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A NEW BOUND FOR THE FIRST EIGENVALUES OF THE BIHARMONIC OPERATOR ON MANIFOLD

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Abstract. Consider (M^n, g) as a complete simply connected Riemannian manifold. Also let N^m for $m \leqslant n$ be a complete noncompact submanifold in M. In this paper, we are going to find a lower bound for the first eigenvalues of famous buckling and clamped plate problems on submanifold N with mean curvature H.

1. Introduction

Studying the eigenvalue of the Laplacian on a given manifold is an important aspect in Riemannian geometry. There has been an increasing interest to study the properties of spectrum of the Laplacian, because of the theory of self-adjoint operators. Many papers such as [4] and [5] are written in the spectrum of the Laplacian on compact manifolds with or without boundary or noncompact complete manifolds, due to in these two cases the linear Laplacian can be uniquely extended to self adjoint operators.

Consider (M^n, g) as an n-dimensional complete simply connected Riemannian manifold with smooth boundary ∂M. We are going to study the buckling problem as

$$\begin{cases} \Delta^2 u = \Lambda \Delta u & \text{in M}, \\ u|_{\partial M} = \frac{\partial u}{\partial \mathfrak{n}} = 0 & \text{on } \partial \text{M}, \end{cases} \tag{1}$$

and a clamped plate problems

$$\begin{cases} \Delta^2 u = \Gamma u & \text{in M}, \\ u|_{\partial M} = \frac{\partial u}{\partial \mathfrak{n}} = 0 & \text{on } \partial \text{M}, \end{cases} \tag{2}$$

where Δ^2 is the biharmonic operator and we denote $\mathfrak{n}$ as a outer unit normal vector on ∂M. Also Λ and Γ are corresponding eigenvalues for buckling and clamped plate problems, respectively.

We introduce the following variational characterization for the first eigenvalues of the buckling and clamped plate problems as

$$\Lambda_1(M) = \min_{u \neq 0} \frac{\int_M (\Delta u)^2 \, dV}{\int_M |\nabla u|^2 dV},$$

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and

$$\Gamma_1(M) = \min_{u\neq 0} \frac{\int_M (\Delta u)^2 dV}{\int_M |u|^2 dV},$$

respectively.

It has proved before these eigenvalues are real and purely discrete and we have

$$0 < \Lambda_1 \leqslant \Lambda_2 \leqslant \Lambda_3 \leqslant \cdots,$$
$$0 < \Gamma_1 \leqslant \Gamma_2 \leqslant \Gamma_3 \leqslant \cdots.$$

Before, Seto and Wei in [11], have studied the first eigenvalue of the p-Laplace under integral curvature condition which is defined as

$$\lambda_{1,p}(M) = \min_{u\neq 0} \frac{\int_M |\nabla u|^p dv}{\int_M |u|^2 dv},$$

for given manifold M.

There are many published papers in this subject. Zhang and Zhao in [14] have found the lower bound of the first eigenvalues for the biharmonic operator on manifolds. Also Prikazhechikov and Klunnik in [9] have studied the perturbed eigenvalues of biharmonic operator. You can find some similar useful results in [1] in a case of p-biharmonic.

Recently, H. Sun et al, in [13] have found lower bound depended mean curvature for the first eigenvalues of the p-Laplace and weighted p-Laplace on given submanifold. In this paper we are going to improve similar results for the first eigenvalues of buckling and clamped plate problems in a same method as [14].

As a quick review, we can characterized our main results as

THEOREM 1. *Consider (M^n, g) as a complete simply connected Riemannian manifold with non-positive sectional curvature bounded above by $-a^2$. Also let N be a complete non-compact submanifold in M with the maen curvature vector H and $Ric_N \geqslant Kg_N$ for some constant K satisfying $||H||_n < \frac{1}{nD(n)}$, then,*

$$\Lambda_1(N) \geqslant \frac{1}{4}\left[1 - nD(n)\,||H||_n\right]^2 (n-1)^2 a^2 + K,$$

where $\Lambda_1(N)$ is the first positive eigenvalue of buckling problem on submanifold N, H is mean curvature and $D(n)$ comes from L^1-Sobolev inequality.

THEOREM 2. *With the same assumptions as theorem 1, we get*

$$\Gamma_1(N) \geqslant \left(\frac{1}{4}\left[1 - nD(n)\,||H||_n\right]^2 (n-1)^2 a^2 + K\right)\lambda_1(N),$$

where $\Gamma_1(N)$ is the first positive eigenvalue of clamped plate problem and $\lambda_1(N)$ is the first eigenvalue of Laplace-Beltrami operator (set $p = 2$ in [11]).

The importance of studying systems including biharmonic operators goes back to 1980s Chen's program of better understanding the Euclidean submanifolds which yields to Chen's conjecture. Chen's conjecture expresses that every biharmonic submanifold in Euclidean space is a minimal one. The surface (hypersurface) $N \subset \mathbb{R}^3$ is called minimal if and only if its mean curvature vanishes for every point. Although Chen's conjecture attracted many efforts around the world, it remains open until now.

The variational characterizations serve as weak formulations for both problems (systems (1) and (2)), allowing eigenvalues to be defined as infima of specific energy functionals. These formulations and related inequalities are essential tools for analyzing the eigenvalues and their dependence on geometry. To get deeper into the eigenvalue problems, first of all, consider the buckling problem (1). Multiply $\Delta^2 u = \Lambda \Delta u$ by a test function $v \in H^2(M)$, where $H^2(M)$ is a Sobolev space related to manifold M. integrate over M, and use the divergence form

$$\int_M \left(\Delta^2 u\right) v dV = \int_M \Lambda\,(\Delta u)\, v dV.$$

Applying integration by parts twice to the left-hand side, and also since the boundary terms vanish because of $u|_{\partial M} = \frac{\partial u}{\partial \mathfrak{n}}|_{\partial M} = 0$, then

$$\int_M \left(\Delta^2 u\right) v dV = \int_M (\Delta u)(\Delta v)\, dV.$$

This is a weak form of the buckling problem. In this case, Λ_1 can be found as a minimization of the Rayleigh quotient

$$\Lambda_1(M) = \min_{u \neq 0} \frac{\int_M (\Delta u)^2\, dV}{\int_M |\nabla u|^2 dV}.$$

Although this process can be seen as a standard process to define the first eigenvalue of the system (1), this minimization can be defined in a different way. As an another point of view, the variational principle allows us to express eigenvalue problems as optimization problems. The energy functional derived from the weak form as

$$E[u] = \int_M (\Delta u)^2\, dV.$$

This represents the bending energy of the u. The eigenvalue Λ_1 corresponds to the minimum value of the functional E subject to a constraint,

$$\Lambda_1 = \min_{u \in H^2(M) \setminus \{0\}} \frac{\int_M (\Delta u)^2\, dV}{\int_M |\nabla u|^2 dV}.$$

The numerator $\int_M (\Delta u)^2\, dV$ measures the biharmonic energy. The dominator $\int_M |\nabla u|^2 dV$ normalizes u, ensuring the solution is non-trivial. The above functional corresponds to solving

$$\forall v \in H^2(M), \quad \int_M (\Delta u)(\Delta v)\, dV = \Lambda \int_M \langle \nabla u, \nabla v \rangle dV.$$

In the Riemannian case, $\langle \nabla u, \nabla v\rangle$ depends on the metric, and curvature effects appear implicitly.

The same process holds for the clamped plate problems. As a brief discussion, consider the system (2). The first eigenvalue Γ_1 is the minimizer of the Rayleigh quotient

$$\Gamma_1 = \min_{u\in H^2(M)\setminus\{0\}} \frac{\int_M (\Delta u)^2 \, dV}{\int_M u^2 dV}.$$

The numerator $\int_M (\Delta u)^2 \, dV$ measures the biharmonic energy of u, where the denominator $\int_M u^2 dV$ ensures normalization. This corresponds to solving

$$\forall v \in H^2(M), \ \int_M (\Delta u)(\Delta v)\, dV = \Gamma \int_M uv dV.$$

Boundary conditions play a critical role in ensuring the functional is well-defined and that the eigenvalues are discrete.

In this paper, the authors wish to study the first nonzero eigenvalues of systems (1) and (2) which are not only important as a mathematical pint of view, but also, they are important as an engineering concept. In this paper, we are going to find lower bounds for the first eigenvalues of systems (1) and (2) based on the submanifold mean curvature vector which is not done before. Also these results can be seen as an improvement of existing results from [14] and [13].

2. Preliminaries

In this section, we are going to recall some preliminary results which are useful for giving proofs. From [10], we have following Reilly's formula

PROPOSITION 1. (Reilly's formula) *Let M be an n-dimensional complete simply connected Riemannian manifold with smooth boundary ∂M. For every smooth function u, we get*

$$\begin{aligned}&\int_M \left[(\Delta u)^2 - |\nabla^2 u|^2 - Ric(\nabla u, \nabla u)\right] dv \\ &= \int_{\partial M} \left[-2\left(\bar{\Delta} u\right)\langle \nabla u, \mathfrak{n}\rangle + (n-1) H \langle \nabla u, \mathfrak{n}\rangle^2 + \Pi\left(\bar{\nabla} u, \bar{\nabla} u\right)\right] dv,\end{aligned}$$

where Δu, ∇u, $\nabla^2 u$ are the Laplacian, gradient and Hessian of u, Ric is the Ricci curvature of M, $\bar{\Delta} u$ and $\bar{\nabla} u$ are the Laplacian and gradient of u in ∂M, also Π and H are the second fundamental form and the mean curvature of ∂M with respect to the inner unit normal vector field $\mathfrak{n}$ on ∂M.

Also our proofs based on L^1-Sobolev inequality due to Hoffman and Spruck [6]

PROPOSITION 2. (L^1-Sobolev inequality) *Consider* (M^n,g) *as a complete manifold has nonpositive sectional curvature, then*

$$\left(\int_M u^{\frac{n}{n-1}}dv\right)^{\frac{n-1}{n}} \leqslant D(n)\int_M (|\nabla u|+n|H|u)\,dv,$$

holds for any $u\in C_0^1(M)$, *where*

$$D(n)=2^n(1+n)^{\frac{n+1}{n}}(n-1)^{-1}\tau_n^{-\frac{1}{n}},$$

and also τ_n *is the volume of the unit ball in* $\mathbb{R}^n$.

Also, we recall

$$||H||_n=\left(\int_M |H|^n dv\right)^{\frac{1}{n}}.$$

There are some other papers which are concerned with lower bounds of the first eigenvalues. According to Schoen and Yau [12], it is an important question to find conditions which implies $\lambda_1(M)>0$. For an n-dimensional complete non-compact simply connected Riemannian manifold whose sectional curvature bounded above by $-a^2$, McKean [8] showed that

$$\lambda_1(M)\geqslant \frac{(n-1)^2a^2}{4}.$$

Also Cheung and Leung [3] proved that if $|H|<\kappa$ for some constant $\kappa<n-1$, then the first eigenvalue of the Laplacian on M (with the Dirichlet boundary condition) satisfies

$$\lambda_1(N)\geqslant \frac{1}{4}(n-1-\kappa)^2,$$

where N is the n-dimensional complete non-compact submanifold in hyperbolic space $H^n(-1)$ with the mean curvature vector H. And, finally, Lin [7] proved that the first eigenvalue of the Laplacian of M satisfies

$$\lambda_1(M)\geqslant \frac{(n-1)^2a^2}{4}(1-nD(n)\,||H||_n)^2.$$

In this paper, we are going to improve the result of Lin for the first eigenvalues of buckling and clamped plate problems.

3. Main proofs

Before, Bessa and Montenegro in [2] have given following theorem for the first eigenvalue of Laplace operator to complete non-compact manifolds in a complete simply connected Riemannian manifold as

THEOREM 3. *Let M^{n+m} be a complete simply connected Riemannian manifold with sectional curvature K_M satisfying $K_M \leqslant -a^2$ for a positive constant $a>0$. Let M^n be a complete non-compact submanifold in M with $n|H| \leqslant b < (n-1)a$. Then*

$$\lambda_1(M) \geqslant \frac{[(n-1)a-b]^2}{4}.$$

Now we are ready enough to give proof for our main theorems. Our proofs are based on method from [2] and [7].

Proof of the Theorem 1. Let $\rho(x)$ be the geodesic distance on M from a fixed point $x_0 \in M \setminus N$ to x. From [2], we have

$$\Delta(\rho \circ i) \geqslant (n-1)a - n|H|,$$

where $i : N \hookrightarrow M$ is an isometric immersion. Set $r = \rho \circ i$, then for $u \in C_0^\infty(M)$, we have

$$\begin{aligned} \operatorname{div}\left(\nabla r|\nabla u|^2\right) &= \langle \operatorname{div}|\nabla u|^2, \nabla r\rangle + |\nabla u|^2 \Delta r \\ &= 2|\nabla u|\nabla r|\nabla u| + |\nabla u|^2 \Delta r \\ &\geqslant -2|\nabla u||\nabla|\nabla u|| + (n-1)a|\nabla u|^2 - n|H||\nabla u|^2. \end{aligned}$$

By using divergence theorem and $\nabla u|_{\partial N} = 0$, we arrive at

$$\int_N \operatorname{div}\left(\nabla r|\nabla u|^2\right) dv = \int_{\partial N} |\nabla u|^2 \langle \nabla r, \mathfrak{n}\rangle dv_{\partial M} = 0,$$

which implies

$$-2\int_N |\nabla u||\nabla|\nabla u||dv + (n-1)a\int_N |\nabla u|^2 dv - n\int_N |H||\nabla u|^2 dv \leqslant 0. \tag{3}$$

Using L^1-Sobolev inequality, we deduce

$$\begin{aligned} n\int_N |H||\nabla u|^2 dv &\leqslant n\left(\int_N |H|^n dv\right)^{\frac{1}{n}} \left(\int_N |\nabla u|^{\frac{2n}{n-1}} dv\right)^{\frac{n-1}{n}} \\ &\leqslant nD(n)\,||H||_n \int_N \left(|\nabla\left(|\nabla u|^2|\right) + n|H||\nabla u|^2\right) dv. \end{aligned}$$

Since $1 - nD(n)\,||H||_n \geqslant 0$, we conclude that

$$n\int_N |H||\nabla u|^2 dv \leqslant \frac{nD(n)\,||H||_n}{1-nD(n)\,||H||_n} \int_N |\nabla\left(|\nabla u|^2\right)|dv,$$

substituting the last equation into (3), we see

$$2(-1-\mathscr{A})\int_N |\nabla u||\nabla|\nabla u||dv + (n-1)a\int_N |\nabla u|^2 dv \leqslant 0,$$

where

$$\mathscr{A} := \frac{nD(n)\,||H||_n}{1 - nD(n)\,||H||_n}.$$

Now using Young's inequality for the term $\int_N |\nabla u||\nabla|\nabla u||dv$, we have

$$|\nabla u||\nabla|\nabla u|| \leqslant \frac{|\nabla|\nabla u||^2}{2\varepsilon^2} + \frac{\varepsilon^2|\nabla u|^2}{2},$$

where ε is arbitrary. So, we obtain

$$\frac{\int_N |\nabla|\nabla u||^2 dv}{\int_N |\nabla u|^2 dv} \geqslant -\varepsilon^4 + \frac{(n-1)\,a}{1+\mathscr{A}}\varepsilon^2,$$

Set the function $g(\varepsilon)$ as

$$g(\varepsilon) = -\varepsilon^4 + \frac{(n-1)\,a}{1+\mathscr{A}}\varepsilon^2.$$

Since

$$g'(\varepsilon) = -4\varepsilon^3 + \frac{2\,(n-1)\,a}{1+\mathscr{A}}\varepsilon,$$

and

$$g''(\varepsilon) = -12\varepsilon^2 + \frac{2\,(n-1)\,a}{1+\mathscr{A}},$$

when $g(\varepsilon) = 0$, we infer

$$\varepsilon_0 = \left(\frac{(n-1)\,a}{2\,(1+\mathscr{A})}\right)^{\frac{1}{2}},$$

and due to $g''(\varepsilon_0) < 0$ it's maximum easily compute as

$$\max g(\varepsilon) = \frac{1}{4}\left(\frac{(n-1)\,a}{1+\mathscr{A}}\right)^2,$$

and also clearly

$$\frac{\int_N |\nabla|\nabla u||^2 dv}{\int_N |\nabla u|^2 dv} \geqslant \frac{1}{4}\left(\frac{(n-1)\,a}{1+\mathscr{A}}\right)^2.$$

Scince from [14] it follows that

$$|\nabla|\nabla u||^2 \leqslant |\nabla^2 u|^2,$$

and

$$\frac{\int_N |\nabla^2 u|^2 dv}{\int_N |\nabla u|^2 dv} \geqslant \frac{1}{4}\left(\frac{(n-1)a}{1+\mathscr{A}}\right)^2.$$

From Reilly's formula and boundary value condition, we conclude

$$\int_N \left[(\Delta u)^2 - |\nabla^2 u|^2 - Ric(\nabla u, \nabla u)\right] dv = 0,$$

and also $Ric \geqslant Kg$ implies that

$$\frac{\int_N (\Delta u)^2 dv}{\int_N |\nabla u|^2 dv} \geqslant \frac{\int_N |\nabla^2 u|^2 dv}{\int_N |\nabla u|^2 dv} + K,$$

which conclude

$$\Lambda_1(N) \geqslant \frac{1}{4}\left(\frac{(n-1)a}{1+\mathscr{A}}\right)^2 + K,$$

by substituting $\mathscr{A}$, we finally get

$$\Lambda_1(N) \geqslant \frac{1}{4}\left[1 - nD(n)\,||H||_n\right]^2 (n-1)^2 a^2 + K. \quad \square$$

We will follow the similar process for next theorem.

Proof of Theorem 2. Immediately, from Reilly's formula, we get

$$\begin{aligned}\int_N (\Delta u)^2 dv &\geqslant \frac{\int_N |\nabla^2 u|^2 dv}{\int_N |\nabla u|^2 dv}\int_N |\nabla u|^2 dv + K\int_N |\nabla u|^2 dv \\ &\geqslant \left(\frac{1}{4}\left(\frac{(n-1)a}{1+\mathscr{A}}\right)^2 + K\right)\int_N |\nabla u|^2 dv,\end{aligned}$$

where $\mathscr{A}$ is as same as theorem 1. Now by dividing both sides of the last equation on $\int_N |u|^2 dv$, we have

$$\Gamma_1(N) \geqslant \left(\frac{1}{4}\left[1 - nD(n)\,||H||_n\right]^2 (n-1)^2 a^2 + K\right)\lambda_1(N). \quad \square$$

REMARK 1. As we mentioned before, Lin in [7] has proved that

$$\lambda_1(N) \geqslant \frac{(n-1)^2 a^2}{4}\left(1 - nD(n)\,||H||_n\right)^2,$$

where $\lambda_1(N)$ is the first eigenvalue of Laplace operator on a given manifold M and it's submanifold N. Clearly we get

$$\Gamma_1(N) \geqslant \left(\frac{1}{4}\left[1 - nD(n)\,||H||_n\right]^2 (n-1)^2 a^2 + K\right)\frac{(n-1)^2 a^2}{4}\left(1 - nD(n)\,||H||_n\right)^2.$$

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