-(\lambda + \frac{1}{q})} \left(\int_{B_\gamma} \prod_{i=1}^m |x|_p^{nq\lambda_i - \beta_i + \alpha(q\lambda_i + \frac{q}{q_i})} |x|_p^{\beta_i} dx \right)^{\frac{1}{q}} \\
 &= C_m \sup_{\gamma \in \mathbb{Z}} \left(\int_{B_\gamma} |x|_p^\alpha dx \right)^{-(\lambda + \frac{1}{q})} \left(\int_{B_\gamma} |x|_p^{nq\lambda + \alpha(q\lambda + 1)} dx \right)^{\frac{1}{q}}.
 \end{aligned}$$

From (2.5), it is calculate that

$$\begin{aligned}
 &\prod_{i=1}^m \|f_i\|_{B^{q_i,\lambda_i}(\mathbb{Q}_p^n, |x|_p^\alpha, |x|_p^{\frac{q_i\lambda_i}{q}})} \\
 &= \sup_{\gamma \in \mathbb{Z}} \prod_{i=1}^m \left(\int_{B_\gamma} |x|_p^\alpha dx \right)^{-(\lambda_i + \frac{1}{q_i})} \left(\int_{B_\gamma} |x|_p^{nq_i\lambda_i + \alpha(q_i\lambda_i + 1)} dx \right)^{\frac{1}{q_i}} \\
 &= \sup_{\gamma \in \mathbb{Z}} \left(\int_{B_\gamma} |x|_p^\alpha dx \right)^{-(\lambda + \frac{1}{q})} \left(\int_{B_\gamma} |x|_p^{nq\lambda + \alpha(q\lambda + 1)} dx \right)^{\frac{1}{q}}.
 \end{aligned}$$

Therefore,

$$\| \mathcal{H}_\Phi^m(f_1, \dots, f_m) \|_{B^{q,\lambda}(\mathbb{Q}_p^n, |x|_p^\alpha, |x|_p^\beta)} = C_m \prod_{i=1}^m \|f_i\|_{B^{q_i,\lambda_i}(\mathbb{Q}_p^n, |x|_p^\alpha, |x|_p^{\frac{q_i\lambda_i}{q}})}.$$

Thus, we have

$$\begin{aligned}
 C_m &\leq \| \mathcal{H}_\Phi^m(f_1, \dots, f_m) \|_{B^{q_1,\lambda_1}(\mathbb{Q}_p^n, |x|_p^\alpha, |x|_p^{\frac{q_1\beta_1}{q}}) \times \cdots \times B^{q_m,\lambda_m}(\mathbb{Q}_p^n, |x|_p^\alpha, |x|_p^{\frac{q_m\beta_m}{q}})} \rightarrow B^{q,\lambda}(\mathbb{Q}_p^n, |x|_p^\alpha, |x|_p^\beta) \\
 &< \infty.
 \end{aligned}$$

This finishes the proof of Theorem 1.5. \square

Proof of Corollary 1.6. This proof is similar to the proof of *Corollary 1.4*, so we omit the details. \square

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