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FRACTIONAL INTEGRAL APPROACH TO PARAMETERIZED INEQUALITIES FOR (s,P)–PREINVEXITY

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Abstract. Fractional calculus is an invaluable tool with significant potential for application in the physical sciences. This paper focuses on addressing parameterized fractional inequalities for (s,P)-preinvex functions. In light of this, we introduce the concept of (s,P)-preinvex functions and investigate their related properties. By considering limited first- and second-order derivative functions, we present two fractional integral identities with a single parameter using exponential kernel fractional integrals. Building upon these identities, we establish parameterized integral inequalities for (s,P)-preinvex functions. To provide a more intuitive display of the results, we also offer illustrative examples with graphs to demonstrate the validity of our theoretical findings.

1. Introduction

Inequality theory, since its very beginning, has been a focal point of scientific research, with myriad applications in various domains, including optimization theory [54], automation control [28], engineering science [46], and even the realm of physics [20]. Recently, with the assistance of generalized convexity, numerous new inequalities have been investigated in the context of Riemann integrals by several researchers. For example, one can refer to Çakmak et al. [12] for h-convex functions, Yaşar et al. [49] for s-convex functions, Eken et al. [17] for p-convex functions, Zhang et al. [57] for (α,m)-convex functions, Ali et al. [2] for coordinated convex functions, Latif [30] for GA-convex and geometrically quasiconvex mappings, Fahad et al. [19] for generalized geometric-arithmetic convex functions, Andrić [3] for $(h,g;m)$-convex functions and so on. For more information about the generalized convexity and inequalities, see [42, 43, 4, 35] and the relevant citations therein.

Fractional calculus, a framework that extends classic differential and integral operations to non-integer orders, surpasses classical calculus in elucidating intricate multiscale phenomena. Due to its practicality, fractional calculus has found widespread applications across diverse fields. For instance, it is employed to tackle challenges encountered in physical systems, including fluid mechanics [52], electromagnetic wave propagation [45], as well as nuclear and particle physics [23]. In particular, within these

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domains, fractional integral inequalities play a pivotal role in deriving energy estimates. These applications have ignited the curiosity of numerous researchers, prompting them to delve into the extension of integral inequalities using various fractional integral operators. As an illustration, Hezenci et al. [24] derived several fractional Simpson-type inequalities involving twice-differentiable convex functions within the context of Riemann-Liouville fractional integrals. Furthermore, Nie et al. [34] presented Simpson-type inequalities with multiple parameters utilizing generalized (α, m)-preinvexity via k-fractional integrals. Building upon these foundations, Mohammed [32] established Hermite-Hadamard's inequalities for convex functions with respect to increasing functions, leveraging generalized Riemann-Liouville fractional integrals, and elucidated their connection with previous findings in specific cases. Meanwhile, Butt et al. derived generalized Hadamard-type inequalities using Atangana–Baleanu fractional integrals in their work [10], and further extended this approach to establish Hadamard–Mercer inequalities in Ref. [11]. By utilizing multiplicative k-Riemann–Liouville fractional integrals, Khan et al. [26] constructed Hermite–Hadamard-type inequalities for multiplicatively (P, m)-superquadratic functions. Additionally, Erden and Sarikaya [18] formulated generalized Bullen-type inequalities employing local fractional integrals. Moreover, Du et al. [15] conducted a study on Bullen-type inequalities through generalized fractional integrals. For more results related to the fractional integral operators, the interested reader is directed to [13, 22, 29, 31] and the references cited therein.

In 2019, Ahmad et al. [1] constructed a novel category of fractional integral operators having exponential kernel. This groundbreaking work has since sparked a flurry of research interest among scholars. For example, Budak et al. [9] formulated Hermite–Hadamard- and Ostrowski-type fractional inequalities through convex functions. Wu et al. [48], on the other hand, delved into the bounds estimation of left- and right-sided fractional Hermite–Hadamard-type inequalities. Considering functions whose absolute values of first derivatives are convex, Yuan et al. [51] further established the parameterized fractional integral inequalities. In Ref. [39], the authors deduced the fractional Hermite–Hadamard-, Hermite–Hadamard–Fejér- and Pachpatte-type inequalities via exponential convexity. In addition, Zhou et al. [56] introduced the interval-valued fractional integrals with exponential kernels, and constructed corresponding fractional integral inclusions. For a comprehensive overview of fractional integrals with exponential kernels, readers are encouraged to consult [25, 44, 47, 50] and the references cited therein.

Inspired by previous research, this paper concentrates on investigating estimation-type outcomes using fractional integral operators with exponential kernels for (s, P)-preinvex functions. The structure of this article is as follows: following the introduction and preliminaries, we introduce the concept of (s, P)-preinvex functions, explore their properties, and establish the fractional Hermite–Hadamard-type inequality for such functions in Sec. 3. In Sec. 4, considering first- and second-order differentiable cases, we derive parameterized integral inequalities for (s, P)-preinvex functions that unify midpoint-, trapezoid-, Simpson-, and Bullen-type inequalities with specific choices of parameters. Finally, we summarize our main findings and offer insights for future research in Sec. 5.

2. Preliminaries

This section states the necessary definitions of convexity and fractional integrals, accompanied by various related results. Throughout this paper, let $I \subseteq \mathbb{R}$ be a real-valued interval and I° be the interior of I.

Let's begin by revisiting the (s,P)-function definitions and the corresponding Hermite–Hadamard-type inequality, formulated originally by Numan and İşcan.

DEFINITION 2.1. [37] It is assumed that for some fixed $s \in (0,1]$, the function $g : I \subseteq \mathbb{R} \to \mathbb{R}$ is called (s,P)-function if the coming inequality

$$g(tx+(1-t)y) \leqslant (t^s+(1-t)^s)[g(x)+g(y)]$$

holds for every $x,y \in I$ and $t \in [0,1]$.

Evidently, setting $s=1$ results in the (s,P)-functions transforming into P-convex functions.

THEOREM 2.1. [37] *It is assumed that* $g : [a,b] \to \mathbb{R}$ *is an* (s,P)*-function with some fixed* $s \in (0,1]$. *If* $g \in L^1([a,b])$, *then the following Hermite–Hadamard-type inequality holds*

$$2^{s-2} g\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b g(\tau)\mathrm{d}\tau \leqslant \frac{2}{s+1}[g(a)+g(b)]. \tag{2.1}$$

The notion of invex set is formulated in the expressions below.

DEFINITION 2.2. [5] Considering the mapping $\eta : A \times A \to \mathbb{R}^n$, the set $A \subseteq \mathbb{R}^n$ is said to be invex with respect to the mapping η, if for every $x,y \in A$ and $t \in [0,1]$,

$$x + t\eta(y,x) \in A.$$

Observe that convex sets inherently possess invex properties under the mapping $\eta(y,x) = y-x$, yet the inverse statement is not universally valid.

DEFINITION 2.3. [7] The function g defined on the invex set $A \subseteq \mathbb{R}^n$, in relation to the mapping $\eta : A \times A \to \mathbb{R}^n$, is said to be preinvex, if it satisfies the following inequality

$$g(x+t\eta(y,x)) \leqslant (1-t)g(x) + tg(y)$$

for every $x,y \in A$ and $t \in [0,1]$.

Preinvexity encompasses convexity, as every convex function is preinvex under the mapping $\eta(y,x) = y-x$, whereas the inverse does not necessarily apply.

Below, we retrospect the concept of P-preinvex functions.

DEFINITION 2.4. [36] Let the set $A \subseteq \mathbb{R}^n$ be invex. The function $g : A \to \mathbb{R}$ is said to be P-preinvex regarding the mapping $\eta : A \times A \to \mathbb{R}^n$, if the following inequality

$$g(x+t\eta(y,x)) \leqslant g(x)+g(y)$$

is true for every $x,y \in A$ and $t \in [0,1]$.

The Condition C introduced by Mohan and Neogy [33] is the following one.

CONDITION C. Let $A \subseteq \mathbb{R}^n$ be an open invex subset with respect to the mapping $\eta : A \times A \to \mathbb{R}$. We say that the mapping η satisfies the Condition C if for every $x,y \in A$ and $t \in [0,1]$,

$$\eta(y,y+t\eta(x,y)) = -t\eta(x,y),$$

and

$$\eta(x,y+t\eta(x,y)) = (1-t)\eta(x,y).$$

For every $x,y \in A$ and $t \in [0,1]$, we can also readily observe that

$$\eta\left(y+t_2\eta(x,y),y+t_1\eta(x,y)\right) = (t_2-t_1)\eta(x,y).$$

The Hermite–Hadamard's inequality is the following.

$$g\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b g(\tau)\mathrm{d}\tau \leqslant \frac{g(a)+g(b)}{2},$$

where $g : I \to \mathbb{R}$ is a convex function defined on the interval I, for any $a,b \in I$ with $a<b$. This inequality provides estimations for the mean value of a continuous convex function $g : [a,b] \to \mathbb{R}$.

The Simpson's inequality is stated by the following way.

$$\left|\frac{1}{6}\left[g(a)+4g\left(\frac{a+b}{2}\right)+g(b)\right]-\frac{1}{b-a}\int_a^b g(\tau)\mathrm{d}\tau\right| \leqslant \frac{1}{2880}\left\|g^{(4)}\right\|_\infty (b-a)^4,$$

where $g : I \to \mathbb{R}$ is a four-order continuously differentiable function on I° with $\left\|g^{(4)}\right\|_\infty = \sup_{\tau\in I^\circ}\left|g^{(4)}(\tau)\right| < \infty$.

Bullen [8] proved the following inequality which is known as the Bullen's inequality for convex functions.

$$\frac{1}{b-a}\int_a^b g(\tau)\mathrm{d}\tau \leqslant \frac{1}{2}\left[g\left(\frac{a+b}{2}\right)+\frac{g(a)+g(b)}{2}\right].$$

The definitions of fractional integral operators with exponential kernels, proposed by Ahmad et al. in 2019, are the subsequent ones.

DEFINITION 2.5. [1] Let $g \in L^1([a,b])$. The fractional integrals operators with exponential kernels, denoted by $\mathscr{I}^\alpha_{a^+}g$ and $\mathscr{I}^\alpha_{b^-}g$ of order $\alpha \in (0,1)$, are defined as the coming expressions, respectively.

$$\mathscr{I}^\alpha_{a^+}g(x) = \frac{1}{\alpha}\int_a^x \exp\left(-\frac{1-\alpha}{\alpha}(x-\tau)\right)g(\tau)\mathrm{d}\tau,\ x>a,$$

and

$$\mathscr{I}_{b^-}^{\alpha} g(x) = \frac{1}{\alpha} \int_x^b \exp\left(-\frac{1-\alpha}{\alpha}(\tau - x)\right) g(\tau)\mathrm{d}\tau,\ x < b.$$

From Definition 2.5, it readily yields that

$$\lim_{\alpha\to 1} \mathscr{I}_{a^+}^{\alpha} g(x) = \int_a^x g(\tau)\mathrm{d}\tau, \quad \lim_{\alpha\to 1} \mathscr{I}_{b^-}^{\alpha} g(x) = \int_x^b g(\tau)\mathrm{d}\tau.$$

Within the same paper, they formulated a fractional Hermite–Hadamard-type inequality with exponential kernels, as specified below.

THEOREM 2.2. [1] *Let $g:[a,b]\to\mathbb{R}$ be a positive convex function with $0\leqslant a<b$. If $g\in L^1([a,b])$, then the following inequality for fractional integrals with exponential kernels holds*

$$g\left(\frac{a+b}{2}\right) \leqslant \frac{1-\alpha}{2(1-e^{-\rho})}\left[\mathscr{I}_{a^+}^{\alpha} g(b) + \mathscr{I}_{b^-}^{\alpha} g(a)\right] \leqslant \frac{g(a)+g(b)}{2},$$

where

$$\rho = \frac{1-\alpha}{\alpha}(b-a).$$

A fractional equality for once-differentiable functions was put forth by Yuan et al. in 2021.

LEMMA 2.1. [51] *Let $g:[a,b]\to\mathbb{R}$ be a differentiable function on (a,b). If $g'\in L^1([a,b])$ and $0\leqslant\lambda\leqslant 1$, then the following identity for fractional integrals holds*

$$\begin{aligned}
&\frac{1-\alpha}{2(1-e^{-\rho})}\left[\mathscr{I}_{a^+}^{\alpha} g(b) + \mathscr{I}_{b^-}^{\alpha} g(a)\right] - (1-\lambda)g\left(\frac{a+b}{2}\right) - \lambda\frac{g(a)+g(b)}{2} \\
&\quad = \frac{b-a}{2(1-e^{-\rho})}\int_0^1 f(t)g'(ta+(1-t)b)\mathrm{d}t,
\end{aligned}$$

where

$$f(t) = \begin{cases} (1-\lambda)(1-e^{-\rho}) + e^{-\rho(1-t)} - e^{-\rho t}, & 0\leqslant t\leqslant \dfrac{1}{2}, \\[2ex] (1-\lambda)(e^{-\rho}-1) + e^{-\rho(1-t)} - e^{-\rho t}, & \dfrac{1}{2} < t \leqslant 1. \end{cases}$$

Another integral identity involving twice-differentiable functions was presented by Zhou et al. as follows.

LEMMA 2.2. [55] *Let $g:[a,b]\to\mathbb{R}$ be a twice differentiable function on (a,b). If $g''\in L^1([a,b])$ and $0\leqslant\lambda\leqslant 1$, then the following identity for fractional integrals* *holds*

$$\frac{1-\alpha}{2(1-e^{-\rho})}\left[\mathscr{I}_{a^+}^{\alpha}g(b)+\mathscr{I}_{b^-}^{\alpha}g(a)\right]-(1-\lambda)g\left(\frac{a+b}{2}\right)-\lambda\frac{g(a)+g(b)}{2}$$
$$=\frac{(b-a)^2}{2}\int_0^1 u(t)g''(ta+(1-t)b)\mathrm{d}t,$$

where

$$u(t)=\begin{cases} t(1-\lambda)-\dfrac{1+e^{-\rho}-e^{-\rho t}-e^{-\rho(1-t)}}{\rho(1-e^{-\rho})}, & 0\leqslant t\leqslant \dfrac{1}{2},\\[2ex] (1-t)(1-\lambda)-\dfrac{1+e^{-\rho}-e^{-\rho t}-e^{-\rho(1-t)}}{\rho(1-e^{-\rho})}, & \dfrac{1}{2}<t\leqslant 1.\end{cases}$$

3. Properties and inequality for (s,P)-preinvex functions

In this section, we introduce the concept of (s,P)-preinvex functions, which draws inspiration from (s,P)-functions and preinvex functions. We subsequently delve into their properties and establish the Hermite–Hadamard-type inequality for this function class.

DEFINITION 3.1. Let $A^*\subseteq\mathbb{R}$ be an open invex subset with respect to the mapping $\eta : A^*\times A^*\to\mathbb{R}$. The function g defined on the invex set A^* is said to be (s,P)-preinvex function with respect to the mapping η, if the following inequality

$$g(x+t\eta(y,x))\leqslant\left(t^s+(1-t)^s\right)\left[g(x)+g(y)\right] \tag{3.1}$$

holds for every $x,y\in A^*$ and $t\in[0,1]$ together with some fixed $s\in(0,1]$.

REMARK 3.1. In Definition 3.1, we consider the following special cases:
(i) Putting $s=1$, we get the concept of P-preinvex functions.
(ii) Putting $\eta(y,x)=y-x$, we obtain the concept of (s,P)-functions.
(iii) Putting $s=1$ and $\eta(y,x)=y-x$, we achieve the concept of P-convex functions.

REMARK 3.2. We note that if g is an (s,P)-preinvex function, then g is nonnegative. In fact, if we take $t=0$ in the inequality (3.1), then the following inequality

$$g(x)\leqslant g(x)+g(y)$$

holds for every $x,y\in A^*$. Thus, we have $g(y)\geqslant 0$ for all $y\in A^*$.

Next, we investigate certain properties about the (s,P)-preinvex functions. Since the proofs of propositions 3.1 and 3.2 are straightforward, we omit them here.

PROPOSITION 3.1. *Let $A^* \subseteq \mathbb{R}$ be an invex set. Assume that $f,g : A^* \to \mathbb{R}$ are both two (s,P)-preinvex functions with regard to the same mapping $\eta : A^* \times A^* \to \mathbb{R}$. Then, we have that*

(i) The function $f+g$ is an (s,P)-preinvex function with regard to the mapping η.

(ii) If $c \in \mathbb{R}^+$, then cf is an (s,P)-preinvex function with regard to the mapping η.

PROPOSITION 3.2. *Every (s,P)-preinvex function is also an h-preinvex function regarding the function $h(t) = t^s + (1-t)^s$.*

We proceed to study the properties of (s,P)-preinvex functions.

PROPOSITION 3.3. *Every P-preinvex function is also an (s,P)-preinvex function.*

Proof. Since the inequalities

$$t \leqslant t^s \text{ and } 1-t \leqslant (1-t)^s$$

are valid for each $t \in [0,1]$ and $s \in (0,1]$, we have that

$$1 \leqslant t^s + (1-t)^s.$$

Making use of the concept of P-preinvexity, we deduce that

$$\begin{aligned} g(x+t\eta(y,x)) &\leqslant g(x)+g(y) \\ &\leqslant \left(t^s+(1-t)^s\right)\left[g(x)+g(y)\right]. \end{aligned}$$

The proof is done. $\square$

PROPOSITION 3.4. *Every nonnegative preinvex function is also an (s,P)-preinvex function.*

Proof. Let g be an arbitrary nonnegative preinvex function on the invex set $A^* \subseteq \mathbb{R}$. For every $x,y \in A^*$, $t \in [0,1]$ and some fixed $s \in (0,1]$, we obtain that

$$\begin{aligned} g(x+t\eta(y,x)) &\leqslant (1-t)g(x)+tg(y) \\ &\leqslant (1-t)^s g(x)+t^s g(y) \\ &\leqslant \left(t^s+(1-t)^s\right)\left[g(x)+g(y)\right]. \end{aligned}$$

This fulfills the proof. $\square$

REMARK 3.3. From the proof of Proposition 3.4, we note that every nonnegative s-preinvex function is also an (s,P)-preinvex function.

PROPOSITION 3.5. *Let $A^* \subseteq \mathbb{R}$ be an invex set. Suppose that the function $g : A^* \to \mathbb{R}$ is bounded pertaining to the mapping $\eta : A^* \times A^* \to \mathbb{R}$. Then, there exists a number $c \in \mathbb{R}$ satisfying that $G(x) = g(x) + c$ is an (s,P)-preinvex function with respect to the mapping η.*

Proof. If we take $c = \sup\limits_{\substack{x,y \in A^* \\ z \in [x,y]}} [g(z) - g(x) - g(y)]$, then the following inequality

$$g(x + t\eta(y,x)) \leqslant g(x) + g(y) + c$$

is true for each $x,y \in A^*$ and $t \in [0,1]$. Let $G(x) = g(x) + c$. For every $x,y \in A^*$, $t \in [0,1]$ and some fixed $s \in (0,1]$, we have that

$$\begin{aligned} G(x + t\eta(y,x)) &= g(x + t\eta(y,x)) + c \\ &\leqslant g(x) + c + g(y) + c \\ &= G(x) + G(y) \\ &\leqslant (t^s + (1-t)^s)[G(x) + G(y)]. \end{aligned}$$

This ends the proof. □

PROPOSITION 3.6. *Let $A^* \subseteq \mathbb{R}$ be an invex set. If the functions $g_1, g_2 : A^* \to \mathbb{R}$ are both (s,P)-preinvex concerning the same mapping $\eta : A^* \times A^* \to \mathbb{R}$, then the function $g = \max\{g_1, g_2\}$ is also (s,P)-preinvex on A^* with respect to the mapping η.*

Proof. By means of the (s,P)-preinvexity, the subsequent inequalities

$$g_1(x + t\eta(y,x)) \leqslant (t^s + (1-t)^s)[g_1(x) + g_1(y)]$$

and

$$g_2(x + t\eta(y,x)) \leqslant (t^s + (1-t)^s)[g_2(x) + g_2(y)]$$

hold for any $x,y \in A^*$, $t \in [0,1]$ and some fixed $s \in (0,1]$. Then, we can derive that

$$\begin{aligned} &g(x + t\eta(y,x)) \\ &= \max\{g_1(x + t\eta(y,x)), g_2(x + t\eta(y,x))\} \\ &\leqslant \max\{(t^s + (1-t)^s)[g_1(x) + g_1(y)], (t^s + (1-t)^s)[g_2(x) + g_2(y)]\} \\ &\leqslant (t^s + (1-t)^s)\max\{g_1(x), g_2(x)\} + (t^s + (1-t)^s)\max\{g_1(y), g_2(y)\} \\ &= (t^s + (1-t)^s)[g(x) + g(y)]. \end{aligned}$$

This accomplishes the proof. □

PROPOSITION 3.7. *Let $A^* \subseteq \mathbb{R}$ be an invex set. Suppose that $g_i : A^* \to \mathbb{R}$ is an arbitrary crowd of (s,P)-preinvex functions regarding the mapping $\eta : A^* \times A^* \to \mathbb{R}$. And let $g(v) = \sup\limits_i g_i(v)$. If $V = \{v \in A^* : g(v) < \infty\}$ is nonempty, then V is an interval and g is an (s,P)-preinvex function on V with regard to the mapping η.*

Proof. If $x,y \in V, s \in (0,1]$ and $t \in [0,1]$, then we have that

$$\begin{aligned}
g(x+t\eta(x,y)) &= \sup_i g_i(x+t\eta(x,y)) \\
&\leqslant \sup_i \{(t^s+(1-t)^s)[g_i(x)+g_i(y)]\} \\
&\leqslant (t^s+(1-t)^s)\left[\sup_i g_i(x)+\sup_i g_i(y)\right] \\
&= (t^s+(1-t)^s)[g(x)+g(y)] < \infty.
\end{aligned}$$

This shows that V is an interval. Since it contains every point between any two of its points, and g is an (s,P)-preinvex function on V. This ends the proof. □

PROPOSITION 3.8. *Let $A^* \subseteq \mathbb{R}$ be an invex set. Suppose that the function $g : A^* \to \mathbb{R}$ is related to the mapping $\eta : A^* \times A^* \to \mathbb{R}$, which meets Condition C. Then, the function g is an (s,P)-preinvex function pertaining to the mapping η if and only if the function $\phi : [0,1] \to \mathbb{R}$ defined by $\phi(t) = g(x+t\eta(x,y))$ is an (s,P)-function concerning the mapping η for all $x,y \in A^*$.*

Proof. "$\Longleftarrow$" Suppose that ϕ is an (s,P)-function on $[0,1]$. Let $\nu_1 = x+\mu_1\eta(y,x)$ and $\nu_2 = x+\mu_2\eta(y,x)$, where $x,y \in A^*$ and $\mu_1,\mu_2 \in [0,1]$. Notice that the mapping η satisfies the Condition C. Then, for every $t \in [0,1]$ and some fixed $s \in (0,1]$, we have that

$$\begin{aligned}
g(\nu_1+t\eta(\nu_2,\nu_1)) &= g\left(x+\mu_1\eta(y,x)+t\eta[x+\mu_2\eta(y,x),x+\mu_1\eta(y,x)]\right) \\
&= g(x+\mu_1\eta(y,x)+t(\mu_2-\mu_1)\eta(y,x)) \\
&= g(x+((1-t)\mu_1+t\mu_2)\eta(y,x)) \\
&= \phi((1-t)\mu_1+t\mu_2) \\
&\leqslant (t^s+(1-t)^s)[\phi(\mu_1)+\phi(\mu_2)] \\
&= (t^s+(1-t)^s)[g(\nu_1)+g(\nu_2)],
\end{aligned}$$

which shows that g is an (s,P)-preinvex function regarding the mapping η.

"$\Longrightarrow$" Suppose that g is an (s,P)-preinvex function concerning the mapping η. Let $x,y \in A^*$ and $\mu_1,\mu_2 \in [0,1]$. Since the mapping η meets the Condition C, for every $t \in [0,1]$ and some fixed $s \in (0,1]$, we deduce that

$$\begin{aligned}
\phi((1-t)\mu_1+t\mu_2) &= g(x+((1-t)\mu_1+t\mu_2)\eta(y,x)) \\
&= g\big(x+\mu_1\eta(y,x)+t\eta\left(x+\mu_2\eta(y,x),x+\mu_1\eta(y,x)\right)\big) \\
&\leqslant (t^s+(1-t)^s)[g(x+\mu_1\eta(y,x))+g(x+\mu_2\eta(y,x))] \\
&= (t^s+(1-t)^s)[\phi(\mu_1)+\phi(\mu_2)],
\end{aligned}$$

which indicates that ϕ is an (s,P)-function on $[0,1]$. This completes the proof. □

Under the assumption of (s,P)-preinvexity, we present the Hermite–Hadamard-type inequality constructed from fractional integrals with exponential kernels.

THEOREM 3.1. *Let $A^* \subseteq \mathbb{R}$ be an open invex subset with respect to the mapping $\eta : A^* \times A^* \to \mathbb{R}^+$ and $a, b \in A^*$. If $g : [a, a+\eta(b,a)] \to (0,\infty)$ is an (s,P)-preinvex function, $g \in L^1([a, a+\eta(b,a)])$ and the mapping η satisfies Condition C, for $\xi \in [0,1]$, the following inequality for fractional integrals holds*

$$\begin{aligned} 2^{s-2} g\left(\frac{2a+\eta(b,a)}{2}\right) &\leqslant \frac{1-\alpha}{2(1-e^{-\rho})}\left[\mathscr{I}^{\alpha}_{a^+} g(a+\eta(b,a)) + \mathscr{I}^{\alpha}_{(a+\eta(b,a))^-} g(a)\right] \\ &\leqslant \frac{2\rho e^{-\rho\xi}}{(s+1)(1-e^{-\rho})}\left[g(a)+g(a+\eta(b,a))\right], \end{aligned} \tag{3.2}$$

where

$$\rho = \frac{1-\alpha}{\alpha}\eta(b,a).$$

Proof. Since g is an (s,P)-preinvex function on $[a, a+\eta(b,a)]$, we have that

$$g\left(x+\frac{\eta(y,x)}{2}\right) \leqslant 2^{1-s}\left[g(x)+g(y)\right]. \tag{3.3}$$

Using the change of variables $x = a+(1-t)\eta(b,a)$ and $y = a+t\eta(b,a)$ in inequality (3.3), we get that

$$\begin{aligned} & 2^{s-1} g\left(a+(1-t)\eta(b,a)+\frac{\eta\big(a+t\eta(b,a), a+(1-t)\eta(b,a)\big)}{2}\right) \\ &= 2^{s-1} g\left(a+(1-t)\eta(b,a)+\frac{(2t-1)\eta(b,a)}{2}\right) \\ &= 2^{s-1} g\left(\frac{2a+\eta(b,a)}{2}\right) \\ &\leqslant g\Big(a+(1-t)\eta(b,a)\Big)+g\Big(a+t\eta(b,a)\Big). \end{aligned} \tag{3.4}$$

Multiplying the both sides of inequality (3.4) by $e^{-\rho t}$, then integrating the resulting inequality with respect to t over $[0,1]$, we obtain that

$$\begin{aligned} & \frac{2^{s-1}(1-e^{-\rho})}{\rho} g\left(\frac{2a+\eta(b,a)}{2}\right) \\ &\leqslant \int_0^1 e^{-\rho t} g\big(a+(1-t)\eta(b,a)\big)\mathrm{d}t + \int_0^1 e^{-\rho t} g\big(a+t\eta(b,a)\big)\mathrm{d}t \\ &= \frac{1}{\eta(b,a)}\left[\int_a^{a+\eta(b,a)} e^{-\frac{1-\alpha}{\alpha}(a+\eta(b,a)-u)} g(u)\mathrm{d}u + \int_a^{a+\eta(b,a)} e^{-\frac{1-\alpha}{\alpha}(u-a)} g(u)\mathrm{d}u\right] \\ &= \frac{\alpha}{\eta(b,a)}\left[\mathscr{I}^{\alpha}_{a^+} g\big(a+\eta(b,a)\big) + \mathscr{I}^{\alpha}_{(a+\eta(b,a))^-} g(a)\right], \end{aligned} \tag{3.5}$$

that is

$$2^{s-2} g\left(\frac{2a+\eta(b,a)}{2}\right) \leqslant \frac{1-\alpha}{2(1-e^{-\rho})}\left[\mathscr{I}^{\alpha}_{a^+} g(a+\eta(b,a)) + \mathscr{I}^{\alpha}_{(a+\eta(b,a))^-} g(a)\right].$$

We finish the proof of the first inequality in inequality (3.2).

For the proof of the second inequality in inequality (3.2), we first note that if g is an (s,P)-preinvex function on $[a,a+\eta(b,a)]$ and the mapping η satisfies Condition C, then for every $t\in[0,1]$ and some fixed $s\in(0,1]$, we have that

$$\begin{aligned} g\big(a+t\eta(b,a)\big) &= g\Big(a+\eta(b,a)+(1-t)\eta\big(a,a+\eta(b,a)\big)\Big) \\ &\leqslant \big(t^s+(1-t)^s\big)\big[g(a)+g(a+\eta(b,a))\big], \end{aligned}$$

and

$$\begin{aligned} g\big(a+(1-t)\eta(b,a)\big) &= g\Big(a+\eta(b,a)+t\eta\big(a,a+\eta(b,a)\big)\Big) \\ &\leqslant \big(t^s+(1-t)^s\big)\big[g(a)+g(a+\eta(b,a))\big]. \end{aligned}$$

By adding these inequalities, we have that

$$g\big(a+t\eta(b,a)\big)+g\big(a+(1-t)\eta(b,a)\big)\leqslant 2\big(t^s+(1-t)^s\big)\big[g(a)+g(a+\eta(b,a))\big]. \tag{3.6}$$

Multiplying the both sides of inequality (3.6) by $e^{-\rho t}$ and integrating the resulting inequality with respect to t over $[0,1]$, we obtain that

$$\begin{aligned} &\int_0^1 e^{-\rho t}\big[g(a+t\eta(b,a))+g(a+(1-t)\eta(b,a))\big]\mathrm{d}t \\ &\quad\leqslant 2\int_0^1 e^{-\rho t}\big(t^s+(1-t)^s\big)\big[g(a)+g(a+\eta(b,a))\big]\mathrm{d}t. \end{aligned}\tag{3.7}$$

According to the mean value theorem of generalized integrals, for $\xi\in[0,1]$, we derive that

$$\begin{aligned} &2\int_0^1 e^{-\rho t}\big(t^s+(1-t)^s\big)\big[g(a)+g(a+\eta(b,a))\big]\mathrm{d}t \\ &\quad= 2e^{-\rho\xi}\big[g(a)+g(a+\eta(b,a))\big]\int_0^1\big(t^s+(1-t)^s\big)\mathrm{d}t \\ &\quad= \frac{4}{s+1}e^{-\rho\xi}\big[g(a)+g(a+\eta(b,a))\big]. \end{aligned}\tag{3.8}$$

Substituting the equality (3.8) into the inequality (3.7), we have that

$$\frac{\alpha}{\eta(b,a)}\Big[\mathscr{I}^{\alpha}_{a^+}g(a+\eta(b,a))+\mathscr{I}^{\alpha}_{(a+\eta(b,a))^-}g(a)\Big]\leqslant\frac{4}{s+1}e^{-\rho\xi}\big[g(a)+g(a+\eta(b,a))\big].$$

By multiplying both sides by $\frac{\rho}{2(1-e^{-\rho})}$, we have that

$$\begin{aligned} &\frac{1-\alpha}{2(1-e^{-\rho})}\Big[\mathscr{I}^{\alpha}_{a^+}g(a+\eta(b,a))+\mathscr{I}^{\alpha}_{(a+\eta(b,a))^-}g(a)\Big] \\ &\quad\leqslant\frac{2\rho e^{-\rho\xi}}{(s+1)(1-e^{-\rho})}\big[g(a)+g(a+\eta(b,a))\big]. \end{aligned}$$

Thus, the proof of Theorem 3.1 is completed. □

REMARK 3.4. In Theorem 3.1, if we take $\alpha \to 1$, i.e., $\rho = \frac{1-\alpha}{\alpha}\eta(b-a) \to 0$, then we have that

$$\lim_{\alpha\to 1} \frac{1-\alpha}{1-e^{-\rho}} = \frac{1}{\eta(b,a)}, \tag{3.9}$$

and

$$\lim_{\rho\to 0} \frac{\rho e^{-\rho\xi}}{1-e^{-\rho}} = 1.$$

Thus, the inequality (3.2) is transformed to

$$2^{s-2} g\left(\frac{2a+\eta(b,a)}{2}\right) \leqslant \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} g(u)\mathrm{d}u \leqslant \frac{2}{s+1}\big[g(a)+g(a+\eta(b,a))\big]. \tag{3.10}$$

Specially, putting $\eta(b,a) = b-a$, the inequality (3.10) turns into the inequality (2.1) established by Numan and İşcan in [37].

4. Parameterized fractional integral inequalities

This section endeavors to establish parameterized inequalities for (s,P)-preinvex functions, tied to fractional integrals involving exponential kernels. Initially, Subsection 4.1 formulates a fractional integral identity with a single parameter for first-order differentiable functions, upon which we derive the desired parameterized inequalities. Similarly, Subsection 4.2 extends this methodology to second-order differentiable functions, presenting a parameterized fractional integral identity and corresponding inequalities.

4.1. Parameterized inequalities for once-differentiable functions

For brevity, we will use the following notation in the sequel:

$$\begin{aligned} &T_g(\lambda,\alpha;a,a+\eta(b,a)) \\ &:= \frac{1-\alpha}{2(1-e^{-\rho})}\left[\mathscr{I}_{a^+}^{\alpha} g(a+\eta(b,a)) + \mathscr{I}_{(a+\eta(b,a))^-}^{\alpha} g(a)\right] \\ &\qquad -(1-\lambda)g\left(\frac{2a+\eta(b,a)}{2}\right) - \lambda\frac{g(a)+g(a+\eta(b,a))}{2}. \end{aligned} \tag{4.1}$$

Now we present the following lemma, which involves a parameter.

LEMMA 4.1. *Let $g:[a,a+\eta(b,a)] \to (0,\infty)$ be a differentiable function on $(a,a+\eta(b,a))$, where the mapping $\eta:\mathbb{R}\times\mathbb{R}\to\mathbb{R}^+$. If the function $g' \in L^1([a,a+\eta(b,a)])$ and $0 \leqslant \lambda \leqslant 1$, then the following identity for fractional integrals with exponential kernels holds*

$$T_g(\lambda,\alpha;a,a+\eta(b,a)) = \frac{\eta(b,a)}{2(1-e^{-\rho})} \int_0^1 w(t) g'(a+t\eta(b,a))\mathrm{d}t, \tag{4.2}$$

where

$$w(t)=\begin{cases}(1-\lambda)(e^{-\rho}-1)+e^{-\rho t}-e^{-\rho(1-t)}, & 0\leqslant t\leqslant \frac{1}{2},\\ (1-\lambda)(1-e^{-\rho})+e^{-\rho t}-e^{-\rho(1-t)}, & \frac{1}{2}<t\leqslant 1.\end{cases}$$

Proof. Considering the right hand side of the identity (4.2), we can write that

$$\begin{aligned}&\frac{\eta(b,a)}{2(1-e^{-\rho})}\int_0^1 w(t)g'(a+t\eta(b,a))\mathrm{d}t\\ &=\frac{\eta(b,a)}{2(1-e^{-\rho})}\left[\begin{array}{l}\int_0^{\frac{1}{2}}\left((1-\lambda)(e^{-\rho}-1)+e^{-\rho t}-e^{-\rho(1-t)}\right)g'(a+t\eta(b,a))\mathrm{d}t\\ \quad+\int_{\frac{1}{2}}^1\left((1-\lambda)(1-e^{-\rho})+e^{-\rho t}-e^{-\rho(1-t)}\right)g'(a+t\eta(b,a))\mathrm{d}t\end{array}\right]\\ &=\frac{\eta(b,a)}{2(1-e^{-\rho})}\left[\begin{array}{l}\int_0^{\frac{1}{2}}(1-\lambda)(e^{-\rho}-1)g'(a+t\eta(b,a))\mathrm{d}t\\ \quad+\int_{\frac{1}{2}}^1(1-\lambda)(1-e^{-\rho})g'(a+t\eta(b,a))\mathrm{d}t\\ \quad+\int_0^1\left(e^{-\rho t}-e^{-\rho(1-t)}\right)g'(a+t\eta(b,a))\mathrm{d}t\end{array}\right].\end{aligned}\tag{4.3}$$

Direct computation yields that

$$\begin{aligned}&\frac{\eta(b,a)}{2(1-e^{-\rho})}\left[\int_0^{\frac{1}{2}}(1-\lambda)(e^{-\rho}-1)g'(a+t\eta(b,a))\mathrm{d}t\right.\\ &\quad\left.+\int_{\frac{1}{2}}^1(1-\lambda)(1-e^{-\rho})g'(a+t\eta(b,a))\mathrm{d}t\right]\\ &=-\frac{(1-\lambda)}{2}\left(g(a+t\eta(b,a))\Big|_0^{\frac{1}{2}}-g(a+t\eta(b,a))\Big|_{\frac{1}{2}}^1\right)\\ &=(1-\lambda)\frac{g(a)+g(a+\eta(b,a))}{2}-(1-\lambda)g\left(\frac{2a+\eta(b,a)}{2}\right).\end{aligned}\tag{4.4}$$

Integrating by parts and changing the variables, we derive that

$$\begin{aligned}&\frac{\eta(b,a)}{2(1-e^{-\rho})}\int_0^1\left(e^{-\rho t}-e^{-\rho(1-t)}\right)g'(a+t\eta(b,a))\mathrm{d}t\\ &=\frac{\eta(b,a)}{2(1-e^{-\rho})}\left[\begin{array}{l}\frac{1}{\eta(b,a)}g(a+t\eta(b,a))\left(e^{-\rho t}-e^{-\rho(1-t)}\right)\Big|_0^1\\ \quad+\int_0^1\frac{\rho}{\eta(b,a)}\left(e^{-\rho t}+e^{-\rho(1-t)}\right)g(a+t\eta(b,a))\mathrm{d}t\end{array}\right]\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2(1-e^{-\rho})}\left[\begin{array}{l} (e^{-\rho}-1)\Big(g(a)+g(a+\eta(b,a))\Big) \\ \quad + \frac{\rho}{\eta(b,a)}\int_a^{a+\eta(b,a)} \left(e^{-\frac{1-\alpha}{\alpha}(\mu-a)} + e^{-\frac{1-\alpha}{\alpha}(a+\eta(b,a)-\mu)}\right) g(\mu)\mathrm{d}\mu \end{array}\right] \\
&= -\frac{g(a)+g(a+\eta(b,a))}{2} + \frac{1-\alpha}{2(1-e^{-\rho})}\left[\mathscr{I}^{\alpha}_{a^+}g(a+\eta(b,a)) + \mathscr{I}^{\alpha}_{(a+\eta(b,a))^-}g(a)\right].
\end{aligned} \tag{4.5}
$$

Applying equations (4.4) and (4.5) to the equality (4.3), we deduce the desired identity. This ends the proof. □

COROLLARY 4.1. *Consider Lemma 4.1, one could see that the next identities are correct clearly:*

(i) For $\lambda = 0$, we have the following midpoint-type equality for fractional integrals with exponential kernels:

$$
\begin{aligned}
&\frac{1-\alpha}{2(1-e^{-\rho})}\left[\mathscr{I}^{\alpha}_{a^+}g(a+\eta(b,a)) + \mathscr{I}^{\alpha}_{(a+\eta(b,a))^-}g(a)\right] - g\left(\frac{2a+\eta(b,a)}{2}\right) \\
&\quad = \frac{\eta(b,a)}{2(1-e^{-\rho})}\int_0^1 \left(k(t)(1-e^{-\rho}) + e^{-\rho t} - e^{-\rho(1-t)}\right) g'(a+t\eta(b,a))\mathrm{d}t,
\end{aligned}
$$

where

$$
k(t) = \begin{cases} -1, & 0 \leqslant t \leqslant \frac{1}{2}, \\ 1, & \frac{1}{2} < t \leqslant 1. \end{cases}
$$

(ii) For $\lambda = 1$, we have the following trapezoid-type equality for fractional integrals with exponential kernels:

$$
\begin{aligned}
&\frac{1-\alpha}{2(1-e^{-\rho})}\left[\mathscr{I}^{\alpha}_{a^+}g(a+\eta(b,a)) + \mathscr{I}^{\alpha}_{(a+\eta(b,a))^-}g(a)\right] - \frac{g(a)+g(a+\eta(b,a))}{2} \\
&\quad = \frac{\eta(b,a)}{2(1-e^{-\rho})}\int_0^1 \left(e^{-\rho t} - e^{-\rho(1-t)}\right) g'(a+t\eta(b,a))\mathrm{d}t.
\end{aligned}
$$

REMARK 4.1. In Lemma 4.1, if we take $\eta(b,a) = b-a$, then we have Lemma 2.1 established by Yuan et al. in [51].

REMARK 4.2. In Lemma 4.1, if we take $\alpha \to 1$, i.e., $\rho = \frac{1-\alpha}{\alpha}\eta(b-a) \to 0$, then we have

$$
\lim_{\rho \to 0} \frac{e^{-\rho t} - e^{-\rho(1-t)}}{1-e^{-\rho}} = 1-2t. \tag{4.6}
$$

Putting $\eta(b,a) = b-a$, and using results (3.9) together with (4.6), the equality (4.2) is transformed to

$$\frac{1}{b-a}\int_a^b g(u)\mathrm{d}u - (1-\lambda)g\left(\frac{a+b}{2}\right) - \lambda\frac{g(a)+g(b)}{2}$$
$$= \frac{b-a}{2}\int_0^1 c(t)g'(ta+(1-t)b)\mathrm{d}t,$$

where

$$c(t) = \begin{cases} 2t-\lambda, & 0 \leqslant t < \frac{1}{2}, \\ \lambda - 2(1-t), & \frac{1}{2} \leqslant t \leqslant 1. \end{cases}$$

In particular, for $\lambda = 0$, we get Lemma 2.1 established by Kirmaci in [27]. If we take $\lambda = 1$, then we obtain the same result presented by Dragomir and Agarwal in [14, Lemma 2.1].

By means of Lemma 4.1, the fractional inequality for once-differentiable functions is given as follows.

THEOREM 4.1. *Let $g : [a, a+\eta(b,a)] \to (0,\infty)$ be a differentiable function on $(a, a+\eta(b,a))$ satisfying $g' \in L^1([a, a+\eta(b,a)])$, where the mapping $\eta : \mathbb{R}\times\mathbb{R} \to \mathbb{R}^+$. For $0 \leqslant \lambda \leqslant 1$, if $|g'|$ is (s,P)-preinvex on $[a, a+\eta(b,a)]$, then the following inequality holds*

$$\left|T_g(\lambda,\alpha;a,a+\eta(b,a))\right|$$
$$\leqslant \eta(b,a)\left\{\begin{array}{c} \frac{2}{s+1}(1-\lambda)\left[(1-\Upsilon)^{s+1} - \Upsilon^{s+1} - \frac{1}{2}\right] \\ +\frac{2^{1-s}}{\rho(1-e^{-\rho})}\left[\left(1+e^{-\frac{\rho}{2}}\right)^2 - 2\sqrt{\Delta}\right] \end{array}\right\}\left(|g'(a)| + |g'(b)|\right), \tag{4.7}$$

where

$$\Upsilon := -\frac{1}{\rho}\ln\frac{(1-\lambda)(1-e^{-\rho})+\sqrt{\Delta}}{2}, \tag{4.8}$$

and

$$\Delta := (1-\lambda)^2(1-e^{-\rho})^2 + 4e^{-\rho}. \tag{4.9}$$

Proof. In the light of Lemma 4.1 and the (s,P)-preinvexity of $|g'|$ on $[a, a+\eta(b,a)]$, we have that

$$\left|T_g(\lambda;\alpha;a,a+\eta(b,a))\right|$$
$$\leqslant \frac{\eta(b,a)}{2(1-e^{-\rho})}\int_0^1 |w(t)|\left|g'(a+t\eta(b,a))\right|\mathrm{d}t$$

$$
\begin{aligned}
&\leqslant \frac{\eta(b,a)}{2(1-e^{-\rho})}\int_0^1 |w(t)|(t^s+(1-t)^s)(|g'(a)|+|g'(b)|)\mathrm{d}t \\
&= \frac{\eta(b,a)}{2(1-e^{-\rho})}(|g'(a)|+|g'(b)|) \\
&\quad\times\left[\begin{array}{l} \int_0^{\frac{1}{2}} \left|(1-\lambda)(e^{-\rho}-1)+e^{-\rho t}-e^{-\rho(1-t)}\right|(t^s+(1-t)^s)\mathrm{d}t \\ \quad +\int_{\frac{1}{2}}^1 \left|(1-\lambda)(1-e^{-\rho})+e^{-\rho t}-e^{-\rho(1-t)}\right|(t^s+(1-t)^s)\mathrm{d}t \end{array}\right].
\end{aligned} \tag{4.10}
$$

It is easy to prove that the following inequality

$$
0 \leqslant (1-\lambda)(e^{-\rho}-1)+e^{-\rho t}-e^{-\rho(1-t)} \leqslant \lambda(1-e^{-\rho}) \tag{4.11}
$$

holds for any $t \in [0,\Upsilon]$, where $\Upsilon := -\frac{1}{\rho}\ln\frac{(1-\lambda)(1-e^{-\rho})+\sqrt{\Delta}}{2}$ together with $\Delta := (1-\lambda)^2(1-e^{-\rho})^2+4e^{-\rho}$. In the same way, for any $t \in \left[\Upsilon, \frac{1}{2}\right]$, we have that

$$
(1-\lambda)(e^{-\rho}-1) \leqslant (1-\lambda)(e^{-\rho}-1)+e^{-\rho t}-e^{-\rho(1-t)} \leqslant 0. \tag{4.12}
$$

As a consequence, we obtain that

$$
\begin{aligned}
&\int_{\frac{1}{2}}^1 \left|(1-\lambda)(1-e^{-\rho})+e^{-\rho t}-e^{-\rho(1-t)}\right|(t^s+(1-t)^s)\mathrm{d}t \\
&\quad= \int_0^{\frac{1}{2}} \left|(1-\lambda)(e^{-\rho}-1)+e^{-\rho t}-e^{-\rho(1-t)}\right|(t^s+(1-t)^s)\mathrm{d}t \\
&\quad= \int_0^{\Upsilon} \left((1-\lambda)(e^{-\rho}-1)+e^{-\rho t}-e^{-\rho(1-t)}\right)(t^s+(1-t)^s)\mathrm{d}t \\
&\qquad+\int_{\Upsilon}^{\frac{1}{2}} \left((1-\lambda)(1-e^{-\rho})-e^{-\rho t}+e^{-\rho(1-t)}\right)(t^s+(1-t)^s)\mathrm{d}t.
\end{aligned} \tag{4.13}
$$

Direct computation yields that

$$
\int_0^{\Upsilon}(t^s+(1-t)^s)\mathrm{d}t = \frac{1}{s+1}\left[\Upsilon^{s+1}-(1-\Upsilon)^{s+1}+1\right], \tag{4.14}
$$

and

$$
\int_{\Upsilon}^{\frac{1}{2}}(t^s+(1-t)^s)\mathrm{d}t = \frac{1}{s+1}\left[(1-\Upsilon)^{s+1}-\Upsilon^{s+1}\right]. \tag{4.15}
$$

Since $t^s+(1-t)^s \leqslant 2^{1-s}$ for all $t \in [0,1]$, we get that

$$
\begin{aligned}
&\int_0^{\Upsilon}\left(e^{-\rho t}-e^{-\rho(1-t)}\right)(t^s+(1-t)^s)\,\mathrm{d}t \\
&\quad\leqslant 2^{1-s}\int_0^{\Upsilon}\left(e^{-\rho t}-e^{-\rho(1-t)}\right)\mathrm{d}t \\
&\quad= \frac{2^{1-s}}{\rho}\left(1+e^{-\rho}-\sqrt{\Delta}\right),
\end{aligned} \tag{4.16}
$$

and

$$\begin{aligned}
&\int_{\Upsilon}^{\frac{1}{2}}\left(e^{-\rho(1-t)}-e^{-\rho t}\right)(t^s+(1-t)^s)\mathrm{d}t\\
&\leqslant 2^{1-s}\int_{\Upsilon}^{\frac{1}{2}}\left(e^{-\rho(1-t)}-e^{-\rho t}\right)\mathrm{d}t\\
&=\frac{2^{1-s}}{\rho}\left(2e^{-\frac{\rho}{2}}-\sqrt{\Delta}\right). \qquad (4.17)
\end{aligned}$$

Applying (4.14)–(4.17) to equality (4.13), we obtain that

$$\begin{aligned}
&\int_0^{\frac{1}{2}}\left|(1-\lambda)(e^{-\rho}-1)+e^{-\rho t}-e^{-\rho(1-t)}\right|(t^s+(1-t)^s)\mathrm{d}t\\
&\leqslant\frac{2}{s+1}(1-\lambda)(1-e^{-\rho})\left[(1-\Upsilon)^{s+1}-\Upsilon^{s+1}-\frac{1}{2}\right]+\frac{2^{1-s}}{\rho}\left[\left(1+e^{-\frac{\rho}{2}}\right)^2-2\sqrt{\Delta}\right].
\end{aligned}$$

As a result, we have that

$$\begin{aligned}
&\int_0^1|w(t)|(t^s+(1-t)^s)\mathrm{d}t\\
&\leqslant 2\left\{\frac{2}{s+1}(1-\lambda)(1-e^{-\rho})\left[(1-\Upsilon)^{s+1}-\Upsilon^{s+1}-\frac{1}{2}\right]+\frac{2^{1-s}}{\rho}\left[\left(1+e^{-\frac{\rho}{2}}\right)^2-2\sqrt{\Delta}\right]\right\}.
\end{aligned} \qquad (4.18)$$

Employing the inequality (4.18) in the inequality (4.10), we obtain the expected result. Thus, the proof is concluded. □

COROLLARY 4.2. *If we choose $s=1$ in Theorem 4.1, then we have the following inequality:*

$$\begin{aligned}
&\left|T_g(\lambda,\alpha;a,a+\eta(b,a))\right|\\
&\leqslant\eta(b,a)\left[(1-\lambda)\left(\frac{1}{2}-2\Upsilon\right)+\frac{\left(1+e^{-\frac{\rho}{2}}\right)^2-2\sqrt{\Delta}}{\rho(1-e^{-\rho})}\right]\left(|g'(a)|+|g'(b)|\right),
\end{aligned} \qquad (4.19)$$

where Υ and Δ are given in (4.8) and (4.9) within Theorem 4.1.

REMARK 4.3. In Corollary 4.2, if we take $\alpha\to 1$, i.e., $\rho=\frac{1-\alpha}{\alpha}\eta(b-a)\to 0$, then we have that

$$\lim_{\rho\to 0}\left(-\frac{1}{\rho}\ln\frac{(1-\lambda)(1-e^{-\rho})+\sqrt{\Delta}}{2}\right)=\frac{\lambda}{2}, \qquad (4.20)$$

and

$$\lim_{\rho\to 0}\frac{\left(1+e^{-\frac{\rho}{2}}\right)^2-2\sqrt{\Delta}}{\rho(1-e^{-\rho})}=\frac{1}{4}-\frac{(1-\lambda)^2}{2}. \qquad (4.21)$$

Putting $\eta(b,a)=b-a$ and using results (3.9), (4.20) as well as (4.21), the inequality (4.19) is transformed to

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b g(u)\mathrm{d}u-(1-\lambda)g\left(\frac{a+b}{2}\right)-\lambda\frac{g(a)+g(b)}{2}\right|\\&\quad\leqslant\frac{b-a}{2}\left(\lambda^2-\lambda+\frac{1}{2}\right)\left(|g'(a)|+|g'(b)|\right).\end{aligned}\tag{4.22}$$

REMARK 4.4. Taking the specific choices of the parameter λ in the inequality (4.22), we can get the following known outcomes.

(i) For $\lambda=0$, we have that

$$\left|\frac{1}{b-a}\int_a^b g(u)\mathrm{d}u-g\left(\frac{a+b}{2}\right)\right|\leqslant\frac{b-a}{4}\left(|g'(a)|+|g'(b)|\right),$$

which is provided by Sarikaya et al. in [41, Theorem 4.8] for the case of $x\to a$ or $x\to b$.

(ii) For $\lambda=\frac{1}{3}$, we get that

$$\left|\frac{1}{6}\left[g(a)+4g\left(\frac{a+b}{2}\right)+g(b)\right]-\frac{1}{b-a}\int_a^b g(u)\mathrm{d}u\right|\leqslant\frac{5(b-a)}{36}\left(|g'(a)|+|g'(b)|\right),$$

which is established by Sarikaya et al. in [41, Theorem 5.8] for the case of $x\to a$ or $x\to b$.

(iii) For $\lambda=1$, we obtain that

$$\left|\frac{1}{b-a}\int_a^b g(u)\mathrm{d}u-\frac{g(a)+g(b)}{2}\right|\leqslant\frac{b-a}{4}\left(|g'(a)|+|g'(b)|\right),$$

which is presented by Sarikaya et al. in [41, Theorem 3.8] for the case of $x\to a$ or $x\to b$.

(iv) For $\lambda=\frac{1}{2}$, we have the following Bullen-type inequality for P-convex functions:

$$\left|\frac{1}{4}\left[g(a)+2g\left(\frac{a+b}{2}\right)+g(b)\right]-\frac{1}{b-a}\int_a^b g(u)\mathrm{d}u\right|\leqslant\frac{b-a}{8}\left(|g'(a)|+|g'(b)|\right).$$

To better understand Theorem 4.1, we give an example with graph.

EXAMPLE 4.1. Considering the function $g(x)=\frac{1}{\ln 2}(\frac{1}{2})^x$ on the interval $(0,\infty)$. Then $|g'(x)|=(\frac{1}{2})^x$ is an (s,P)-preinvex function with regard to $\eta(y,x)=2y-x$ for $s\in(0,1]$.

$$\begin{aligned}\left|g'(x+t\eta(y,x))\right|&=\left(\frac{1}{2}\right)^{x+t(2y-x)}\\&=\left(\frac{1}{2}\right)^{(1-t)x+2ty}\\&\leqslant(1-t)\left(\frac{1}{2}\right)^x+t\left(\frac{1}{2}\right)^{2y}\end{aligned}$$

$$\begin{aligned}
&\leqslant (1-t)\left(\frac{1}{2}\right)^x + t\left(\frac{1}{2}\right)^y \\
&\leqslant (1-t)^s\left(\frac{1}{2}\right)^x + t^s\left(\frac{1}{2}\right)^y \\
&\leqslant \left[(1-t)^s + t^s\right]\left[\left(\frac{1}{2}\right)^x + \left(\frac{1}{2}\right)^y\right].
\end{aligned}$$

If we take $a=0$, $b=1$, $s=\frac{1}{2}$, $\lambda=\frac{1}{3}$, then all assumptions in Theorem 4.1 are satisfied. Clearly, $\rho = \frac{1-\alpha}{\alpha}\eta(b,a) = \frac{2(1-\alpha)}{\alpha}$. The left side of inequality (4.7) is

$$\left|T_g\left(\frac{1}{3},\alpha;0,2\right)\right| = \left|\frac{1-\alpha}{2\alpha\left(1-e^{-\frac{2(1-\alpha)}{\alpha}}\right)\ln 2}\int_0^2\left(e^{-\frac{1-\alpha}{\alpha}(2-x)}+e^{-\frac{1-\alpha}{\alpha}x}\right)\left(\frac{1}{2}\right)^x \mathrm{d}x - \frac{13}{24\ln 2}\right|.$$

Meanwhile, the right side of inequality (4.7) can be written as

$$R(\alpha) = \frac{8}{3}\left[(1-\Upsilon^*)^{\frac{3}{2}} - \Upsilon^{*\frac{3}{2}} - \frac{1}{2}\right] + \frac{3\sqrt{2}\alpha}{2(1-\alpha)\left(1-e^{-\frac{2(1-\alpha)}{\alpha}}\right)}\left[\left(1+e^{-\frac{1-\alpha}{\alpha}}\right)^2 - 2\sqrt{\Delta^*}\right],$$

where

$$\Upsilon^* := -\frac{\alpha}{2(1-\alpha)}\ln\frac{\frac{2}{3}\left(1-e^{-\frac{2(1-\alpha)}{\alpha}}\right)+\sqrt{\Delta^*}}{2},$$

and

$$\Delta^* := \frac{4}{9}\left(1-e^{-\frac{2(1-\alpha)}{\alpha}}\right)^2 + 4e^{-\frac{2(1-\alpha)}{\alpha}}.$$

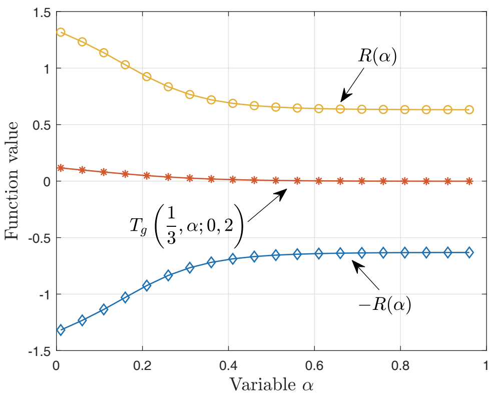

Figure 4.1: *An example to the inequality* (4.7)

We put the variable $\alpha \in [0.01, 0.99]$ and plot the graphical depiction of the functions $T_g\left(\frac{1}{3}, \alpha; 0, 2\right)$, $R(\alpha)$ and $-R(\alpha)$ in Figure 4.1. It is not laborious to observe that $-R(\alpha) < T_g\left(\frac{1}{3}, \alpha; 0, 2\right) < R(\alpha)$, which shows the consistency with the outcome established in Theorem 4.1.

With the help of the Hölder's inequality, the following theorem is given.

THEOREM 4.2. *Let $g : [a, a+\eta(b,a)] \to (0,\infty)$ be a differentiable function on $(a, a+\eta(b,a))$ satisfying $g' \in L^1([a, a+\eta(b,a)])$, where the mapping $\eta : \mathbb{R}\times\mathbb{R} \to \mathbb{R}^+$. For $0 \leqslant \lambda \leqslant 1$, $q > 1$ with $p^{-1}+q^{-1} = 1$, if $|g'|^q$ is (s,P)-preinvex on $[a, a+\eta(b,a)]$, then the following inequality for fractional integrals holds*

$$\begin{aligned}&\left|T_g(\lambda,\alpha;a,a+\eta(b,a))\right|\\&\quad\leqslant 2^{\frac{1}{p}-1}\eta(b,a)\left(\frac{2}{s+1}\right)^{\frac{1}{q}}\left[\Upsilon\lambda^p+\left(\frac{1}{2}-\Upsilon\right)(1-\lambda)^p\right]^{\frac{1}{p}}\left(|g'(a)|^q+|g'(b)|^q\right)^{\frac{1}{q}},\end{aligned}$$

where Υ is defined by (4.8) *in Theorem* 4.1.

Proof. Utilizing Lemma 4.1, the definition of $w(t)$, and the Hölder's inequality, we obtain that

$$\begin{aligned}&\left|T_g(\lambda,\alpha;a,a+\eta(b,a))\right|\\&\leqslant \frac{\eta(b,a)}{2(1-e^{-\rho})}\int_0^1 |w(t)||g'(a+t\eta(b,a))|\mathrm{d}t\\&\leqslant \frac{\eta(b,a)}{2(1-e^{-\rho})}\left(\int_0^1 |w(t)|^p\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^1 |g'(a+t\eta(b,a))|^q\mathrm{d}t\right)^{\frac{1}{q}}\\&= \frac{\eta(b,a)}{2(1-e^{-\rho})}\left(\int_0^{\frac{1}{2}} |w_1(t)|^p\mathrm{d}t+\int_{\frac{1}{2}}^1 |w_2(t)|^p\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^1 |g'(a+t\eta(b,a))|^q\mathrm{d}t\right)^{\frac{1}{q}},\end{aligned} \tag{4.23}$$

where

$$w_1(t) = (1-\lambda)(e^{-\rho}-1)+e^{-\rho t}-e^{-\rho(1-t)}, \quad t \in \left[0, \frac{1}{2}\right],$$

and

$$w_2(t) = (1-\lambda)(1-e^{-\rho})+e^{-\rho t}-e^{-\rho(1-t)}, \quad t \in \left(\frac{1}{2}, 1\right].$$

Taking advantage of the inequalities (4.11) and (4.12), we have that

$$\begin{aligned}\int_{\frac{1}{2}}^1 |w_2(t)|^p\mathrm{d}t &= \int_0^{\frac{1}{2}} |w_1(t)|^p\mathrm{d}t\\&= \int_0^{\Upsilon}\left((1-\lambda)(e^{-\rho}-1)+e^{-\rho t}-e^{-\rho(1-t)}\right)^p\mathrm{d}t\\&\quad+\int_{\Upsilon}^{\frac{1}{2}}\left((1-\lambda)(1-e^{-\rho})-e^{-\rho t}+e^{-\rho(1-t)}\right)^p\mathrm{d}t\end{aligned}$$

$$\leqslant \int_0^{\Upsilon} \left(\lambda(1-e^{-\rho})\right)^p \mathrm{d}t + \int_{\Upsilon}^{\frac{1}{2}} \left((1-\lambda)(1-e^{-\rho})\right)^p \mathrm{d}t$$
$$= \Upsilon\left(\lambda(1-e^{-\rho})\right)^p + \left(\frac{1}{2}-\Upsilon\right)\left((1-\lambda)(1-e^{-\rho})\right)^p. \tag{4.24}$$

Since $|g'|^q$ is (s,P)-preinvex on $[a,a+\eta(b,a)]$, we get that

$$\int_0^1 |g'(a+t\eta(b,a))|^q \mathrm{d}t \leqslant \int_0^1 (t^s+(1-t)^s)(|g'(a)|^q+|g'(b)|^q)\mathrm{d}t$$
$$= \frac{2}{s+1}(|g'(a)|^q+|g'(b)|^q). \tag{4.25}$$

A combination of inequalities (4.23)–(4.25) gives the required result. Thus, the proof of Theorem 4.2 is end. $\square$

Through application of the power-mean inequality method, we arrive at the theorem stated below.

THEOREM 4.3. *Let $g:[a,a+\eta(b,a)]\to(0,\infty)$ be a differentiable function on $(a,a+\eta(b,a))$ satisfying $g'\in L^1([a,a+\eta(b,a)])$, where the mapping $\eta:\mathbb{R}\times\mathbb{R}\to\mathbb{R}^+$. For $0\leqslant\lambda\leqslant 1$, if $|g'|^q$ is (s,P)-preinvex on $[a,a+\eta(b,a)]$ with $q>1$, then the following inequality holds*

$$\begin{aligned}
&\left|T_g(\lambda,\alpha;a,a+\eta(b,a))\right| \\
&\leqslant \eta(b,a)\left(|g'(a)|^q+|g'(b)|^q\right)^{\frac{1}{q}}\left[(1-\lambda)\left(\frac{1}{2}-2\Upsilon\right)+\frac{\left(1+e^{-\frac{\rho}{2}}\right)^2-2\sqrt{\Delta}}{\rho(1-e^{-\rho})}\right]^{1-\frac{1}{q}} \\
&\times\left\{\frac{2}{s+1}(1-\lambda)\left[(1-\Upsilon)^{s+1}-\Upsilon^{s+1}-\frac{1}{2}\right]+\frac{2^{1-s}}{\rho(1-e^{-\rho})}\left[\left(1+e^{-\frac{\rho}{2}}\right)^2-2\sqrt{\Delta}\right]\right\}^{\frac{1}{q}},
\end{aligned} \tag{4.26}$$

where Υ and Δ are given in (4.8) and (4.9) of Theorem 4.1.

Proof. Using Lemma 4.1 and the power-mean inequality, we have that

$$\begin{aligned}
&\left|T_g(\lambda,\alpha;a,a+\eta(b,a))\right| \\
&\leqslant \frac{\eta(b,a)}{2(1-e^{-\rho})}\int_0^1 |w(t)|\left|g'(a+t\eta(b,a))\right|\mathrm{d}t \\
&\leqslant \frac{\eta(b,a)}{2(1-e^{-\rho})}\left(\int_0^1 |w(t)|\mathrm{d}t\right)^{1-\frac{1}{q}}\left(\int_0^1 |w(t)|\left|g'(a+t\eta(b,a))\right|^q\mathrm{d}t\right)^{\frac{1}{q}}.
\end{aligned} \tag{4.27}$$

Utilizing the definition of $w(t)$ and necessary calculation, we obtain that

$$\begin{aligned}
\int_0^1 |w(t)|\mathrm{d}t &= \int_0^{\frac{1}{2}} |w_1(t)|\mathrm{d}t + \int_{\frac{1}{2}}^1 |w_2(t)|\mathrm{d}t \\
&= 2\int_0^{\frac{1}{2}} |w_1(t)|\mathrm{d}t \\
&= 2\left[\begin{array}{l} \int_0^{\Upsilon} \left((1-\lambda)\left(e^{-\rho}-1\right)+e^{-\rho t}-e^{-\rho(1-t)}\right)\mathrm{d}t \\ \quad +\int_{\Upsilon}^{\frac{1}{2}} \left((1-\lambda)\left(1-e^{-\rho}\right)-e^{-\rho t}+e^{-\rho(1-t)}\right)\mathrm{d}t \end{array}\right] \\
&= 2\left[(1-\lambda)\left(1-e^{-\rho}\right)\left(\frac{1}{2}-2\Upsilon\right)+\frac{\left(1+e^{-\frac{\rho}{2}}\right)^2-2\sqrt{\Delta}}{\rho}\right]. \qquad (4.28)
\end{aligned}$$

Using the (s,P)-preinvexity of $|g'|^q$ on $[a,a+\eta(b,a)]$ and the inequality (4.18), we get that

$$\begin{aligned}
&\int_0^1 |w(t)|\left|g'(a+t\eta(b,a))\right|^q \mathrm{d}t \\
&\leqslant \int_0^1 |w(t)|(t^s+(1-t)^s)(|g'(a)|^q+|g'(b)|^q)\mathrm{d}t \\
&\leqslant 2\left\{\begin{array}{l} \frac{2}{s+1}(1-\lambda)(1-e^{-\rho})\left[(1-\Upsilon)^{s+1}-\Upsilon^{s+1}-\frac{1}{2}\right] \\ \quad +\frac{2^{1-s}}{\rho}\left[\left(1+e^{-\frac{\rho}{2}}\right)^2-2\sqrt{\Delta}\right] \end{array}\right\}(|g'(a)|^q+|g'(b)|^q).
\end{aligned} \qquad (4.29)$$

Making use of (4.28) and (4.29) in (4.27), we get the desired outcome. Thus, the proof is done. □

COROLLARY 4.3. *If we choose $s=1$ in Theorem 4.3, then we have the following inequality:*

$$\begin{aligned}
&\left|T_g(\lambda,\alpha;a,a+\eta(b,a))\right| \\
&\leqslant \eta(b,a)\left[(1-\lambda)\left(\frac{1}{2}-2\Upsilon\right)+\frac{(1+e^{-\frac{\rho}{2}})^2-2\sqrt{\Delta}}{\rho(1-e^{-\rho})}\right]\left(|g'(a)|^q+|g'(b)|^q\right)^{\frac{1}{q}},
\end{aligned} \qquad (4.30)$$

where Υ and Δ are defined in (4.8) and (4.9) of Theorem 4.1.

REMARK 4.5. In Corollary 4.3, putting $\eta(b,a)=b-a$ and letting $\alpha\to 1$, together with using results (3.9), (4.20) and (4.21), the inequality (4.30) is transformed to

$$\left|\frac{1}{b-a}\int_a^b g(u)\mathrm{d}u-(1-\lambda)g\left(\frac{a+b}{2}\right)-\lambda\frac{g(a)+g(b)}{2}\right|$$
$$\leqslant\frac{b-a}{2}\left(\lambda^2-\lambda+\frac{1}{2}\right)\left(|g'(a)|^q+|g'(b)|^q\right)^{\frac{1}{q}}.$$

In particular, for $\lambda=1$, we get Theorem 2.3 established by Barani in [6].

4.2. Parameterized inequalities for the twice-differentiable functions

The lemma that follows is introduced to establish inequalities that are parameterized and based on the (s,P)-preinvexity of twice-differentiable functions.

LEMMA 4.2. *Let $g:[a,a+\eta(b,a)]\to(0,\infty)$ be a twice-differentiable function on $(a,a+\eta(b,a))$, where the mapping $\eta:\mathbb{R}\times\mathbb{R}\to\mathbb{R}^+$. For $0\leqslant\lambda\leqslant 1$, if the function $g''\in L^1([a,a+\eta(b,a)])$, then the following identity for fractional integrals with exponential kernels holds*

$$T_g(\lambda,\alpha;a,a+\eta(b,a))=\frac{\eta^2(b,a)}{2}\int_0^1 h(t)g''(a+t\eta(b,a))\mathrm{d}t, \tag{4.31}$$

where

$$h(t)=\begin{cases}t(1-\lambda)-\dfrac{1+e^{-\rho}-e^{-\rho t}-e^{-\rho(1-t)}}{\rho(1-e^{-\rho})}, & 0\leqslant t\leqslant\dfrac{1}{2},\\[2ex] (1-t)(1-\lambda)-\dfrac{1+e^{-\rho}-e^{-\rho t}-e^{-\rho(1-t)}}{\rho(1-e^{-\rho})}, & \dfrac{1}{2}<t\leqslant 1,\end{cases}$$

and $T_g(\lambda,\alpha;a,a+\eta(b,a))$ is defined in (4.1).

Proof. The proof is analogous to that of Lemma 4.1 and we omit the details here. □

COROLLARY 4.4. *Consider Lemma 4.2, we can get the following known results.*

(i) For $\lambda=0$, we have the following midpoint-type equality for fractional integrals with exponential kernels:

$$\frac{1-\alpha}{2(1-e^{-\rho})}\left[\mathscr{I}^{\alpha}_{a^+}g\big(a+\eta(b,a)\big)+\mathscr{I}^{\alpha}_{(a+\eta(b,a))^-}g\big(a\big)\right]-g\left(\frac{2a+\eta(b,a)}{2}\right)$$
$$=\frac{\eta^2(b,a)}{2}\int_0^1 m(t)g''(a+t\eta(b,a))\mathrm{d}t,$$

where

$$m(t)=\begin{cases}t-\dfrac{1+e^{-\rho}-e^{-\rho t}-e^{-\rho(1-t)}}{\rho(1-e^{-\rho})}, & 0\leqslant t\leqslant\dfrac{1}{2},\\[2ex] (1-t)-\dfrac{1+e^{-\rho}-e^{-\rho t}-e^{-\rho(1-t)}}{\rho(1-e^{-\rho})}, & \dfrac{1}{2}<t\leqslant 1.\end{cases}$$

(ii) For $\lambda=1$, we obtain the following trapezoid-type equality for fractional integrals with exponential kernels:

$$\begin{aligned}&\frac{1-\alpha}{2(1-e^{-\rho})}\left[\mathscr{I}_{a^+}^{\alpha}g\big(a+\eta(b,a)\big)+\mathscr{I}_{(a+\eta(b,a))^-}^{\alpha}g(a)\right]-\frac{g(a)+g(a+\eta(b,a))}{2}\\&\quad=-\frac{\eta^2(b,a)}{2\rho(1-e^{-\rho})}\int_0^1\left(1+e^{-\rho}-e^{-\rho t}-e^{-\rho(1-t)}\right)g''(a+t\eta(b,a))\mathrm{d}t.\end{aligned}$$

REMARK 4.6. In Lemma 4.2, putting $\eta(b,a)=b-a$, we acquire Lemma 2.2 established by Zhou et al. in [55].

REMARK 4.7. In Lemma 4.2, if we take $\alpha\to 1$, i.e., $\rho=\frac{1-\alpha}{\alpha}\eta(b-a)\to 0$, then we get that

$$\lim_{\rho\to 0}\frac{1+e^{-\rho}-e^{-\rho t}-e^{-\rho(1-t)}}{\rho(1-e^{-\rho})}=-t^2+t. \tag{4.32}$$

Putting $\eta(b,a)=b-a$ and using results (3.9) together with (4.32), the equality (4.31) is transformed to

$$\begin{aligned}&\frac{1}{b-a}\int_a^b g(u)\mathrm{d}u-(1-\lambda)g\left(\frac{a+b}{2}\right)-\lambda\frac{g(a)+g(b)}{2}\\&\quad=\frac{(b-a)^2}{2}\int_0^1 s(t)g''(ta+(1-t)b)\mathrm{d}t,\end{aligned}$$

where

$$s(t)=\begin{cases}t^2-\lambda t, & 0\leqslant t<\frac{1}{2},\\ (1-t)(1-\lambda-t), & \frac{1}{2}\leqslant t\leqslant 1,\end{cases}$$

which is recorded by Sarikaya and Aktan in [40, Lemma 2].

Before giving the parameterized-type inequalities for the twice-differentiable functions, we recall that hyperbolic tangent function is defined by

$$\tanh(x)=\frac{\sinh(x)}{\cosh(x)}=\frac{e^x-e^{-x}}{e^x+e^{-x}}.$$

THEOREM 4.4. *Let $g:[a,a+\eta(b,a)]\to(0,\infty)$ be a twice-differentiable function on $(a,a+\eta(b,a))$ satisfying $g''\in L^1([a,a+\eta(b,a)])$, where the mapping $\eta:\mathbb{R}\times\mathbb{R}\to\mathbb{R}^+$. For $0\leqslant\lambda\leqslant 1$, if $|g''|$ is (s,P)-preinvex on $[a,a+\eta(b,a)]$, then the following inequality holds*

$$\begin{aligned}&\left|T_g(\lambda,\alpha;a,a+\eta(b,a))\right|\\&\quad\leqslant\frac{\eta^2(b,a)}{2}\left[\frac{2\tanh(\frac{\rho}{4})}{\rho(s+1)}+\frac{2-2^{-s}}{(s+1)(s+2)}(1-\lambda)\right]\left(|g''(a)|+|g''(b)|\right).\end{aligned}$$

Proof. Using Lemma 4.2 and the definition of $h(t)$, we have that

$$
\begin{aligned}
&\left|T_g(\lambda,\alpha;a,a+\eta(b,a))\right| \\
&\leqslant \frac{\eta^2(b,a)}{2}\left[\begin{array}{l}\displaystyle\int_0^{\frac{1}{2}}\left|t(1-\lambda)-\frac{1+e^{-\rho}-e^{-\rho t}-e^{-\rho(1-t)}}{\rho(1-e^{-\rho})}\right|\left|g''(a+t\eta(b,a))\right|\mathrm{d}t \\
\displaystyle\quad+\int_{\frac{1}{2}}^1\left|(1-t)(1-\lambda)-\frac{1+e^{-\rho}-e^{-\rho t}-e^{-\rho(1-t)}}{\rho(1-e^{-\rho})}\right| \\
\quad\times\left|g''(a+t\eta(b,a))\right|\mathrm{d}t\end{array}\right] \\
&\leqslant \frac{\eta^2(b,a)}{2}\left[\begin{array}{l}\displaystyle\int_0^{\frac{1}{2}}t(1-\lambda)\left|g''(ta+(1-t)b)\right|\mathrm{d}t \\
\displaystyle\quad+\int_{\frac{1}{2}}^1(1-t)(1-\lambda)\left|g''(a+t\eta(b,a))\right|\mathrm{d}t \\
\displaystyle\quad+\int_0^1\left|\frac{1+e^{-\rho}-e^{-\rho t}-e^{-\rho(1-t)}}{\rho(1-e^{-\rho})}\right|\left|g''(a+t\eta(b,a))\right|\mathrm{d}t\end{array}\right]. \qquad (4.33)
\end{aligned}
$$

Since $|g''|^q$ is (s,P)-preinvex on $[a,a+\eta(b,a)]$, we get that

$$
\begin{aligned}
&\int_0^{\frac{1}{2}}t(1-\lambda)\left|g''(a+t\eta(b,a))\right|\mathrm{d}t+\int_{\frac{1}{2}}^1(1-t)(1-\lambda)\left|g''(a+t\eta(b,a))\right|\mathrm{d}t \\
&\quad\leqslant \int_0^{\frac{1}{2}}t(1-\lambda)(t^s+(1-t)^s)\left(|g''(a)|+|g''(b)|\right)\mathrm{d}t \\
&\qquad+\int_{\frac{1}{2}}^1(1-t)(1-\lambda)(t^s+(1-t)^s)\left(|g''(a)|+|g''(b)|\right)\mathrm{d}t \\
&\quad=2(1-\lambda)\left(|g''(a)|+|g''(b)|\right)\int_0^{\frac{1}{2}}(t^{s+1}+t(1-t)^s)\mathrm{d}t \\
&\quad=\frac{2-2^{-s}}{(s+1)(s+2)}(1-\lambda)\left(|g''(a)|+|g''(b)|\right). \qquad (4.34)
\end{aligned}
$$

Since $2e^{-\frac{\rho}{2}}\leqslant e^{-\rho t}+e^{-\rho(1-t)}\leqslant 1+e^{-\rho}$ for any $t\in[0,1]$ and $|g''|$ is (s,P)-preinvex on $[a,a+\eta(b,a)]$, we obtain that

$$
\begin{aligned}
&\int_0^1\left|\frac{1+e^{-\rho}-e^{-\rho t}-e^{-\rho(1-t)}}{\rho(1-e^{-\rho})}\right|\left|g''(a+t\eta(b,a))\right|\mathrm{d}t \\
&\quad\leqslant\int_0^1\frac{1+e^{-\rho}-2e^{-\frac{\rho}{2}}}{\rho(1-e^{-\rho})}(t^s+(1-t)^s)\left(|g''(a)|+|g''(b)|\right)\mathrm{d}t \\
&\quad=\frac{2\tanh(\frac{\rho}{4})}{\rho(s+1)}\left(|g''(a)|+|g''(b)|\right). \qquad (4.35)
\end{aligned}
$$

A combination of inequalities (4.33)–(4.35) yields the desired result. Thus, the proof is done. □

Below, we present a theorem which is proved by invoking the Hölder's inequality.

THEOREM 4.5. *Let $g:[a,a+\eta(b,a)]\to(0,\infty)$ be a twice-differentiable function on $(a,a+\eta(b,a))$ satisfying $g''\in L^1([a,a+\eta(b,a)])$, where the mapping $\eta:\mathbb{R}\times\mathbb{R}\to\mathbb{R}^+$. For $0\leqslant\lambda\leqslant 1$, $q>1$ with $p^{-1}+q^{-1}=1$, if $|g''|^q$ is (s,P)-preinvex on $[a,a+\eta(b,a)]$, then the following inequalities for fractional integrals hold*
(i) For $0\leqslant\lambda<1$, we have that

$$\begin{aligned}&\left|T_g(\lambda,\alpha;a,a+\eta(b,a))\right|\\&\leqslant\frac{\eta^2(b,a)(1-\lambda)}{2}\left(\frac{2}{p+1}\right)^{\frac{1}{p}}\left(\frac{2}{s+1}\right)^{\frac{1}{q}}\\&\times\left[\left(\frac{1}{2}+\frac{\tanh(\frac{\rho}{4})}{\rho(1-\lambda)}\right)^{p+1}-\left(\frac{\tanh(\frac{\rho}{4})}{\rho(1-\lambda)}\right)^{p+1}\right]^{\frac{1}{p}}\left(|g''(a)|^q+|g''(b)|^q\right)^{\frac{1}{q}}.\end{aligned}\tag{4.36}$$

(ii) For $\lambda=1$, we have that

$$\begin{aligned}&\left|\frac{1-\alpha}{2(1-e^{-\rho})}\left[\mathscr{I}^{\alpha}_{a^+}g(a+\eta(b,a))+\mathscr{I}^{\alpha}_{(a+\eta(b,a))^-}g(a)\right]-\frac{g(a)+g(a+\eta(b,a))}{2}\right|\\&\quad\leqslant\frac{\eta^2(b,a)\tanh(\frac{\rho}{4})}{2\rho}\left(\frac{2}{s+1}\right)^{\frac{1}{q}}\left(|g''(a)|^q+|g''(b)|^q\right)^{\frac{1}{q}}.\end{aligned}$$

Proof. First, suppose that $0\leqslant\lambda<1$. Utilizing Lemma 4.2, the definition of $h(t)$, and the Hölder's inequality, we obtain that

$$\begin{aligned}&\left|T_g(\lambda,\alpha;a,a+\eta(b,a))\right|\\&\leqslant\frac{\eta^2(b,a)}{2}\int_0^1|h(t)||g''(a+t\eta(b,a))|\mathrm{d}t\\&\leqslant\frac{\eta^2(b,a)}{2}\left(\int_0^1|h(t)|^p\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^1|g''(a+t\eta(b,a))|^q\mathrm{d}t\right)^{\frac{1}{q}}\\&=\frac{\eta^2(b,a)}{2}\left(\int_0^{\frac{1}{2}}|h_1(t)|^p\mathrm{d}t+\int_{\frac{1}{2}}^1|h_2(t)|^p\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^1|g''(a+t\eta(b,a))|^q\mathrm{d}t\right)^{\frac{1}{q}},\end{aligned}\tag{4.37}$$

where

$$h_1(t)=t(1-\lambda)-\frac{1+e^{-\rho}-e^{-\rho t}-e^{-\rho(1-t)}}{\rho(1-e^{-\rho})},\quad t\in\left[0,\frac{1}{2}\right],$$

and

$$h_2(t)=(1-t)(1-\lambda)-\frac{1+e^{-\rho}-e^{-\rho t}-e^{-\rho(1-t)}}{\rho(1-e^{-\rho})},\quad t\in\left(\frac{1}{2},1\right].$$

Owing to $2e^{-\frac{\rho}{2}} \leqslant e^{-\rho t}+e^{-\rho(1-t)} \leqslant 1+e^{-\rho}$ for any $t \in [0,1]$, we have that

$$\begin{aligned}
\int_{\frac{1}{2}}^{1} |h_2(t)|^p \mathrm{d}t &= \int_0^{\frac{1}{2}} |h_1(t)|^p \mathrm{d}t \\
&\leqslant \int_0^{\frac{1}{2}} \left(t(1-\lambda) + \frac{1+e^{-\rho}-e^{-\rho t}-e^{-\rho(1-t)}}{\rho(1-e^{-\rho})} \right)^p \mathrm{d}t \\
&\leqslant \int_0^{\frac{1}{2}} \left(t(1-\lambda) + \frac{1+e^{-\rho}-2e^{-\frac{\rho}{2}}}{\rho(1-e^{-\rho})} \right)^p \mathrm{d}t \\
&= (1-\lambda)^p \int_0^{\frac{1}{2}} \left(t + \frac{\left(1-e^{-\frac{\rho}{2}}\right)^2}{\rho(1-e^{-\rho})(1-\lambda)} \right)^p \mathrm{d}t \\
&= (1-\lambda)^p \frac{1}{p+1} \left[\left(\frac{1}{2} + \frac{\tanh(\frac{\rho}{4})}{\rho(1-\lambda)} \right)^{p+1} - \left(\frac{\tanh(\frac{\rho}{4})}{\rho(1-\lambda)} \right)^{p+1} \right].
\end{aligned}$$

As a result, it follows that

$$\int_0^1 |h(t)|^p \mathrm{d}t \leqslant (1-\lambda)^p \frac{2}{p+1} \left[\left(\frac{1}{2} + \frac{\tanh(\frac{\rho}{4})}{\rho(1-\lambda)} \right)^{p+1} - \left(\frac{\tanh(\frac{\rho}{4})}{\rho(1-\lambda)} \right)^{p+1} \right]. \tag{4.38}$$

Since $|g''|^q$ is (s,P)-preinvex on $[a,a+\eta(b,a)]$, we get that

$$\begin{aligned}
\int_0^1 |g''(a+t\eta(b,a))|^q \mathrm{d}t &\leqslant \int_0^1 (t^s+(1-t)^s)(|g''(a)|^q+|g''(b)|^q) \mathrm{d}t \\
&= \frac{2}{s+1}(|g''(a)|^q+|g''(b)|^q).
\end{aligned} \tag{4.39}$$

Applying inequalities (4.38) and (4.39) to inequality (4.37), we obtain the required result (4.36). Thus, this ends the proof for this case.

Now, suppose that $\lambda = 1$. The remainder of the argument is analogous to that of part one in Theorem 4.5 and we omit the details. Thus, the proof of Theorem 4.5 is completed. □

THEOREM 4.6. *Let $g : [a,a+\eta(b,a)] \to (0,\infty)$ be a twice-differentiable function on $(a,a+\eta(b,a))$ satisfying $g'' \in L^1([a,a+\eta(b,a)])$, where the mapping $\eta : \mathbb{R}\times\mathbb{R} \to \mathbb{R}^+$. For $0 \leqslant \lambda \leqslant 1$, if $|g''|^q$ is (s,P)-preinvex on $[a,a+\eta(b,a)]$ with $q>1$, then the following inequality holds*

$$\begin{aligned}
&\left|T_g(\lambda,\alpha;a,a+\eta(b,a))\right| \\
&\quad \leqslant 2^{\frac{1-s-q}{q}} \eta^2(b,a) \left(\frac{1-\lambda}{4} + \frac{\rho+\rho e^{-\rho}+2e^{-\rho}-2}{\rho^2(1-e^{-\rho})} \right) \left(|g''(a)|^q+|g''(b)|^q\right)^{\frac{1}{q}}.
\end{aligned} \tag{4.40}$$

Proof. Utilizing Lemma 4.2, the definition of $h(t)$, and the power-mean inequality, we have that

$$\begin{aligned}
&\left|T_g(\lambda,\alpha;a,a+\eta(b,a))\right| \\
&\quad\leqslant \frac{\eta^2(b,a)}{2}\int_0^1 |h(t)|\left|g''(a+t\eta(b,a))\right|\mathrm{d}t \\
&\quad\leqslant \frac{\eta^2(b,a)}{2}\left(\int_0^1 |h(t)|\mathrm{d}t\right)^{1-\frac{1}{q}}\left(\int_0^1 |h(t)|\left|g''(a+t\eta(b,a))\right|^q\mathrm{d}t\right)^{\frac{1}{q}}. \qquad (4.41)
\end{aligned}$$

Using the properties of the module and direct computation, we obtain that

$$\begin{aligned}
\int_0^1 |h(t)|\mathrm{d}t &= \int_0^{\frac{1}{2}} |h_1(t)|\mathrm{d}t + \int_{\frac{1}{2}}^1 |h_2(t)|\mathrm{d}t \\
&\leqslant \int_0^{\frac{1}{2}}\left(t(1-\lambda)+\frac{1+e^{-\rho}-e^{-\rho t}-e^{-\rho(1-t)}}{\rho(1-e^{-\rho})}\right)\mathrm{d}t \\
&\quad+\int_{\frac{1}{2}}^1\left((1-t)(1-\lambda)+\frac{1+e^{-\rho}-e^{-\rho t}-e^{-\rho(1-t)}}{\rho(1-e^{-\rho})}\right)\mathrm{d}t \\
&= \frac{1-\lambda}{4}+\frac{\rho+\rho e^{-\rho}+2e^{-\rho}-2}{\rho^2(1-e^{-\rho})}. \qquad (4.42)
\end{aligned}$$

Since $|g''|^q$ is (s,P)-preinvex on $[a,a+\eta(b,a)]$ and $t^s+(1-t)^s \leqslant 2^{1-s}$ for all $t \in [0,1]$, we get that

$$\begin{aligned}
&\int_0^1 |h(t)|\left|g''(a+t\eta(b,a))\right|^q\mathrm{d}t \\
&\quad\leqslant \int_0^1 |h(t)|(t^s+(1-t)^s)(|g''(a)|^q+|g''(b)|^q)\mathrm{d}t \\
&\quad\leqslant 2^{1-s}\left(|g''(a)|^q+|g''(b)|^q\right)\int_0^1 |h(t)|\mathrm{d}t \\
&\quad\leqslant 2^{1-s}\left[\frac{1-\lambda}{4}+\frac{\rho+\rho e^{-\rho}+2e^{-\rho}-2}{\rho^2(1-e^{-\rho})}\right](|g''(a)|^q+|g''(b)|^q). \qquad (4.43)
\end{aligned}$$

Employing inequalities (4.42) and (4.43) in inequality (4.41), we obtain the desired result. Thus, the proof is accomplished. □

REMARK 4.8. In Theorem 4.6, if we take $\alpha \to 1$, i.e., $\rho = \frac{1-\alpha}{\alpha}\eta(b-a) \to 0$, then we have that

$$\lim_{\rho\to 0}\frac{\rho+\rho e^{-\rho}+2e^{-\rho}-2}{\rho^2(1-e^{-\rho})} = \frac{1}{6}. \qquad (4.44)$$

Using results (3.9) and (4.44), the inequality (4.40) is transformed to

$$\left|\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} g(u)\mathrm{d}u-(1-\lambda)g\left(\frac{2a+\eta(b,a)}{2}\right)-\lambda\frac{g(a)+g(a+\eta(b,a))}{2}\right| \leqslant 2^{\frac{1-s-q}{q}}\eta^2(b,a)\left(\frac{5}{12}-\frac{\lambda}{4}\right)\left(|g''(a)|^q+|g''(b)|^q\right)^{\frac{1}{q}}. \tag{4.45}$$

To illustrate the result of Theorem 4.6 more intuitively, we offer an example here.

EXAMPLE 4.2. Considering the function $g(x)=\frac{q^2}{s^2}e^{\frac{s}{q}x}$ for $q>1$ and $s\in(0,1]$ on the interval $x\in(0,\infty)$. Then $|g''(x)|^q=e^{sx}$ is an (s,P)-preinvex function with regard to $\eta(y,x)=\frac{1}{2}y-x$ for $s\in(0,1]$.

$$\begin{aligned}\left|g''(x+t\eta(y,x))\right|^q &= e^{s(x+t(\frac{1}{2}y-x))}\\ &= e^{s(1-t)x+\frac{1}{2}sty}\\ &\leqslant (1-t)e^{sx}+te^{\frac{1}{2}sy}\\ &\leqslant (1-t)e^{sx}+te^{sy}\\ &\leqslant [(1-t)^s+t^s][e^{sx}+e^{sy}].\end{aligned}$$

If we take $a=2$, $b=6$, $\lambda=\frac{1}{2}$, $\alpha=\frac{1}{2}$, then all assumptions in Theorem 4.6 are satisfied. Clearly, $\rho=\frac{1-\alpha}{\alpha}\eta(b,a)=1$. The left part of inequality (4.40) is

$$\begin{aligned}|L(s,q)| &= \left|T_g\left(\frac{1}{2},\frac{1}{2};2,3\right)\right|\\ &= \left|\frac{q^2}{2s^2(1-e^{-1})}\left(\int_2^3\left(e^{x-3}+e^{2-x}\right)e^{\frac{sx}{q}}\mathrm{d}x\right)-\frac{q^2}{2s^2}e^{\frac{5s}{2q}}-\frac{q^2}{4s^2}\left(e^{\frac{2s}{q}}+e^{\frac{3s}{q}}\right)\right|\\ &= \left|\frac{q^2}{2s^2(1-e^{-1})}\left[\frac{q}{q+s}\left(e^{\frac{3s}{q}}-e^{\frac{2s-q}{q}}\right)+\frac{q}{s-q}\left(e^{\frac{3s-q}{q}}-e^{\frac{2s}{q}}\right)\right]\right.\\ &\quad\left.-\frac{q^2}{2s^2}e^{\frac{5s}{2q}}-\frac{q^2}{4s^2}\left(e^{\frac{2s}{q}}+e^{\frac{3s}{q}}\right)\right|.\end{aligned}$$

The right part of inequality (4.40) is identified as

$$R(s,q)=2^{\frac{1-s-q}{q}}\left(\frac{1}{8}+\frac{3e^{-1}-1}{1-e^{-1}}\right)\left(e^{2s}+e^{6s}\right)^{\frac{1}{q}}.$$

We take $s\in[0.2,1]$ and $q\in[1.1,5]$ as variables to plot $|L(s,p)|$ and $R(s,p)$ in Figure 4.2. It is obvious that $|L(s,p)|\leqslant R(s,p)$, which agrees with the result stated in Theorem 4.6.

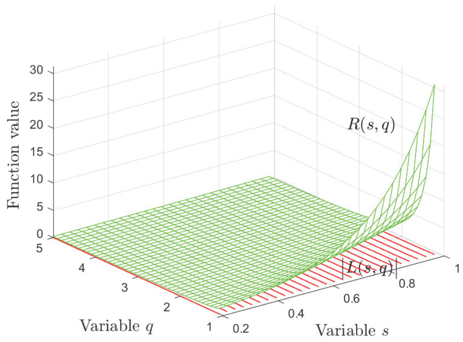

Figure 4.2: *An example to the inequality* (4.40)

5. Conclusions

In this research, we commence by proposing a novel class of (s,P)-preinvex functions along with their properties, from which we subsequently derive the fractional Hermite–Hadamard-type inequality. Taking advantage of fractional integrals with exponential kernels, we establish a comprehensive series of parameterized inequalities tailored for (s,P)-preinvex functions. In particular, the inequalities derived in this study extend and generalize previous results presented by Numan and İşcan [37], Yuan et al. [51], Zhou et al. [55], and Sarikaya et al. [41]. Furthermore, these inequalities demonstrate significant potential in addressing challenges in mathematical physics.

Building upon the parameterization methodology developed in this paper, we envision the construction of analogous inequalities through various types of multiplicative fractional integrals, including the multiplicative Riemann-Liouville fractional integrals [16], multiplicative k-Riemann–Liouville fractional integrals [53], multiplicative fractional integrals with exponential kernels [38], and multiplicative tempered fractional integrals [21]. This represents a promising and innovative direction for future research endeavors.

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