

## A NEW PROOF OF THE SHARP WEIGHTED POWER MEAN BOUNDS FOR THE SCHWAB–BORCHARDT MEAN

TIE-HONG ZHAO AND MIAO-KUN WANG\*

**Abstract.** Let  $SB(x, y)$  be Schwab-Borchardt mean of two positive numbers  $x$  and  $y$ . In this paper, by using hypergeometric function theory and some new analytical techniques, we provide a new proof of the sharp weighted power mean bounds for the Schwab-Borchardt mean, that is, the double inequality

$$\left(\frac{1}{3}x^p + \frac{2}{3}y^p\right)^{1/p} < SB(x, y) < \left(\frac{1}{3}x^q + \frac{2}{3}y^q\right)^{1/q}$$

holds for  $0 < x < y$  if and only if  $p \leq 4/5$  and  $q \geq \log_{\pi/2}(3/2) = 0.8978 \dots$ , and it holds for  $x > y > 0$  if and only if  $p \leq 0$  and  $q \geq 4/5$ .

**Mathematics subject classification (2020):** 33E05, 26E06.

**Keywords and phrases:** Schwab-Borchardt mean, weighted power mean, Gaussian hypergeometric function, inequalities.

### REFERENCES

- [1] M. ABRAMOWITZ, I. S. STEGUN, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, U.S. Government Printing Office, Washington, 1964.
- [2] M. BIERNACKI, J. KRZYZ, *On the monotonicity of certain functionals in the theory of analytic functions*, Ann. Univ. Mariae Curie-Skłodowska Sect. A, **9** (1955): 135–147.
- [3] B. C. CARLSON, *Algorithms involving arithmetic and geometric means*, Amer. Math. Monthly, **78** (1971): 496–505.
- [4] B. C. CARLSON, *The logarithmic mean*, Amer. Math. Monthly, **79** (1972): 615–618.
- [5] B. C. CARLSON, *Special Functions of Applied Mathematics*, Academic Press, New York, 1977.
- [6] Y. M. CHU, M. K. WANG, W. M. GONG, *Two sharp double inequalities for Seiffert mean*, J. Inequal. Appl., **2011** (2011), Paper No. 44, 7 pages.
- [7] E. B. LEACH, M. C. SHOLANDER, *Extended mean values II*, J. Math. Anal. Appl., **92** (1983): 207–223.
- [8] E. NEUMAN, *On some means derived from the Schwab-Borchardt mean*, J. Math. Inequal., **8**, 1 (2014): 171–183.
- [9] E. NEUMAN, *On some means derived from the Schwab-Borchardt mean II*, J. Math. Inequal., **8**, 2 (2014): 359–368.
- [10] E. NEUMAN, *A note on a certain bivariate mean*, J. Math. Inequal., **6**, 4 (2012): 637–643.
- [11] E. NEUMAN, J. SÁNDOR, *On the Schwab-Borchardt mean*, Math. Pannonica, **14**, 2 (2003): 253–266.
- [12] S. PONNUSAMY, M. VUORINEN, *Asymptotic expansions and inequalities for hypergeometric functions*, Mathematika, **44**, 2 (1997): 278–301.
- [13] F. QI, W. H. LI, *A logarithmically completely monotonic function involving the ratio of gamma functions*, J. Appl. Anal. Comput., **5**, 4 (2015): 626–634.
- [14] E. D. RAINVILLE, *Special functions*, Chelsea Publishing Company, New York, 1960.
- [15] J. SÁNDOR, *On certain inequalities for means III*, Arch. Math., **76**, 1 (2001), 34–40.
- [16] H. J. SEIFFERT, *Problem 887*, Nieuw. Arch. Wisk., **11** (1993): 176.
- [17] H. J. SEIFFERT, *Aufgabe 16*, Würzel, **29** (1995): 87.

- [18] Z. H. YANG, *A new way to prove L'Hospital monotone rules with applications*, preprint, arXiv:1409.6408, 2014, 19 pages, <https://doi.org/10.48550/arXiv.1409.6408>.
- [19] Z. H. YANG, *Recurrence relations of coefficients involving hypergeometric function with an application*, preprint, arXiv:2204.04709, 2022, 17 pages, <https://doi.org/10.48550/arXiv.2204.04709>.
- [20] Z. H. YANG, Y. M. CHU, M. K. WANG, *Monotonicity criterion for the quotient of power series with applications*, J. Math. Anal. Appl., **428**, 1 (2015): 587–604.
- [21] Z. H. YANG, J. F. TIAN, M. K. WANG, *A positive answer to Bhatia-Li conjecture on the monotonicity for a new mean in its parameter*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat., **114**, 3 (2020), Paper No. 126, 22 pp.
- [22] J. F. TIAN, Z. H. YANG, M. H. HA, H. J. XING, *A family of high order approximations of Ramanujan type for perimeter of an ellipse*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat., **115**, 2 (2021), Paper No. 85, 20 pp.
- [23] Z. H. YANG, *Sharp bounds for Seiffert mean in terms of weighted power means of arithmetic mean and geometric mean*, Math. Inequal. Appl., **17**, 2 (2014): 499–511.
- [24] Z. H. YANG, J. F. TIAN, *Sharp inequalities for the generalized elliptic integrals of the first kind*, Ramanujan J., **48**, 1 (2019): 91–116.
- [25] T. H. ZHAO, M. K. WANG, *Sharp bounds for the lemniscatic mean by the weighted Hölder mean*, Rev. Real Acad. Cienc. Exactas Fís. Nat. Ser. A-Mat., **117**, 3 (2023), Paper No. 96, 19 pages.
- [26] L. ZHU, *A source of inequalities for circular functions*, Comput. Math. Appl., **58**, 10 (2009): 1998–2004.
- [27] L. ZHU, *Inequalities for hyperbolic functions and their applications*, J. Inequal. Appl., **2010** (2010), Art. ID 130821, 10 pages.