

A NOTE ON INEQUALITIES FOR GAUSSIAN HYPERGEOMETRIC FUNCTIONS

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Abstract. We establish new inequalities for Gaussian hypergeometric functions that are not zero-balanced. In the literature, such inequalities are referred to as Landen-type inequalities, as they generalize the classical Landen identity for the complete elliptic integral of the first kind. We also discuss comparisons, including numerical simulations, between our results and existing similar ones in the literature.

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REFERENCES

- [1] G. ALMKVIST AND B. BERNDT, *Gauss, Landen, Ramanujan, the arithmetic-geometric mean, ellipses, π , and the Ladies diary*, Amer. Math. Monthly **95** (1988), no. 7, 585–608, doi:10.2307/2323302, MR0966232.
- [2] Á. BARICZ, *Landen inequalities for special functions*, Proc. Amer. Math. Soc. **142** (2014), no. 9, 3059–3066, doi:10.1090/S0002-9939-2014-12016-8, MR3223362.
- [3] M. BIERNACKI, IECZYŚLAW AND J. KRZYŻ, *On the monotonicity of certain functionals in the theory of analytic functions*, Ann. Univ. Mariae Curie-Skłodowska Sect. A **9** (1955), 135–147 (1957) (English, with Russian and Polish summaries), MR0089903.
- [4] X.-Y. MA AND T.-R. HUANG, *Inequalities for Gaussian hypergeometric functions*, J. Math. Inequal. **15** (2021), no. 1, 1–8, doi:10.7153/jmi-2021-15-01, MR4364621.
- [5] F. W. J. OLVER, A. B. OLDE DAALHUIS, D. W. LOZIER, B. I. SCHNEIDER, R. F. BOISVERT, C. W. CLARK, B. R. MILLER, B. V. SAUNDERS, H. S. COHL AND M. A. MCCLAIN (eds.), *NIST Digital Library of Mathematical Functions*, <https://dlmf.nist.gov/>, Release 1.2.2 of 2024-09-15.
- [6] S.-L. QIU, X.-Y. MA AND Y.-M. CHU, *Sharp Landen transformation inequalities for hypergeometric functions, with applications*, J. Math. Anal. Appl. **474** (2019), no. 2, 1306–1337, doi:10.1016/j.jmaa.2019.02.018, MR3926168.
- [7] S.-L. QIU AND M. VUORINEN, *Landen inequalities for hypergeometric functions*, Nagoya Math. J. **154** (1999), 31–56, doi:10.1017/S0027763000025290, MR1689171.
- [8] S. SIMIĆ AND M. VUORINEN, *Landen inequalities for zero-balanced hypergeometric functions*, Abstr. Appl. Anal., posted on 2012, Art. ID 932061, 11, doi:10.1155/2012/932061, MR2898059.
- [9] M. K. WANG, Y. M. CHU AND Y. P. JIANG, *Ramanujan's cubic transformation inequalities for zero-balanced hypergeometric functions*, Rocky Mountain J. Math. **46** (2016), no. 2, 679–691, doi:10.1216/RMJ-2016-46-2-679, MR3529087.
- [10] Z.-H. YANG, Y.-M. CHU AND M.-K. WANG, *Monotonicity criterion for the quotient of power series with applications*, J. Math. Anal. Appl. **428** (2015), no. 1, 587–604, doi:10.1016/j.jmaa.2015.03.043, MR3327005.
- [11] T.-H. ZHAO, M.-K. WANG, G.-J. HAI AND Y.-M. CHU, *Landen inequalities for Gaussian hypergeometric function*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **116** (2022), no. 1, Paper No. 53, 23, doi:10.1007/s13398-021-01197-y, MR4356744.