

INTEGRAL FORMS AND FUNCTIONAL BOUNDS FOR CERTAIN EXTENDED EXTON'S DOUBLE HYPERGEOMETRIC FUNCTIONS

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Abstract. We extend and systematically investigate some particular Exton's double hypergeometric function $X_{C:D:D'}^{A:B:B'}[x, y]$, which is motivated by the recent integrated version of the Euler's Beta integral form with a Macdonald function $K_\nu(z)$ in the integrand. The newly introduced extended Exton's double hypergeometric functions $X_{C:D:D'}^{A:B:B'}[x, y; p, q, \nu, \lambda]$ is then represented by a number of integral representations of the Euler and Laplace types, including several further representations involving Bessel $J_\nu(z)$ and modified Bessel functions $I_\nu(z)$ of the first kind along with recurrence formulae. Using existing functional bounds for extended Euler's Beta function, various functional upper bounds are derived for particular extended Exton's double hypergeometric functions $X_{C:D:D'}^{A:B:B'}[x, y; p, q, \nu, \lambda]$. Also, plethora of bounding inequalities are established by virtue of Luke's, von Lommel's, Minakshisundaram and Szász and Olenko's bounds. The exposition ends with a newly introduced probability distribution applying extended Kummer and of Horn functions, for which moment inequalities of Turán type are proved.

Mathematics subject classification (2020): Primary 26D15, 33C20, 33C65; Secondary 33C70, 33C99, 60E05.

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REFERENCES

- [1] L. EULER, *De serie Lambertiana plurimisque eius insignibus proprietatibus*, Acta Academiae Scientiarum Imperialis Petropolitanae II, (1779–1783), 29–51; reprinted in *Opera Omnia Ser. I*, vol. 6. (Teubner, Leipzig, 1921), 350–369, (in Latin).
- [2] H. EXTON, *Multiple Hypergeometric Functions and Applications*, Halsted Press, Ellis Horwood, Chichester, New York, NY, USA; London, UK; Sydney, Australia; Toronto, QN, Canada, 1976.
- [3] H. EXTON, *Reducible double hypergeometric functions and associated integrals*, Anais da Faculdade de Ciências. Universidade do Porto **LXIII** (1982), no. 1–4, 137–143.
- [4] K. GÓRSKA, A. HORZELA, D. JANKOV MAŠIREVIĆ AND T. K. POGÁNY, *Observations on the McKay I_ν Bessel distribution*, J. Math. Anal. Appl. **516** (2022), Article No. 126481.
- [5] J. HORN, *Hypergeometrische Funktionen zweier Veränderlichen*, Math. Ann. **105** (1931), no. 1, 381–407.
- [6] E. K. IFANTIS AND P. D. SIAFARIKAS, *Inequalities involving Bessel and modified Bessel functions*, J. Math. Anal. Appl. **147** (1990), 214–227.
- [7] D. JANKOV MAŠIREVIĆ, *On new formulae for the cumulative distribution function of the noncentral Chi-square distribution*, Mediterr. J. Math. **14** (2017), Article No. 66.
- [8] D. JANKOV MAŠIREVIĆ AND T. K. POGÁNY, *On new formulae for cumulative distribution function for McKay Bessel distribution*, Comm. Statist. Theory Methods **50** (2021), no. 1, 143–160.
- [9] D. JANKOV MAŠIREVIĆ AND T. K. POGÁNY, *CDF of non-central χ^2 distribution revisited. Incomplete hypergeometric type functions approach*, Indag. Math. **32** (2021), no. 4, 901–915.
- [10] D. JANKOV MAŠIREVIĆ AND T. K. POGÁNY, *Functional bounds for Exton's double hypergeometric X function*, J. Math. Inequal. **17** (2023), no. 1, 259–267.

- [11] D. JANKOV MAŠIREVIĆ, T. K. POGÁNY AND N. UJIĆ, *Observations on the the McKay I_ν Bessel distribution II*, J. Math. Anal. Appl. **550** (2025), no. 2, Article No. 129569.
- [12] C. KRATTENTHALER AND K. SRINIVASA RAO, *Automatic generation of hypergeometric identities by the beta integral method*, in R. JAGANNATHAN, S. KANEMITSU, G. VAN DEN BERGHE AND V. VAN ASSCHE (Eds.) *Proceedings of the International Conference on Special Functions and their Applications* (Chennai, 2002); J. Comput. Appl. Math. **160** (2003), (1–2), 159–173.
- [13] L. LANDAU, *Monotonicity and bounds on Bessel functions*, in *Proceedings of the Symposium on Mathematical Physics and Quantum Field Theory* (Berkeley, CA, 1999), 147–154; Electron. J. Differ. Equ. Conf. **4** (Southwest Texas State Univ., San Marcos, TX, 2000).
- [14] E. C. J. VON LOMMEL, *Die Beugungserscheinungen einer kreisrunden Öffnung und eines kreisrunden Schirmchens theoretisch und experimentell bearbeitet*, Abh. der math. phys. Classe der k. b. Akad. der Wiss. (München) **15** (1884–1886), 229–328.
- [15] E. C. J. VON LOMMEL, *Die Beugungserscheinungen geradlinig begrenzter Schirme*, Abh. der math. phys. Classe der k. b. Akad. der Wiss. (München) **15** (1884–1886), 529–664.
- [16] E. LUKACS, *Characteristic Functions*, translated from the English and with a preface by Zolotarev. (V. M. Nauka, Moscow, 1979) (in Russian).
- [17] Y. L. LUKE, *Inequalities for generalized hypergeometric functions*, J. Approx. Theory **5** (1974), no. 1, 41–65.
- [18] A. M. MATHAI AND H. J. HAUBOLD, *Special Functions for Applied Scientists*, Springer, New York, NY, 2008.
- [19] G. V. MILOVANOVIĆ, R. K. PARMAR AND T. K. POGÁNY, *Bounds for the (p, v) -extended Beta function and certain consequences*, J. Math. Ineq. **17** (2023), no. 4, 1433–1441.
- [20] S. MINAKSHISUNDARAM AND O. SZÁSZ, *On absolute convergence of multiple Fourier series*, Trans. Amer. Math. Soc. **61** (1947), no. 1, 36–53.
- [21] A. YA. OLENKO, *Upper bound on $\sqrt{x}J_\nu(x)$ and its applications*, Integral Transforms Spec. Funct. **17** (2006), no. 6, 455–467.
- [22] F. W. J. OLVER, D. W. LOZIER, R. F. BOISVERT AND C. W. CLARK (Eds.), *NIST Handbook of Mathematical Functions*, Cambridge University Press, Cambridge, 2010.
- [23] R. K. PARMAR, R. B. PARIS AND P. CHOPRA, *On an extension of extended beta and hypergeometric functions*, J. Classical Anal. **11** (2017), no. 1, 91–106.
- [24] R. K. PARMAR AND T. K. POGÁNY, *Bounds for novel extended beta and hypergeometric functions and related results*, J. Inequal. Appl. **2024** (2024), Article No. 77.
- [25] R. K. PARMAR, T. K. POGÁNY AND S. PIRIVINA, *Generalized Horn function $H_{4,p,q,\nu}^\lambda$ and related bounding inequalities with applications to statistics*, Rad Hrvat. Akad. Znan. Umjet. Mat. Znan. (2025) (to appear).
- [26] R. K. PARMAR, T. K. POGÁNY AND L. TEOFANOV, *On extended beta function and related inequalities*, Mathematics (Basel) **12** (2024), Article No. 2709.
- [27] T. K. POGÁNY, *Functional inequalities for Bessel and hypergeometric type functions via probabilistic approach*, Math. Newsl. **35** (2024), no. 1, 1–16.
- [28] L. J. SLATER, *Confluent Hypergeometric Functions*, Cambridge University Press, Cambridge, London, and New York, 1960.
- [29] H. M. SRIVASTAVA AND M. C. DAOUST, *A note on the convergence of Kampé de Fériet double hypergeometric series*, Math. Nachr. **53** (1972), 151–159.
- [30] H. M. SRIVASTAVA AND P. W. KARLSSON, *Multiple Gaussian Hypergeometric Series*, Halsted Press – Ellis Horwood Limited, Chichester, John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1985.
- [31] G. N. WATSON, *A Treatise on the Theory of Bessel Functions*, second edition, Cambridge University Press, Cambridge, London and New York 1944.